Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Sayı: 14 Aralık 2007 ISSN - 1302 - 3055

3 BOYUTLU LORENTZ UZAYINDA KÜRESEL ALANLAR İÇİN GEOMETRİK BİR İNVARYANT

Gülay KORU YÜCEKAYA*

*Gazi University Kirsehir Faculty of Arts and Sciences, Department of Mathematics, 40100, Kirsehir, Turkey, gkoru@gazi.edu.tr

ÖZET

Bu çalışmada, 3-boyutlu Lorentz uzayında 1-parametreli Lorentz küresel hareketi tanımlanarak, bu hareket sırasında space-like birim Lorentz küresi üzerinde elde edilen küresel alanlar üzerindeki bağıntı verilmiş ve bu alanlar yardımıyla geometrik bir invaryant elde edilmiştir.

Anahtar Kelimeler : *Lorentz küresi, Lorentz küresel alanı, Kapalı Lorentz küresel hareketi, 1-parametreli kapalı küresel hareket.*

A GEOMETRICAL INVARIANT FOR SPHERICAL AREAS IN LORENTZ 3-SPACE L^3

ABSTRACT

In this study, by defining the one-parameter closed spherical Lorentz motion in 3-dimensional Lorentz space, we give the relation between spherical areas, generated by this motion on space-like unit Lorentzian sphere.

Key Words: *Lorentz sphere, Lorentz spherical area, Motion of closed Lorentz sphere, One Parameter closed spherical motion.*

1. INTRODUCTION

A 3–dimensional vector space $L = L_1^3$ with scalar product $\langle\ ,\ \rangle_L$ of index 1 is called a Lorentzian vector space. A vector X of L_1^3 is said to be space-like if $\langle X, X\rangle_L > 0$, time-like if $\langle X, X\rangle_L < 0$ and light-like or null if $\langle X, X\rangle_L$ and $X \neq 0$.

A curve in L_1^3 is called space-like (time-like or null, respectively) if its tangent vector is space-like (time-like or null, respectively).

Let $X = (X_i)$ and $Y = (Y_i)$ be the vectors in a 3–dimensional Lorentz vector space L_1^3, then the scalar product of *X* and *Y* is defined by

$$\langle X, X\rangle_L = X_1Y_1 + X_2Y_2 - X_3Y_3,$$

which is called a Lorentzian product. Furthermore, a Lorentzian cross product XxY is given by

$$X \wedge_L Y = (-X_2Y_3 + X_3Y_2, X_3Y_1 - X_1Y_3, X_1Y_2 - X_2Y_1).$$

For $X \in L_1^3$, the norm of *X* is defined by $\|X\|_L = \sqrt{\left|\langle X, X\rangle_L\right|}$, *and* X is called a unit vector if $\|X\|_L = 1$

Lorentzian motion $B' = K/K'$ of the moving unit Lorentz sphere K with the fixed center O with respect to the fixed unit Lorentz sphere of the same center defines a direct unique Lorentzian motion about the fixed point O. Hence, Lorentzian spherical motion is a spatial motion in Lorentz 3 – space . During the Lorentz motion $B' = K/K'$, which leaves the center O fixed, the orthonormal, positive directed two coordinate system

$$\{O; \vec{E}_1, \vec{E}_2, \vec{E}_3\}, \{O; \vec{E}_1', \vec{E}_2', \vec{E}_3'\}$$

represent respectively moving K and fixed K' Lorentzian sphere. These two coordinate systems depend invariantly to unit Lorentz spheres K and K'. Let's denote the matrices

$$E = \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix}, E' = \begin{bmatrix} \vec{E}_1' \\ \vec{E}_2' \\ \vec{E}_3' \end{bmatrix}$$

Since these two systems are orthonornal, for A being an orthogonal Lorentzian matrix, we have

$$E = AE',$$

where

$$A^{-1} = \varepsilon A^T \varepsilon$$

and

$$\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

is the sign matrix in Lorentzian 3 – space. Let the matrices E, E' and A be differentiable functions in sufficient order of a parameter $t \in R$. Here A is not periodic, but during the Lorentzian spherical motion, the spherical curve (X) drawn on K' by the point X taken from the moving sphere K is periodic. Thus the motion $B' = K/K'$ of moving Lorentzian motion K with respect to fixed Lorentz sphere K' is called as one parameter closed spherical motion.

In this study, $\vec{E}_1', \vec{E}_2', \vec{E}_3'$ and $\vec{E}_2'$ are taken to be space-like vektors and $\vec{E}_3, \vec{E}_3'$ are taken to be timelike vektors. The directional Lorentzian 2-manifold, we consider is the surface of space like Lorentzian sphere

$$x^2 + y^2 - z^2 = r^2, \quad r = const.$$

Which has the time-like vector as normal. The curves on the surface are uniform closed spacelike curves. Computatin are made for the upper half of the space-like Lorentzian sphere. The area of upper half part of unit Lorentzian sphere K is 1 [1].

Let X be an arbitrary point on K moving Lorentz sphere. Then, the point X draw the closed (X) Lorentz curve during the one parameter closed Lorentz spherical motion $B' = K/K'$.

The spherical area bounded by the closed Lorentzian curve (X) is

$$F_{X_i} = 2\pi + \Lambda_{\vec{X}_i}$$

[2,3].

