ON CURVES OF CONSTANT WIDTH DUE TO THE BISHOP FRAME OF TYPE-2 IN DUAL EUCLIDEAN SPACE D^3

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Abstract

In this work, we study dual constant width curves due to the Bishop frame of type-2 in dual space D^3. We obtain a differential equation which characterizes these curves in D^3. For special solutions of this differential equation system, we obtain some results in D^3.

Keywords: Dual curves, dual constant width curves, dual Bishop frame of type-2, dual Euclidean space.

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D^3 DUAL ÖKLİDYEN UZAYDA 2. ÇEŞİT BISHOP ÇATISINA GÖRE SABİT GENİŞLİKLİ DUAL EĞRİLER

Özet

Bu çalışmada, D^3 dual Öklid uzayında 2. tip dual Bishop çatısına göre sabit genişlikli dual eğrileri inceliyoruz. D^3 de bu tip eğrileri karakterize eden bir diferensiyel denklem elde diyoruz. D^3 de bu diferensiyel denklem sisteminin çözümü için, bazı sonuçlar elde ediyoruz.

Anahtar Kelimeler: Dual eğriler, Sabit genişlikli dual eğriler, 2. tip dual Bishop çatısı, Dual Öklid uzayı

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1. INTRODUCTION

The introduction of dual numbers were proposed by William Kingdon Clifford as the results of his geometrical researches. Then dual numbers and vectors had been used on line geometry and kinematics by Eduard Study. Some special curves have been studied in Euclidean space and its ambient spaces such as Lorentzian, and Galilean spaces, (see [2], [6], [7], [9]).

Fujivara obtained the solution of problem determining whether or not constant width curves are possible for space curves, and he define the concept "width" of space curves on a constant width surface, (see [2], [3], [4], [5], [10], [11], [12], [13], [14], [15]).

It is also observed that special curves such as spherical curves, Bertrand curves, spherical indicatrices, curves of constant breadth, involutes and evolutes are studied by obtaining solutions of special differential equations characterizing them, (see, [5], [6], [7], [9], [12]). Such curves have been intensively researched, for details (see, [5], [6], [12]).

Bishop frame put forward as alternative frame of curves had been offered by L.R. Bishop in 1975 by using parallel transporting vector fields, see, [1]. Later, there are lots of papers concerning this concept of frame. In general sense, for the researches in Euclidean and Minkowski spaces, one can look at [1], [8], and also for the treatises in dual space, see, [9].

In this paper, using the vector fields known as dual tangent, normal, and binormal vectors of Frenet-Serret frame, we give dual Bishop frame of type-2 of regular dual curves in D^3. Thereafter we characterize dual constant width curves due to the mentioned frame in D^3. We also give some properties of dual constant width curves due to that frame in D^3.

2. PRELIMINARIES

Let E^3 be 3-dimensional Euclidean space, that is, 3-dimensional real vector space E^3 with the metric

$$\langle dx, dx\rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in E^3. An arbitrary vector x of E^3 is said to be $\langle x, x\rangle > 0$ or $x = 0$. For $x \in E^3$ the norm is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ where the vector x is called a positive definite.

Dual numbers are given by the set

$$D=\left\{\hat{x}=x+\varepsilon x^{*} \,\middle|\, x,x^{*}\in E\right\},$$

where the symbol ε designates the dual unit with the property $\varepsilon^{k}=0$ for $\varepsilon\neq 0$. Dual angle is defined as $\hat{\theta}=\theta+\varepsilon\theta^{*}$, where θ is the projected angle between two spears and θ^{*} is the shortest distantce between them. The set D is a commutative ring under the operations $(+)$ and $(\cdot)$ [7].

The set

$$D^{3}=D\times D\times D=\left\{\hat{\vartheta}=\vartheta+\varepsilon\vartheta^{*} \,\middle|\, \vartheta,\vartheta^{*}\in E^{3}\right\}$$

is a module over the ring D [7].

