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UNIT FAV-JERRY DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract

This paper introduces a novel bounded statistical distribution and explores its key characteristics, including cumulative distribution, probability density, and hazard rate functions, illustrated using graphical representations. The study examines mathematical properties such as moments, skewness, kurtosis, the Bonferroni and Lorenz curves, and order statistics. Estimators for the unknown parameter of new models are assessed, with performance evaluated via bias, mean square errors, average absolute bias, and mean relative error in Monte Carlo simulations. Finally, the practical utility of the new model is demonstrated through two real data analyses.
Keywords: Bounded Statistical Distribution, Bonferroni and Lorenz Curves, Moment Properties, Point Estimation, Total Milk Production Proportion Data

BİRİM FAV-JERRY DAĞILIMI: ÖZELLİKLER VE UYGULAMALAR

Özet

Bu makale yeni bir sınırlı istatistiksel dağılım sunmakta ve grafik gösterimler kullanılarak gösterilen kümülatif dağılım, olasılık yoğunluğu ve tehlike oranı fonksiyonları dahil olmak üzere temel özelliklerini araştırmaktadır. Çalışmada momentler, çarpıklık, basıklık, Bonferroni ve Lorenz eğrileri ve sıra istatistikleri gibi matematiksel özellikler incelenmektedir. Yeni modellerin bilinmeyen parametrelerine ilişkin tahminciler, Monte Carlo simülasyonlarında performans bias, ortalama karesel hatalar, ortalama mutlak bias ve ortalama bağıl hata üzerinden değerlendiriliyor. Son olarak, yeni modelin pratik uygulanabilirliği, iki gerçek veri analizi aracılığıyla gösterilmektedir.
Anahtar Kelimeler: Sınırlı İstatistiksel Dağılım, Bonferroni ve Lorenz Eğrileri, Moment Özellikleri, Nokta Tahmini, Toplam Süt Üretimi Oranı Verisi

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1. Introduction

Modeling data in areas like mortality, recovery, and education often relies on statistical distributions, especially those bounded within the (0,1) interval. Despite their importance, the bounded distributions available in the literature are usually inadequate for capturing the complexity of such data. Therefore, lifetime distributions are transformed to (0,1) intervals with various transformations, such as

$$X = \exp(-Y),\ X = Y/(Y+1),\ X = 1/(Y+1),\ X = 1-\exp(-Y),$$

to the existing random variable (Y) to derive a new bounded random variable (X). In recent years, several unit distributions are proposed, including those presented by [1], [2], [3], [4], [5], [6] and [7]. Building upon the Fav-Jerry (FJ) distribution introduced by [8], this study proposes a novel unit distribution tailored for modeling rates and proportions. It serves as an alternative to traditional models like the Kumaraswamy and beta distributions.

The remainder of the study is organized as follows: The new unit distribution is introduced and some of the new unit distribution statistical properties are investigated in Section 2. In Section 3, point estimation is performed on two parameters of the Unit Fav-Jerry distribution with several estimators. In Section 4, the performances of the estimators are evaluated with a Monte Carlo simulation. In Section 5, two data analyses are performed to show the applicability of the Unit Fav-Jerry distribution. The article concludes with Section 6.

2. Unit Fav-Jerry Distribution

Let the random variable Y follow the Fav-Jerry (FJ) distribution, with the cumulative distribution function (cdf) and probability density function (pdf) defined, respectively, as

$$F_{FJ}(y;\theta) = 1 - \left(1 + \frac{\theta^3 y}{\theta^2 + 2}\right)\exp(-\theta y), \quad y > 0, \qquad (1)$$

and

$$f_{FJ}(y;\theta) = \frac{\theta}{\theta^2 + 2}\left(2 + \theta^3 y\right)\exp(-\theta y), \quad y > 0, \qquad (2)$$

where $\theta > 0$. If the random variable Y has the pdf provided in Equation (2), then the cdf and pdf of the random variables $Z = \exp(-Y)$ and $X = \exp(-Y/\beta)$ can be derived, respectively, as

$$F_{UFJ_1}(z;\theta) = \frac{\left(\theta^2 + 2 - \theta^3 \log(z)\right) z^{\theta}}{\theta^2 + 2}, \quad z > 0, \qquad (3)$$

$$f_{UFJ_1}(z;\theta) = \frac{\theta\left(2 - \theta^3 \log(z)\right) z^{\theta-1}}{\theta^2 + 2}, \quad z > 0, \qquad (4)$$

