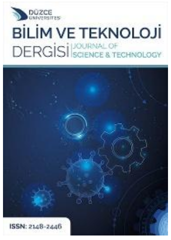

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Research paper

Numerical Solutions of Fractional Order Ratio-Dependent Food Chain Model with Caputo Derivative

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ABSTRACT

The study of life within an ecosystem reveals a complex system. Ecosystems are characterised by the presence of all the elements that give rise to chaotic dynamics. Although chaos is often predicted by mathematical models, there is currently only limited evidence of its existence in nature, with the proof of its occurrence remaining scarce and uncertain. Despite the apparent simplicity of food chains, they exhibit highly complex dynamics. Models created several years ago have confirmed that food chains have complex dynamics. In this study, a fractional order ratio-dependent food chain model is considered. This model consists of three compartments: prey population density (X), predator population density (Y) and top predator density (Z). The fractional derivative is employed in accordance with the Caputo sense. A mathematical analysis of the fractional order ratio-dependent food chain model is conducted. Numerical results are obtained with the aid of the Euler method and the graphs are interpreted.

Keywords:*Fractional Order Food Chain Model, Mathematical Modeling, Euler Method, Caputo Derivative*

I. INTRODUCTION

The term "ratio-dependent predation" was first coined by (Arditi & Ginzburg, 1989) to describe situations in which the predator's feeding rate (functional response) depends not only on prey density, as in classical models, but also on the ratio of state variables, namely prey and predator density. One advantage of ratio-dependent models is that they do not have the paradoxes of biological control and enrichment predicted by classical models. Experimental investigations by (Berryman et al., 1995) showed that prey-dependent models are valid in homogeneous situations, while rate-dependent models are valid in heterogeneous situations. Building on their work, (Ginzburg & Akçakaya, 1992) and (McCarthy et al., 1995) concluded that systems in nature are closer to ratio dependence than prey dependence.

Analyses by Hsu, Hwang and Kuang show that ratio-dependent models can produce much richer and more biologically realistic dynamics. In particular, these models will not produce the paradoxes of enrichment and biological control. Moreover, ratio-dependent models allow for bilateral extinction as a possible outcome of a given predator-prey interaction (Hsu et al., 2001). The primary aim of mathematical modelling is to mathematically represent real-life issues and their mechanisms. It is essential to have the ability to govern the modelled processes. Mathematical models have been crafted to illuminate underlying systems, examine the impact of various components, and anticipate their outcomes. Mathematical modelling is employed not only for epidemics, but also for a diverse range of dynamic models (Podlubny, 1999).

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Fractional derivative models provide better results in control theory of dynamic systems with various physical and biological processes compared to integer derivative models. The use of fractional operators is particularly appropriate for explaining the memory and hereditary properties of many substances and processes, as such properties are ignored in integer derivatives. The future state of a population in population models is dependent on its past state. This is referred to as the memory effect. By adding a delay term or using fractional derivatives in the model, the memory effect of the population can be analyzed (Linda, 2007; Hethcote et al., 2002; Brauer et al., 2008; Joy, 1997; Hsu et al., 2003; Öztürk et al., 2003a; Ws et al., 2012). Mathematical problems, including fractional differential equations, play a significant role in analysis and modelling. Equations play a significant role in the analysis and modelling of various scientific processes, such as damping laws, electrical circuits, fluid mechanics and relaxation processes, since the fractional derivative is a non-local operator (Öztürk et al., 2023b; Kermack & Mckendrick, 1927; Yaro et al., 2015; Hsu et al., 2003; Öztürk et al., 2022; Hastings & Powell, 1991; Kara & Can, 2006; Öztürk et al., 2024a; Hastings & Klebanoff, 1994; Gakkar & Naji, 2003; Öztürk et al., 2024b; Sevindir et al., 2021; Çetinkaya & Demir, 2021; Çetinkaya et al., 2021; Çetinkaya & Demir, 2023).