For any dual vectors $\hat{a}=a+\varepsilon a^{*},\hat{b}=+\varepsilon b^{*}\in D^{3}$ the Euclidean inner product of $\hat{a}$ and $\hat{b}$ is defined by

$$\left\langle \hat{a},\hat{b}\right\rangle=\langle a,b\rangle+\varepsilon\left(\left\langle a^{*},b\right\rangle+\left\langle a,b^{*}\right\rangle\right),$$

thus the dual Euclidean space is the dual space D^{3} together with Euclidean inner product, and denoted by D^{3}, and also for $\hat{\vartheta}\neq 0$ the norm is defined as

$$\left\|\hat{\vartheta}\right\|=\sqrt{\left\langle \hat{\vartheta},\hat{\vartheta}\right\rangle}.$$

3. MAIN RESULT

In this section, we study constant width curves due to dual Bishop frame of type-2 in D^{3}. It is also shown that dual curves of constant width are dual slant helix in some special cases due to dual Bishop frame of type-2 in D^{3}.

Let $\hat{\alpha}=\hat{\alpha}(s)$ be a dual unit speed regular curve in D^{3}. The dual Bishop frame of type-2 formula of the dual curve $\hat{\alpha}(s)$ is defined by

$$\begin{bmatrix} \hat{\xi}_1' \\ \hat{\xi}_2' \\ \hat{B}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\hat{\epsilon}_1 \\ 0 & 0 & -\hat{\epsilon}_2 \\ \hat{\epsilon}_1 & \hat{\epsilon}_2 & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{B} \end{bmatrix}. \tag{1}$$

The relation matrix between dual Frenet frame and dual Bishop frame of type-2 is given as

$$\begin{bmatrix} \hat{T} \\ \hat{N} \\ \hat{B} \end{bmatrix} = \begin{bmatrix} \sin\hat{\theta}(s) & -\cos\hat{\theta}(s) & 0 \\ \cos\hat{\theta}(s) & \sin\hat{\theta}(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{B} \end{bmatrix}, \tag{2}$$

here, the dual Bishop curvatures of type-2 are given by

$$\hat{\epsilon}_1(s) = -\hat{\tau}\cos\hat{\theta}(s), \quad \hat{\epsilon}_2(s) = -\hat{\tau}\sin\hat{\theta}(s), \tag{3}$$

where $\hat{\tau}$ is dual Frenet torsion. Also it can be deduced that dual Frenet curvature is as follows

$$\hat{\theta}' = \hat{\kappa} = \frac{\left(\dfrac{\hat{\epsilon}_2}{\hat{\epsilon}_1}\right)'}{1+\left(\dfrac{\hat{\epsilon}_2}{\hat{\epsilon}_1}\right)^2} \tag{4}$$

in terms of dual Bishop curvature of type-2.

The frame $\left\{\hat{\xi}_1, \hat{\xi}_2, \hat{B}\right\}$ is properly oriented, and the dual angle is also

$$\hat{\theta}(s) = \int_0^s \hat{\kappa}(s)ds.$$

Definition 3.1. Let $(\hat{C}_1)$ be a dual curve due to dual Bishop frame of type-2 in D^3. If $(\hat{C}_1)$ has parallel tangents in the reverse directions at the corresponding points $\hat{\alpha}(s)$ and $\hat{\alpha}^*(s^*)$ and the distance between the points remains always constant, then $(\hat{C}_1)$ is said to be a dual constant width curve due to dual Bishop frame of type-2 in D^3.