and

$$F_{UFJ_2}(x;\theta,\beta) = \frac{\left(\theta^2 + 2 - \theta^3 \log(x)\beta\right) x^{\theta\beta}}{\theta^2 + 2}, \quad x > 0, \qquad (5)$$

$$f_{UFJ_2}(x;\theta,\beta) = \frac{\theta\left(2 - \theta^3 \log(x)\beta\right)\beta x^{\theta\beta-1}}{\theta^2 + 2}, \quad x > 0, \qquad (6)$$

where $0 < z, x < 1$ and $\theta > 0, \beta > 0$ are the model parameters. The new distribution, which cdf given in Equation (5) and pdf in Equation (6) is referred to as the unit Fav-Jerry (UFJ) distribution, and it is denoted by $UFJ(\theta,\beta)$. For the rest of the paper the second type of the unit Fav-Jerry distribution is used with pdf in Equation (6). The hazard rate function (hrf) of the UFJ distribution can be written as

$$h_{UFJ}(x;\theta,\beta) = \frac{\theta\left(2 - \theta^3 \log(x)\beta\right)\beta x^{\theta\beta-1}}{\left(\theta^2 + 2\right)\left(1 - \frac{\left(\theta^2 + 2 - \theta^3 \log(x)\beta\right) x^{\theta\beta}}{\theta^2 + 2}\right)}.$$

The pdf and hrf of the UFJ distribution for various parameter choices are illustrated in Figure 1. As observed from Figure 1, pdf and hrf of the UFJ distribution exhibit distinct patterns; specifically, the pdf demonstrates decreasing, increasing, unimodal, and decreasing-increasing behaviors, while the hrf displays decreasing, increasing and unimodal shapes.

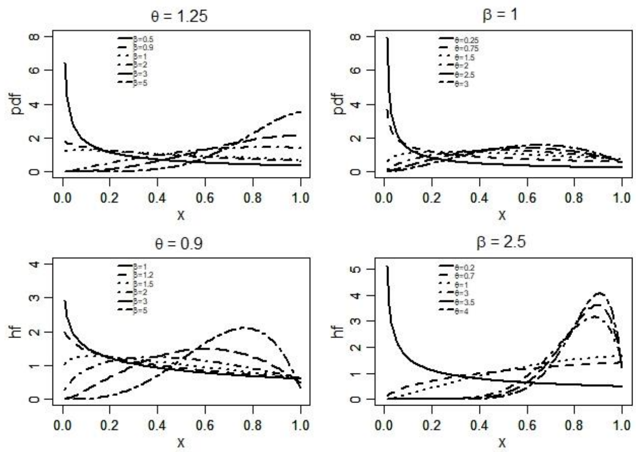

Figure 1. The pdf and hrf of UFJ distribution for various parameter choices

2.1. Moments

The *r*-th moment of the UFJ distribution obtained as

$$\begin{aligned} E\left(X^r\right) &= \int_0^1 x^r f(x)\,dx \\ &= \int_0^1 \frac{\theta\left(2 - \theta^3 \log(x)\beta\right)\beta x^{\theta\beta-1+r}}{\theta^2 + 2}dx \qquad (7) \\ &= \frac{\left(2r\theta\beta + 2\theta^2\beta^2 + \theta^4\beta^2\right)}{\left(\theta^2 + 2\right)\left(r + \theta\beta\right)^2}. \end{aligned}$$

By taking $r = 1, 2, 3$ and 4 in the function in Equation (7), the first four moments can be derived as follows, respectively,

$$E(X) = \frac{\left(2\theta\beta + 2\theta^2\beta^2 + \theta^4\beta^2\right)}{\left(\theta^2 + 2\right)\left(1 + \theta\beta\right)^2}, \qquad (8)$$

$$E\left(X^2\right) = \frac{\left(4\theta\beta + 2\theta^2\beta^2 + \theta^4\beta^2\right)}{\left(\theta^2 + 2\right)\left(2 + \theta\beta\right)^2}, \qquad (9)$$

$$E\left(X^3\right) = \frac{\left(6\theta\beta + 2\theta^2\beta^2 + \theta^4\beta^2\right)}{\left(\theta^2 + 2\right)\left(3 + \theta\beta\right)^2}, \qquad (10)$$

and

$$E\left(X^4\right) = \frac{\left(8\theta\beta + 2\theta^2\beta^2 + \theta^4\beta^2\right)}{\left(\theta^2 + 2\right)\left(4 + \theta\beta\right)^2}. \qquad (11)$$