This paper consists of four parts. In the first part, the importance of fractional order mathematical modelling and information about the food chain are given. In the second part, the formation of the fractional order food chain model is given. The system is mathematically analysed and the Generalised Euler Method is given. In the third section, a new application of the fractional order food chain model is made and numerical results are obtained and graphs are interpreted. In the fourth section, conclusions are given.

II. FRACTIONAL DERIVATIVE AND FRACTIONAL ORDER FOOD CHAIN MODEL

A. *Fractional Derivative*

In this study, Caputo derivative operator is preferred and modelled since it is easily applicable to classical initial conditions and provides ease of calculation. The definition of Caputo fractional derivative is given.

A.1. Definition.

Let $f(t)$ be a function that can be continuously differentiable n times. The value of the function $f(t)$ for the value of α that satisfies the condition $n-1<\alpha<n$. The Caputo fractional derivative of α-th order $f(t)$ is defined ${}_{a}^{C}Dt^{\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_{\alpha}^{t}(t-x)^{n-\alpha-1}f^{n}(x)dx$ (Podlubny, 1999).

B. *Fractional Order Food Chain Model*

The fractional order food chain model consists of three compartments. The first is the prey population density (X), the second is the predator population density (Y) and the third is the top predator population density (Z). The expression of the model as a system of fractional differential equations is as follows.

$$\frac{d^{\alpha}X}{dt^{\alpha}}=rX(1-\frac{X}{K})-\frac{1}{\mu_1}\frac{m_1XY}{a_1Y+X}$$

$$\frac{d^{\alpha}Y}{dt^{\alpha}}=\frac{m_1XY}{a_1Y+X}-d_1Y-\frac{1}{\mu_2}\frac{m_2YZ}{a_2Z+Y} \qquad (1)$$

$$\frac{d^{\alpha}Z}{dt^{\alpha}}=\frac{m_2YZ}{a_2Z+Y}-d_2Z.$$

Here $\frac{d^{\alpha}}{dt^{\alpha}}$ is the Caputo fractional derivative of α-th order with respect to time t. All compartments and parameters are shown in Table 1 and Table 2. The initial values are defined as,

$X(0)=X_0$, $Y(0)=Y_0$, $Z(0)=Z_0$, $0<\alpha\leq 1$.

Since fractional order models have a memory feature in time-dependent events, they produce more realistic and accurate results than integer order models. For this reason, the model was constructed as fractional order (Öztürk et al., 2023a; Kermack & Mckendrick, 1927; Yaro et al., 2015; Hsu et al., 2003; Öztürk et al., 2022; Hastings & Powell, 1991; Kara & Can, 2006; Öztürk et al., 2024a; Hastings & Klebanoff, 1994; Gakkar & Naji, 2003; Öztürk et al., 2024b; Sevindir et al., 2021; Çetinkaya & Demir, 2021; Çetinkaya et al., 2021; Çetinkaya & Demir, 2023). By taking $\alpha = 1$ in this system (1), the fractional order differential equation is reduced to a full order differential equation. All computations and parameters are shown in Table 1 and Table 2.

Table 1. Variables used in the model and their meanings

Variables used in the systems	**Meaning**
$X(t)$	Prey population density at time t
$Y(t)$	Predator population density at time t
$Z(t)$	Top predator population density at time t

Table 2. Parameters and their meanings

Parameters	**Meaning**
r	Prey growth rate
K	Transport capacity
μ_1, μ_2	Product constant
m_1, m_2	Maximal growth rate of the predator
a_1, a_2	Half saturation constant
d_1, d_2	Hunter mortality rate

The simple relationship between these three species is as follows: Z feeds only on Y and Y feeds on X, thus producing a simple food chain. A distinctive feature of this simple food chain is that it has a domino effect, which means that when one of the species in the chain dies out, species at a higher tropical level also die out. To make the system simpler, the system will be dimensionlessized with the following ratios.