A simple closed dual constant width curve is represented due to dual Bishop frame of type-2 in D^3 can be written as

$$\hat{\alpha}^*(s^*) = \hat{\alpha}(s) + \hat{m}_1(s)\hat{\xi}_1(s) + \hat{m}_2(s)\hat{\xi}_2(s) + \hat{m}_3\hat{B}, \tag{5}$$

where $\hat{m}_1(s), \hat{m}_2(s), \hat{m}_3$ are arbitrary functions of s. Differentiating (5) gives

$$\hat{\xi}_1^* \frac{ds^*}{ds} = (1 + \hat{m}_1' + \hat{\in}_1 \hat{m}_3)\hat{\xi}_1 + (\hat{m}_2' + \in_2 \hat{m}_3)\hat{\xi}_2 + (-\hat{\in}_1 \hat{m}_1 - \hat{\in}_2 m_2 + m_3')\hat{B}. \tag{6}$$

Considering $\hat{T}^* = -\hat{T}$ by Definition 3.1, we obtain the system of differential equations as follows

$$\begin{aligned}
\frac{d\hat{m}_1}{ds} &= -\frac{ds^*}{ds} - \hat{m}_3 \hat{\in}_1 - 1, \\
\frac{d\hat{m}_2}{ds} &= -\hat{\in}_2 \hat{m}_3, \\
\frac{d\hat{m}_3}{ds} &= \hat{\in}_1 \hat{m}_1 + \hat{\in}_2 \hat{m}_2 .
\end{aligned} \tag{7}$$

Let us denote the angle between the tangent of curve at the point $\hat{\alpha}(s)$ and a given direction as $\hat{\varphi}$, and also take

$$\hat{\theta}' = \hat{\kappa} = \frac{\left(\dfrac{\hat{\in}_2}{\hat{\in}_1}\right)'}{1 + \left(\dfrac{\hat{\in}_2}{\hat{\in}_1}\right)^2}, \text{ and } \hat{\theta}^* = \hat{\kappa}^* = \frac{\left(\dfrac{\hat{\in}_2^*}{\hat{\in}_1^*}\right)'}{1 + \left(\dfrac{\hat{\in}_2^*}{\hat{\in}_1^*}\right)^2}$$

into consideration, the equation (7) turns into

$$\frac{d\hat{m}_1}{d\hat{\varphi}} = -f(\hat{\varphi}) - \hat{\in}_1 \hat{\rho}\hat{m}_3,$$
$$\frac{d\hat{m}_2}{d\hat{\varphi}} = -\hat{\in}_2 \hat{\rho}\hat{m}_3, \tag{8}$$
$$\frac{d\hat{m}_3}{d\hat{\varphi}} = \hat{\in}_1 \hat{\rho}\hat{m}_1 + \hat{\in}_2 \hat{\rho}\hat{m}_2,$$

where $f(\hat{\varphi}) = \hat{\rho} + \hat{\rho}^* \,;\; \hat{\rho} = \frac{1}{\hat{\kappa}},\; \hat{\rho}^* = \frac{1}{\hat{\kappa}^*}$.

Using $\hat{\rho} = \frac{1}{\hat{\kappa}}$ and the system (8) we obtain the dual differential equation of third order as follows

$$\frac{d^3\hat{m}_1}{d\hat{\varphi}^3} = -f''(\hat{\varphi}) + \left\{-\hat{\in}_1''\hat{\rho} - 2(\hat{\in}_1\hat{\rho}')'\right\}\hat{m}_3 - 2(\hat{\in}_1\hat{\rho})'\hat{m}_3'. \tag{9}$$

Corollary 3.1. The third order differential equation in (9) is a characterization of the simple closed curve $\hat{\alpha}$ due to dual Bishop frame of type-2 in D^3.

Since position vetor of a simple closed curve is determined by solution of (9), let us examine the solution of the equation (9) within the special cases as follows. Let $\hat{\in}_1$ and $f(\hat{\varphi})$ be a constants, then the equation (9) has the from

$$\frac{d^3\hat{m}_1}{d\hat{\varphi}^3} = 0. \tag{10}$$

Solution of equation (10) yields the components

$$\hat{m}_1 = \hat{A}_1 + \hat{A}_2\hat{\varphi} + \hat{A}_3\hat{\varphi}^2,$$
$$\hat{m}_2 = \int \frac{\hat{\in}_2}{\hat{\in}_1}\frac{d}{d\hat{\varphi}}(\hat{A}_1 + \hat{A}_2\hat{\varphi} + \hat{A}_3\hat{\varphi}^2)d\hat{\varphi}, \tag{11}$$