The variance of the UFJ distribution is derived using Equations (9) and (8) as

$$V(X)=\left[\begin{array}{l}\dfrac{\left(4\theta\beta+2\theta^{2}\beta^{2}+\theta^{4}\beta^{2}\right)}{\left(\theta^{2}+2\right)\left(2+\theta\beta\right)^{2}}\\ -\left\{\dfrac{\left(2\theta\beta+2\theta^{2}\beta^{2}+\theta^{4}\beta^{2}\right)}{\left(\theta^{2}+2\right)\left(1+\theta\beta\right)^{2}}\right\}^{2}\end{array}\right]$$

$$=\frac{\theta\beta\left(8+20\theta^{3}\beta+\theta^{5}\beta+4\theta^{3}\beta^{3}+8\theta^{5}\beta^{3}+4\theta^{6}\beta^{2}\right)}{\left(\theta^{2}+2\right)\left(2+\theta\beta\right)^{2}\left(1+\theta\beta\right)^{4}} \quad (12)$$
$$+\frac{\theta\beta\left(2\theta^{7}\beta^{3}+24\theta^{4}\beta^{2}+20\theta\beta+16\theta^{2}\beta^{2}+4\theta^{2}\right)}{\left(\theta^{2}+2\right)\left(2+\theta\beta\right)^{2}\left(1+\theta\beta\right)^{4}}.$$

The skewness (S) and kurtosis (K) of the UFJ distribution can be determined, respectively, using Equations (8)–(12), as follows:

$$S=\frac{E\left(X^{3}\right)-3E\left(X\right)E\left(X^{2}\right)+2\left\{E\left(X\right)\right\}^{3}}{\left\{V\left(X\right)\right\}^{3/2}} \quad (13)$$

and

$$K=\frac{E\left(X^{4}\right)-4E\left(X\right)E\left(X^{3}\right)+6E\left(X^{2}\right)\left\{E\left(X\right)\right\}^{2}-3\left\{E\left(X\right)\right\}^{4}}{\left\{V\left(X\right)\right\}^{2}}. \quad (14)$$

The expected value, variance, S, and K are provided in Table 1 for selected values of θ and β. As the parameters θ and β increase, the expected value rises, while the variance, S, and K decrease. Additionally, for small values of θ and β, the distribution is skewed to the right, while for larger values, it becomes skewed to the left. It is observed that the distribution is always platykurtic based on the K values.

Table 1. Expected value, variances, skewness, and kurtosis for various choices of θ and β.

θ	β	$E(X)$	$V(X)$	S	K
0.1	0.7	0.0651	0.0294	3.3042	13.8783
0.1	1	0.0905	0.0392	2.6691	9.6157
0.5	0.9	0.2866	0.0849	0.8618	2.4999
0.5	1.5	0.4014	0.0896	0.3610	1.8715
0.4	0.1	0.0357	0.0169	4.7025	26.4679
0.8	0.1	0.0574	0.0262	3.5700	15.9419
1.5	0.5	0.2989	0.0784	0.8345	2.5423
2	0.5	0.3333	0.0741	0.6889	2.3850

2.2. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are introduced by [9] and used in many areas such as economics, income inequality analysis, risk assessment in insurance, reliability engineering, and population studies. Now present the Lorenz and Bonferroni curves based on the proposed distribution. Let the random variable X follow the $UFJ(\theta,\beta)$ distribution with the pdf given by Equation (6). The Lorenz and Bonferroni curves are given respectively, by

$$L(u)=\frac{\int_{0}^{u}xf(x)dx}{\mu}$$
$$=\frac{\left(2+\theta^{3}\beta+2\theta\beta-\left(\theta^{4}\beta^{2}+\theta^{3}\beta\right)\log(u)\right)u^{(\theta\beta+1)}}{2+2\theta\beta+\theta^{3}\beta}$$

and

$$B(p)=\frac{1}{p}\int_{0}^{p}L(u)du$$
$$\left[\begin{array}{l}\dfrac{p^{(1+\theta\beta)}\left(2\theta^{4}\beta^{2}-\theta^{5}\beta^{3}\log(p)-3\theta^{4}\beta^{2}\log(p)\right)}{\left(2+2\theta\beta+\theta^{3}\beta\right)\left(\theta^{2}\beta^{2}+4\theta\beta+4\right)}\\ +\dfrac{p^{(1+\theta\beta)}\left(2\theta^{2}\beta^{2}+3\theta^{3}\beta-2\theta^{3}\beta\log(p)+6\theta\beta+4\right)}{\left(2+2\theta\beta+\theta^{3}\beta\right)\left(\theta^{2}\beta^{2}+4\theta\beta+4\right)}\end{array}\right],$$

where $\mu=E(X)$ and $f(x)$ is the pdf in Equation (6).