$t \to rt, X \to \frac{X}{K}, Y \to \frac{a_1}{K}Y, Z \to \frac{a_z a_1}{K}Z, m_1 \to \frac{m_1}{r}, m_2 \to \frac{m_2}{r}, d_1 \to \frac{d_1}{r}, d_2 \to \frac{d_2}{r}$

The new form of the fractional order food chain model is written as follows.

$$
\begin{aligned}
D^\alpha x(t) &= x(1-x) - \frac{c_1 xy}{x+y} \\
D^\alpha y(t) &= \frac{m_1 xy}{x+y} - d_1 y - \frac{c_2 yz}{y+z} \qquad (2) \\
D^\alpha z(t) &= \frac{m_2 yz}{y+z} - d_2 z.
\end{aligned}
$$

Here $c_1 = \frac{m_1}{\mu_{1a_1r}}, c_2 = \frac{m_2}{\mu_{2a_2r}}$.

C. Existence, Uniqueness and Non-Negativity of the System

We investigate the existence and uniqueness of the solutions of the fractional-order system (2) in the region $B \times [t_0, T]$ where

B={$(x, y, z) \in R_{+}^{3}$:max{ $|x|, |y|, |z|$ }$\leq \varphi$,min{ $|x|, |y|, |z|$ }$\geq \varphi_0$} (3)

and $T < +\infty$.

C.1.Theorem

For each H_0= $(x_0 , y_0 , z_0) \in B$ there exists a unique solution $x(t) \in B$ of the fractional-order system (2) with initial condition x_0, which is defined for all $t \geq 0$.

Proof: We denote $H = (x, y, z)$ and $\bar{H} = (\bar{x}, \bar{y}, \bar{z})$. Consider a mapping

$M(H) = (M_1(H), M_2(H), M_3(H))$ and

$$M_1(H) = x(1-x) - \frac{c_1 xy}{x+y}$$

$$M_2(H) = \frac{m_1 xy}{x+y} - d_1 y - \frac{c_2 yz}{y+z} \qquad (4)$$

$$M_3(H) = \frac{m_2 yz}{y+z} - d_2 z.$$

For any $H, \bar{H} \in B$ it follows from equation (4) that

$$\|M(H) - M(\bar{H})\| = |M_1(H) - M_1(\bar{H})| + |M_2(H) - M_2(\bar{H})| + |M_3(H) - M_3(\bar{H})| \qquad (5)$$

$$|M_1(H) - M_1(\bar{H})| = \left| x(1-x) - \frac{c_1 xy}{x+y} - \bar{x}(1-\bar{x}) + \frac{c_1 \bar{x}\bar{y}}{\bar{x}+\bar{y}} \right|$$

$$= \left| (x-\bar{x}) - (x^2 - \bar{x}^2) - c_1\left(\frac{xy}{x+y} - \frac{\bar{x}\bar{y}}{\bar{x}+\bar{y}}\right) \right|$$

$$\leq | x - \bar{x}| + 2\varphi| x - \bar{x}| + c_1| x - \bar{x}| + c_1| y - \bar{y}|$$

$$|M_2(H) - M_2(\bar{H})| = \left| \frac{m_1 xy}{x+y} - d_1 y - \frac{c_2 yz}{y+z} - \frac{m_1 \bar{x}\bar{y}}{\bar{x}+\bar{y}} + d_1 \bar{y} + \frac{c_2 \bar{y}\bar{z}}{\bar{y}+\bar{z}} \right|$$

$$= \left| m_1 \left(\frac{xy}{x+y} - \frac{\bar{x}\bar{y}}{\bar{x}+\bar{y}} \right) - d_1(y-\bar{y}) - c_2 \left(\frac{yz}{y+z} - \frac{\bar{y}\bar{z}}{\bar{y}+\bar{z}} \right) \right|$$