$$\hat{m}_3 = \left[\frac{d}{d\hat{\varphi}} (\hat{A}_1 + \hat{A}_2 \hat{\varphi} + \hat{A}_3 \hat{\varphi}^2) + f(\hat{\varphi}) \right] . \frac{-1}{\hat{\in}_1 \hat{\rho}},$$

where $\hat{A}_1$, $\hat{A}_2$, and $\hat{A}_3$ are constant dual numbers.

Corollary 3.2. The position vector of a simple closed dual constant width curve with constant curvature and constant torsion is found as

$$\hat{\alpha}^*(s^*) = \hat{\alpha}(s) + (\hat{A}_1 + \hat{A}_2 \hat{\varphi}(s) + \hat{A}_3 \hat{\varphi}^2(s)) \hat{\xi}_1(s) + \left(\int \frac{\hat{\in}_2}{\hat{\in}_1} \frac{d}{d\hat{\varphi}} (\hat{A}_1 + \hat{A}_2 \hat{\varphi} + \hat{A}_3 \hat{\varphi}^2) d\hat{\varphi}\right) \hat{\xi}_2(s)$$
$$+ \left(\left[\frac{d}{d\hat{\varphi}} (\hat{A}_1 + \hat{A}_2 \hat{\varphi} + \hat{A}_3 \hat{\varphi}^2) + f(\hat{\varphi}) \right] . \frac{-1}{\hat{\in}_1 \hat{\rho}} \right) \hat{B}$$

in terms of the values of $\hat{m}_1$, $\hat{m}_2$, and $\hat{m}_3$ in the equation (11).

Given the distance between opposite points of $\hat{\alpha}^*$, $\hat{\alpha}$ be constant, then we can write that

$$\left\| \hat{\alpha}^* - \hat{\alpha} \right\| = \hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2 = \text{constant}. \quad (12)$$

Differentiating (12) with respect to $\hat{\varphi}$

$$\hat{m}_1 \frac{d\hat{m}_1}{d\hat{\varphi}} + \hat{m}_2 \frac{d\hat{m}_2}{d\hat{\varphi}} + \hat{m}_3 \frac{d\hat{m}_3}{d\hat{\varphi}} = 0. \quad (13)$$

By virture of (8), the differential equation (14) yields

$$\hat{m}_1 \left(\frac{d\hat{m}_1}{d\hat{\varphi}} + \hat{m}_3 \hat{\in}_1 \hat{\rho}\right) = 0. \quad (14)$$

There are two cases for the equation (14), so we study these cases as follows :

Case 1: If $\hat{m}_1 = 0$, then we have the components

$$\hat{m}_2 = \int \left(\frac{\hat{\in}_2}{\hat{\in}_1} f(\hat{\varphi}) \right) d\varphi \ , \ \hat{m}_3 = \frac{-f(\hat{\varphi})}{\hat{\in}_1 \hat{\rho}}. \tag{15}$$

Using the values of (15) in (5), we have the invariant of dual curves of constant width as

$$\hat{\alpha}^* = \hat{\alpha} + \left[\int \left(\frac{\hat{\in}_2}{\hat{\in}_1} f(\hat{\varphi}) d\hat{\varphi} \right) \right] \xi_2 - \frac{f(\hat{\varphi})}{\hat{\in}_1 \hat{\rho}} \xi_3. \tag{16}$$

<u>Case 2:</u> If $\dfrac{d\hat{m}_1}{d\hat{\varphi}} = -\hat{m}_3 \hat{\in}_1 \hat{\rho}$, that is, $f(\hat{\varphi}) = 0$, then a relation among radii of curvatures is obtaine as $\dfrac{1}{\hat{\kappa}} + \dfrac{1}{\hat{\kappa}^*} = 0$.

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