2.3 Order Statistics

In this subsection, we explore results related to the order statistics of the UFJ distribution. Let $X_1,X_2,\ldots,X_n$ be a random sample from the UFJ distribution and $X_{(1)},X_{(2)},\ldots,X_{(n)}$ indicate the corresponding order statistics. The cdf and pdf of the $X_{(r)}$ are presented in general form, respectively, by

$$F_{X_{(r)}}(x;\theta,\beta)=\sum_{i=r}^{n}\binom{n}{i}F(x;\theta,\beta)^{i}\left\{1-F(x;\theta,\beta)\right\}^{n-i}$$
$$=\sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^{j}\binom{n}{i}\binom{n-i}{j}F(x;\theta,\beta)^{i+j},$$

and

$$f_{X_{(r)}}(x;\theta,\beta)=\frac{1}{B(r,n-r+1)}F(x;\theta,\beta)^{r-1}\left\{1-F(x;\theta,\beta)\right\}^{n-r}f(x;\theta,\beta)$$
$$=\frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^{i}\binom{n-r}{i}F(x;\theta,\beta)^{r+i-1}f(x;\theta,\beta),$$

where $r=1,2,\ldots,n$ and $B(.;.)$ is the classical beta function. The cdf and pdf of the $X_{(r)}$ order statistic of the UFJ distribution is also derived by

$$F_{X_{(r)}}(x;\theta,\beta)=\sum_{i=r}^{n}\sum_{j=0}^{n-i}(-1)^{j}\binom{n}{i}\binom{n-i}{j}\times\left\{\frac{\left(\theta^{2}+2-\theta^{3}\log(x)\beta\right)x^{\theta\beta}}{\theta^{2}+2}\right\}^{i+j},$$

and

$$f_{X_{(r)}}(x;\theta,\beta)=\frac{\theta\left(2-\theta^{3}\log(x)\beta\right)\beta x^{\theta\beta-1}}{\left(\theta^{2}+2\right)B(r,n-r+1)}\times\sum_{i=0}^{n-r}(-1)^{i}\binom{n-r}{i}\left\{\frac{\left(\theta^{2}+2-\theta^{3}\log(x)\beta\right)x^{\theta\beta}}{\theta^{2}+2}\right\}^{r+i-1}.$$

When $r=1$ and $r=n$, the cdf and pdf of $X_{(1)}=\min\{X_1,X_2,\ldots,X_n\}$ and $X_{(n)}=\max\{X_1,X_2,\ldots,X_n\}$ are obtained.

3. Point Estimation

In this section, six different estimation methods are employed to determine the unknown parameters of the UFJ distribution: maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér–von Mises (CvM), Anderson-Darling (AD), and maximum product spacing (MPS). Consider a random sample $X_1,X_2,\ldots,X_n$ from the $UFJ(\theta,\beta)$ distribution and $x_1,x_2,\ldots,x_n$ the observed values of this sample. Let $X_{(1)},X_{(2)},\ldots,X_{(n)}$ represent the order statistics of the sample, with the observed realization being $x_{(1)},x_{(2)},\ldots,x_{(n)}$. The likelihood and log-likelihood functions are given as follows:

$$L(\Xi)=\prod_{i=1}^{n}\left(\frac{\theta\left(2-\theta^{3}\log(x_i)\beta\right)\beta x^{\theta\beta-1}}{\theta^{2}+2}\right)\quad(15)$$

and

$$\ell(\Xi)=\left\{n\log(\theta)+\sum_{i=1}^{n}\log(2-\theta^{3}\log(x_i)\beta)+n\log(\beta)+(\theta\beta-1)\sum_{i=1}^{n}\log(x_i)-n\log(\theta^{2}+2)\right\},\quad(16)$$

where $\Xi=(\theta,\beta)$. Then the ML of $\hat{\Xi}=\left(\hat{\theta},\hat{\beta}\right)$ for Ξ given as

$$\Xi_1=\arg\max\ell(\Xi)$$

$$(\theta,\beta)\in(0,\infty)\times(0,\infty).$$

We proceed by considering the following five functions to derive additional estimators.