$$\leq m_1| x - \bar{x}| + m_1| y - \bar{y}| + d_1| y - \bar{y}| + c_2| y - \bar{y}| + c_2| z - \bar{z}|$$

$$|M_3(H) - M_3(\bar{H})| = \left| \frac{m_2 yz}{y+z} - d_2 z - \frac{m_2 \bar{y}\bar{z}}{\bar{y}+\bar{z}} + d_2 \bar{z} \right|$$

$$= \left| m_2 \left(\frac{yz}{y+z} - \frac{\bar{y}\bar{z}}{\bar{y}+\bar{z}} \right) - d_2(z-\bar{z}) \right|$$

$$\leq m_2| y - \bar{y}| + m_2| z - \bar{z}| + d_2| z - \bar{z}|$$

Then equation (5) becomes,

$$\|M(H)-M(\bar{H})\| \leq |x-\bar{x}| + 2\varphi|x-\bar{x}| + c_1|x-\bar{x}| + c_1|y-\bar{y}| + m_1|x-\bar{x}| + m_1|y-\bar{y}|$$

$$+d_1|y-\bar{y}| + c_2|y-\bar{y}| + c_2|z-\bar{z}| + m_2|y-\bar{y}| + m_2|z-\bar{z}| + d_2|z-\bar{z}|$$

$$\leq (1+2\varphi+c_1+m_1)|x-\bar{x}| + (c_1+m_1+d_1+c_2+m_2)|y-\bar{y}| + (c_2+m_2+d_2)|z-\bar{z}|$$

$\|M(H)-M(\bar{H})\| \leq L\|H-\bar{H}\|$ where

$L = \max(1+2\varphi+c_1+m_1, c_1+m_1+d_1+c_2+m_2, c_2+m_2+d_2)$.

Therefore M(H) obeys Lipschitz condition which implies the existence and uniqueness of solution of the fractional-order system (2).

C.2. Theorem

$\forall t \geq 0$, $X(0) = X_0 \geq 0$, $Y(0) = Y_0 \geq 0$, $Z(0) = Z_0 \geq 0$ the solutions of the system in (1) with initial conditions $\left(X(t), Y(t), Z(t)\right) \in R_+^3$ are not negative (Brauer et al., 2008; Podlubny, 1999).

Proof: (Generalized Mean Value Theorem) Let $f(x) \in C[a,b]$ and $D^\alpha f(x) \in C[a,b]$ for $0 < \alpha \leq 1$. Then we have

$$f(x) = f(\alpha) + \frac{1}{\Gamma(\alpha)} D^\alpha f(\epsilon)\,(x-a)^\alpha \qquad (6)$$

with $0 \leq \epsilon \leq x$, $\forall x \in (a,b]$.

The existence and and uniqueness of the solution (1) in $(0,\infty)$ can be obtained via Generalized Mean Value Theorem. The field R_+^3 must be positively invariant. Since

$$D^\alpha X = rX\left(1-\frac{X}{K}\right) - \frac{1}{\mu_1}\frac{m_1XY}{a_1Y+X} \geq 0$$

$$D^\alpha Y = \frac{m_1XY}{a_1Y+X} - d_1Y - \frac{1}{\mu_2}\frac{m_2YZ}{a_2Z+Y} \geq 0$$

$$D^\alpha Z = \frac{m_2YZ}{a_2Z+Y} - d_2Z \geq 0$$

on each hyperplane bounding the nonnegative orthant, the vector field points into R_+^3.

D. Generalized Euler Method

In this paper, we used the Generalized Euler method to solve the initial value problem with the Caputo fractional derivative. Many of the mathematical models consist of nonlinear systems and finding solutions to these systems can be quite difficult. In most cases, analytical solutions cannot be found and a numerical approach should be considered. One of these approaches is the Generalized Euler method (Yaro et al., 2015).