2. A GEOMETRICAL INVARIANT FOR SPHERICAL AREAS IN LORENTZIAN 3-SPACE L^3

Theorem 2.1. Two constant points M and N on moving unit Lorentzian sphere K usually draw two closed curves on constant unit Lorentz sphere K' during the closed Lorentz motion with one parameter $B' = K/K'$. Let these curves be (M) and (N), and the Lorentzian spherical areas bounded by these two curves be F_M and F_N. Consider another point X of K on $\overset{\frown}{MN}$ arc with constant length of great Lorentzian circle of Lorentzian sphere K. During the same motion, X also draws another closed curve (X) on constant Lorentzian sphere K'. Let F_X be the area bounded by the closed curve (X). In this, the relation between these areas are;

$$F_X = \frac{1}{2}\left\{F_M + F_N + \wedge_{\vec{MX}+\vec{NX}}\right\}$$

Proof. From the formula in [3] we have,

$$\begin{aligned} F_M &= 2\pi + \wedge_{\vec{M}} \\ F_N &= 2\pi + \wedge_{\vec{N}} \\ F_X &= 2\pi + \wedge_{\vec{X}} \end{aligned} \quad [2.1]$$

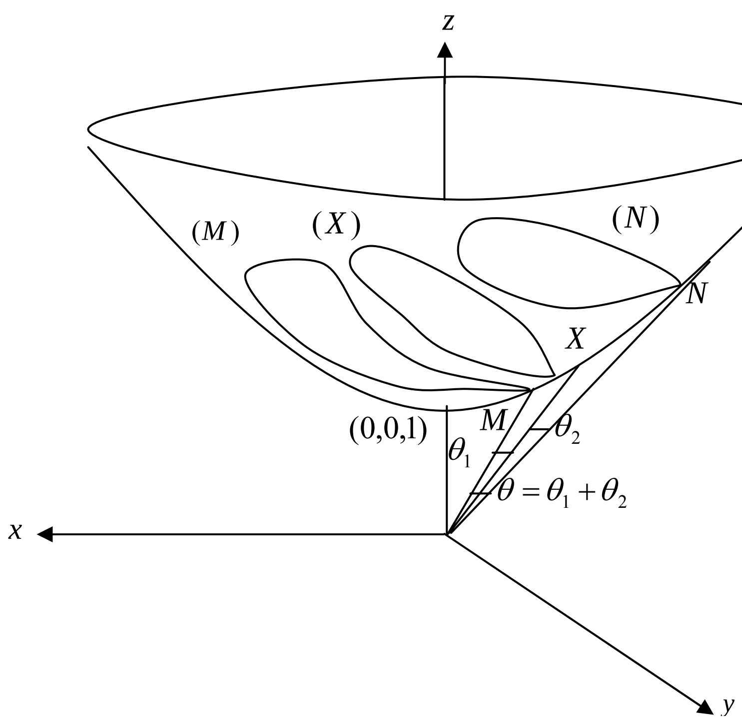

Figure 2.1 The are $\overset{\frown}{MN}$, taken on great Lorentzian circle on Lorentzian sphere $(z > 0)$.

$\vec{M}$, $\vec{N}$ and $\vec{X}$ are the position vectors of M, N and X respectively, and from Figure 2.1, We have

$$\overrightarrow{N} = \overrightarrow{M} + \overrightarrow{MN},\ \overrightarrow{X} = \overrightarrow{M} + \overrightarrow{MX},\ \overrightarrow{X} = \overrightarrow{N} + \overrightarrow{NX} \qquad [2.2]$$

From [2.1] and [2.2], we may write that

$$\begin{aligned} F_N &= F_M + \wedge_{\vec{MN}} \\ F_X &= F_M + \wedge_{\vec{MX}} \end{aligned} \qquad [2.3]$$

and

$$F_X = F_N + \wedge_{\vec{NX}} \qquad [2.4]$$

From the last two terms of F_X we

$$F_X = \frac{1}{2}\left\{ F_M + F_N + \wedge_{\vec{MX}+\vec{NX}} \right\}$$

This equation is equivalent to Holditch theorem. That is, the relation between the closed spherical Lorentzian areas bounded by the closed Lorentzian curves (M),(N) and (X) are independent from the closed Lorentzian motion $B' = K/K'$.

An important result of Holditch theorem is the special case of $F_M = F_N$. The end points M and N either draw the curves having equal area or the same curve (γ) on K' Lorentz sphere. So from [2.3], we have

$$\wedge_{\vec{MN}} = 0 \qquad [2.5]$$

is obtained. Here $\vec{S}$ is a Steiner vector.

Corollary 2.1. During the closed Lorentzian spherical motion with one-parameter, $\wedge_{\vec{MN}} = 0$ iff the end points M and N draw the same spherical curve.

Corollary 2.2. Let's consider the areas bounded by some different points not on the same great circle of moving Lorentzian sphere K. These areas are equal iff the points belong to same curve.

REFERENCES

[1] Liu, H., "Translation surfaces with constant mean curvature in 3-dimensional spaces", 64: 141-149 (1999)

[2] Jee, D.J., "Gauss-Bonnet Formula for general Lorentzian surfaces", 15: 215-231 (1984)

[3] Özyılmaz, E., Yaylı, Y.,Bonnet, O., "Integral formula and some theorems in Minkowski space", Hardronic Journal Institute for Basic Research USA, 15(4): 397-414 (2000)