$$LS(\Xi)=\sum_{i=1}^{n}\left(\frac{\left(\theta^{2}+2-\theta^{3}\log\left(x_{(i)}\right)\beta\right)x_{(i)}{}^{\theta\beta}}{\theta^{2}+2}-\frac{i}{n+1}\right)^{2},\quad(17)$$

$$WLS(\Xi)=\sum_{i=1}^{n}\frac{(n+2)(n+1)^{2}}{i(n-i+1)}\times\left(\frac{\left(\theta^{2}+2-\theta^{3}\log\left(x_{(i)}\right)\beta\right)x_{(i)}{}^{\theta\beta}}{\theta^{2}+2}-\frac{i}{n+1}\right)^{2},\quad(18)$$

$$AD(\Xi)=-n-\sum_{i=1}^{n}\frac{2i-1}{n}\times\left[\begin{array}{l}\log\left\{\frac{\left(\theta^{2}+2-\theta^{3}\log\left(x_{(i)}\right)\beta\right)x_{(i)}{}^{\theta\beta}}{\theta^{2}+2}\right\}\\+\log\left\{1-\frac{\left(\theta^{2}+2-\theta^{3}\log\left(x_{(n+i-1)}\right)\beta\right)x_{(n+i-1)}{}^{\theta\beta}}{\theta^{2}+2}\right\}\end{array}\right],\quad(19)$$

$$CvM(\Xi)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{\left(\theta^{2}+2-\theta^{3}\log\left(x_{(i)}\right)\beta\right)x_{(i)}{}^{\theta\beta}}{\theta^{2}+2}-\frac{2i-1}{2n}\right]^{2},\quad(20)$$

and

$$MPS(\Xi)=\frac{1}{n+1}\sum_{i=1}^{n}\log\left[F\left(x_{(i)}\mid\Xi\right)-F\left(x_{(i-1)}\mid\Xi\right)\right],\quad(21)$$

where $F(x_{0:n}\mid\theta,\beta)=0, F(x_{n+1:n}\mid\theta,\beta)=1$, and F is the cdf of the UFJ distribution in Equation (5). The LS, WLS, AD, CvM, and MPS are calculated by minimizing or maximizing Equations (17)-(21), respectively, as follows:

$$\Xi_2=\arg\min LS(\Xi)$$

$$(\theta,\beta)\in(0,\infty)\times(0,\infty)$$

$$\Xi_3=\arg\min WLS(\Xi)$$

$$(\theta,\beta)\in(0,\infty)\times(0,\infty)$$

$$\Xi_4=\arg\min CvM(\Xi)$$

$$(\theta,\beta)\in(0,\infty)\times(0,\infty)$$

$$\Xi_5=\arg\min AD(\Xi)$$

$$(\theta,\beta)\in(0,\infty)\times(0,\infty)$$

$$\Xi_6 = \arg\max MPS(\Xi)$$
$$(\theta,\beta) \in (0,\infty)\times(0,\infty).$$

These equations cannot be solved for minimization or maximization analytically. Therefore the Nelder-Mead or BFGS methods in the R program are used and executed with the optim function [10] in R.

4. Monte Carlo Simulation Study

In this section, a Monte Carlo simulation experiment is conducted with 5000 trials. The bias, mean square errors (MSE), average absolute bias (ABB), and mean relative error (MRE) of ML, LS, WLS, AD, CvM, and MPS are simulated for the two parameters of the new model to evaluate their performance and accuracy under varying sample sizes and parameter settings. The sample sizes chosen for the experiment are n=25,50,100,250,500 and 1000. The Supplementary Material presents the simulation results in Tables 2-9. From these tables, it is concluded that the bias, MSE, ABB, and MRE values of all estimators decrease and converge to zero as the sample size increases. Based on the simulation results obtained for the θ parameter, it is observed that the best estimation methods are ML, WLS, and MPS for the bias criterion; MPS for the MSE criterion, and ML for the ABB and MRE criteria. Based on the simulation results for the β parameter, it is observed that the best estimation methods are ML, CvM, and MPS for the bias criterion, ML and MPS for the MSE, ABB, and MRE criteria. Additionally, Table 10 presents the sum of rank and overall rank values for all parameter conditions based on bias, MSE, ABB, and MRE measurements. According to Table 10, the best estimation method based on the bias criterion is AD for the θ parameter and CvM for the β parameter. For the MSE criterion, MPS is the most suitable estimation method for both the θ and β parameters. Based on the ABB criterion, the best estimation method is ML for the θ parameter and CvM for the β parameter. Finally, according to the MRE criterion, LS is the best estimation method for the θ parameter, while MPS is the most appropriate method for the β parameter.