$D^\alpha y(t) = f(t, y(t)), y(0) = y_0$, $0 < \alpha \leq 1$, $0 < t < a$ for the initial value problem $h = \frac{a}{n}$ impending $\left[t_j, t_{j+1}\right]$ is divided into n sub with $j = 0,1,\ldots,n-1$. Suppose that $y(t), D^\alpha y(t)$ and $D^{2\alpha} y(t)$ are continuous in range $[0,a]$ and using the generalized Taylor's formula, the following equation is obtained (Yaro, D. et al., 2015).

$$y(t_1) = y(t_0) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_0, y(t_0)).$$

This process will be repeated to create an array. Let. $t_{j+1} = t_j + h$ such that

$$y(t_{j+1}) = y(t_j) + \frac{h^\alpha}{\Gamma(\alpha+1)} f(t_j, y(t_j)),\ j = 0,1,\dots,n-1$$

the generalized formula in the form is obtained. For every $k = 0,1,\dots,n-1$

$$x(k+1) = x(k) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(x(k)(1-x(k)) - \frac{c_1 x(k) y(k)}{x(k)+y(k)}\right)$$

$$y(k+1) = y(k) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(\frac{m_1 x(k) y(k)}{x(k)+y(k)} - d_1 y(k) - \frac{c_2 y(k) z(k)}{y(k)+z(k)}\right) \qquad (7)$$

$$z(k+1) = z(k) + \frac{h^\alpha}{\Gamma(\alpha+1)}\left(\frac{m_2 y(k) z(k)}{y(k)+z(k)} - d_2 z(k)\right).$$

III. NUMERICAL SIMULATION OF FRACTIONAL ORDER FOOD CHAIN MODEL

In this section, we will investigate the sensitive dependence of the system on initial conditions and parameter values using the Generalized Euler Method. For this purpose $x = 0.7$, $y = 0.3$, $z = 0.3$, $m_1 = 10$, $m_2 = 2$, $d_1 = 1$, $d_2 = 1$, $c_1 = 1$, $c_2 = 11$ step size is taken as $h = 0.1$ and the dynamics of the system is analyzed (Hsu et al., 2003).

Table 3. Values of X, Y and Z at time t for α = 1.

t	$x(t)$	$y(t)$	$z(t)$
0	0,70	0,30	0,30
1	0,66	0,24	0,36
2	0,63	0,048	0,43
3	0,64	-0,42	0,51
4	0,75	-1,46	0,62
5	1,00	-3,68	0,74
6	1,79	-8,25	0,89
7	3,30	-17,58	1,07
8	6,06	-36,46	1,28
9	10,28	-74,47	1,54
10	15,62	-150,81	1,85
11	22,92	-303,85	2,22

Table 3 (cont). Values of X, Y and Z at time t for α = 1.

12	33,42	-610,38	2,67
13	47,11	-1223,98	3,20
14	74,62	-2451,82	3,85

Table 4. Values of X, Y and Z at time t for α = 0.9

t	$x(t)$	$y(t)$	$z\ (t)$
0	0,70	0,30	0,30
1	0,64	0,22	0,37
2	0,62	-0,083	0,47
3	0,67	-0,94	0,60
4	0,94	-3,12	0,76
5	1,77	-8,40	0,95
6	3,79	-20,91	1,21
7	7,88	-50,19	1,52
8	13,92	-118,28	1,92
9	21,34	-276,15	2,43
10	36,81	-641,45	3,06
11	32,17	-1485,91	3,87
12	289,87	-3437,00	4,88
13	-97,08	-7943,61	6,16
14	-125,74	-18351,23	7,77

Table 5. Values of X, Y and Z at time t for $\alpha = 0.8$

t	$x(t)$	$y(t)$	$z(t)$
0	0,70	0,30	0,30
1	0,63	0,19	0,40
2	0,60	-0,28	0,53
3	0,74	-1,87	0,72
4	1,41	-6,53	0,96
5	3,53	-19,64	1,29
6	8,69	-55,71	1,73
7	16,27	-114,08	2,33
8	26,42	-321,03	3,12
9	55,40	-743,87	4,18
10	-68,20	-1568,89	5,61
11	183,21	-3383,60	7,52
12	-26,30	-6764,95	10,08
13	-118,33	-12853,42	13,51
14	-239,20	-25513,06	18,11

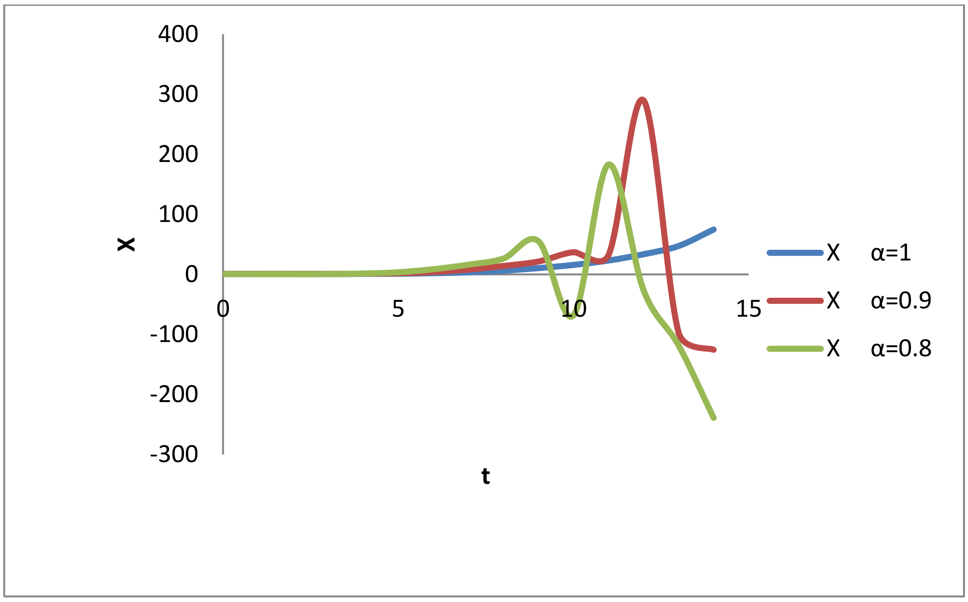

Figure 1. The graph of change of the *X* compartment model.

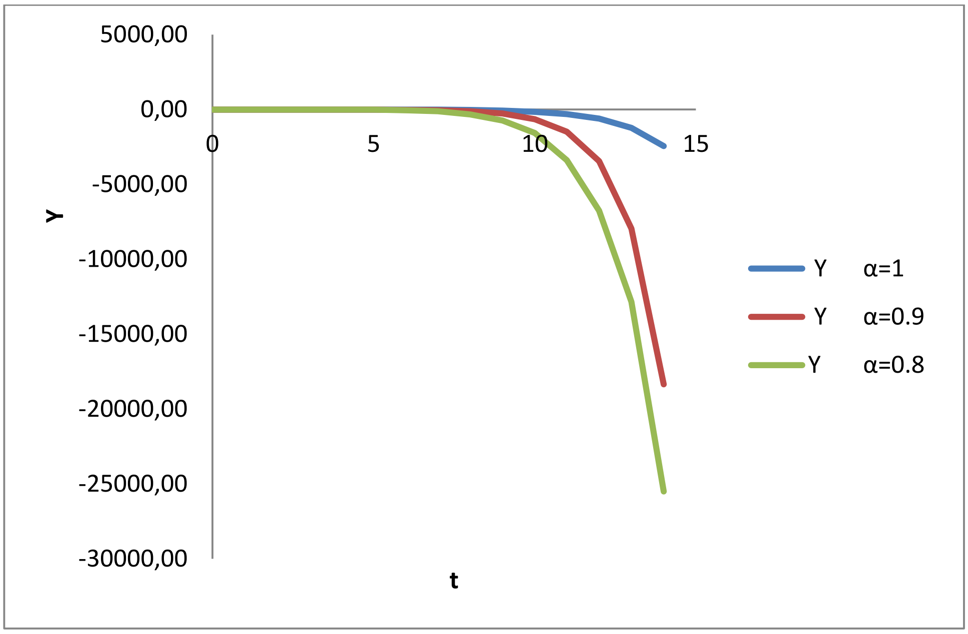

Figure 2. The graph of change of the *Y* compartment model

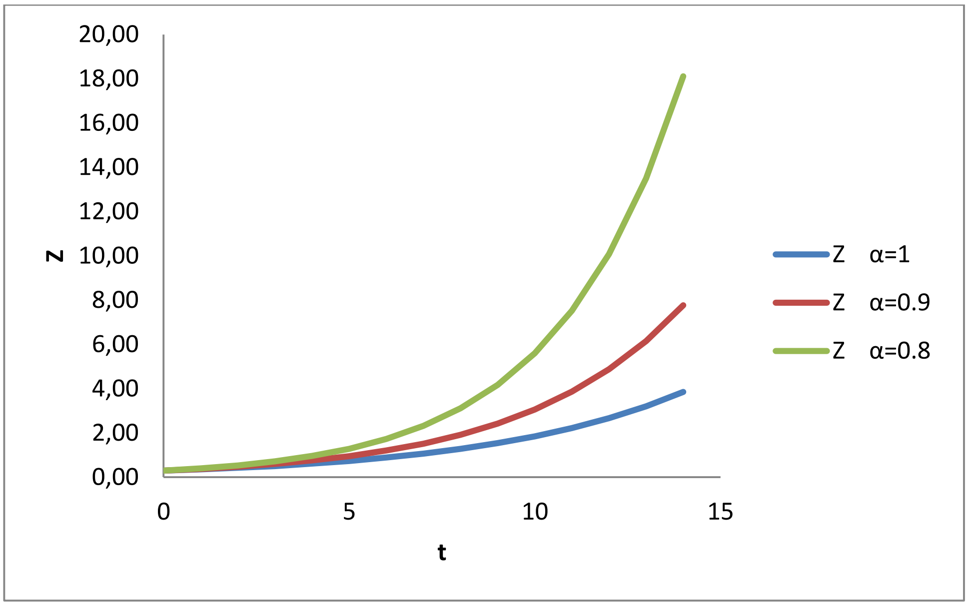

Figure 3. The graph of change of the Z compartment model

In Table 3, Table 4 and Table 5, show the changes of X, Y and Z for different cases of α. According to the graphs above, we can make the following comments.

* It has been observed that, the density of the prey population exhibits a chaotic pattern of increase and decrease over time (Figure-1).

* It is observed that, the hunter population density decreases rapidly over time (Figure-2).

* It is observed that, the top predator population density increases over time (Figure-3).

IV. CONCLUSION

The present study analyses a rate-dependent food chain model. If a minor alteration is made to the initial conditions of a system and this alteration results in substantial consequences in the system's behaviour, one may describe the system's sensitivity to the initial conditions. The relationship between cause and effect is obscured by the intricate web of interactions. It follows that nonlinear systems demonstrate an inability to make long-term predictions. The reinforcing effect of small movements within the system results in a reorientation towards new directions, ultimately leading to unpredictable and unanticipated outcomes. In this study, a fractional order rate-based food chain modelling system is considered. The existence, uniqueness and non-negativity of the system are analysed mathematically. The obtained graphs show a chaotic increase and decrease in prey population density over time, while predator population density decreases rapidly and top predator population density increases over time.

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Plagiarism Statement: This article has been evaluated for plagiarism and no instances of plagiarism were detected.

Availability of Data and Materials: Data sharing is not applicable to this study.

Use of AI Tools: The author declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

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