Table 10. Rank and rank score of bias, MSE, ABB, and MRE for the parameters θ and β.

Bias	θ						β					
Ξ	ML	LS	WLS	AD	CvM	MPS	ML	LS	WLS	AD	CvM	MPS
$(0.5,0.9)$	27	27	15	14	36	7	26	18	29	33	9	11
$(1,1.5)$	14	22	22	15	20	33	19	21	32	28	12	14
$(1.5,2)$	23	19	16	17	23	32	19	22	27	18	7	33
$(2,3)$	32	14	15	15	32	15	15	26	22	21	7	35
$(2.5,4)$	32	19	21	18	30	6	12	24	19	14	21	36
Sum of rank	**128**	**101**	**89**	**79**	**141**	**93**	**91**	**111**	**129**	**114**	**56**	**129**
Overall rank	**5**	**4**	**2**	**1**	**6**	**3**	**2**	**3**	**5.5**	**4**	**1**	**5.5**
MSE	θ						β					
Ξ	ML	LS	WLS	AD	CvM	MPS	ML	LS	WLS	AD	CvM	MPS
$(0.5,0.9)$	13	27	26	18	36	6	32	19	27	29	13	6
$(1,1.5)$	14	29	29	18	30	6	28	20	34	23	13	8
$(1.5,2)$	19	23	29	20	31	6	18	25	30	26	17	10
$(2,3)$	21	23	30	20	28	6	19	24	31	23	19	11
$(2.5,4)$	21	21	30	20	28	6	18	23	31	23	22	9
Sum of rank	**88**	**123**	**144**	**96**	**153**	**30**	**115**	**111**	**153**	**124**	**84**	**44**
Overall rank	**2**	**4**	**5**	**3**	**6**	**1**	**4**	**3**	**6**	**5**	**2**	**1**
ABB	θ						β					
Ξ	ML	LS	WLS	AD	CvM	MPS	ML	LS	WLS	AD	CvM	MPS

$(0.5, 0.9)$	9	32	12	26	16	32	14	36	22	29	6	19
$(1, 1.5)$	7	28	14	28	15	34	12	34	19	30	6	25
$(1.5, 2)$	8	18	21	31	18	30	15	29	26	28	6	22
$(2, 3)$	12	21	13	29	18	33	12	32	23	31	6	22
$(2.5, 4)$	17	19	12	28	23	27	15	31	22	31	6	21
Sum of rank	**53**	**118**	**72**	**142**	**90**	**156**	**68**	**162**	**112**	**149**	**30**	**109**
Overall rank	**1**	**4**	**2**	**5**	**3**	**6**	**2**	**6**	**4**	**5**	**1**	**3**
MRE	θ						β					
Ξ	ML	LS	WLS	AD	CvM	MPS	ML	LS	WLS	AD	CvM	MPS
$(0.5, 0.9)$	13	28	17	13	23	32	23	34	31	20	10	8
$(1, 1.5)$	8	24	18	25	19	32	18	32	28	25	8	15
$(1.5, 2)$	20	9	28	16	29	24	26	22	35	15	15	13
$(2, 3)$	27	7	30	15	32	16	30	16	36	16	18	10
$(2.5, 4)$	28	7	29	15	32	15	31	16	35	19	16	9
Sum of rank	**96**	**75**	**122**	**84**	**135**	**116**	**128**	**120**	**165**	**95**	**67**	**55**
Overall rank	**3**	**1**	**5**	**2**	**6**	**4**	**5**	**4**	**6**	**3**	**2**	**1**
Cumulative sum of rank	**365**	**417**	**427**	**401**	**519**	**395**	**402**	**504**	**559**	**482**	**237**	**337**
Final rank	**1**	**4**	**5**	**3**	**6**	**2**	**3**	**5**	**6**	**4**	**1**	**2**

5. Real Data Examples

In this section, two real data analyses are performed to demonstrate the applicability of the proposed distribution. The UFJ distribution is compared to the Favv-Jery (FJ) [8], unit Weibull (UW) [11], unit Lindley (UL) [12], unit Teissier (UT) [13], unit Muth (UM) [2], and unit Improved Second-Degree Lindley (UISDL) distributions [14]. The $\hat{\ell}$, standard error (se) Akaike's information criteria (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS) statistics, and KS p-values are calculated both data sets.

5.1 Real Data Example 1

The first data set contains information about the total milk production proportion for the first lactation of 107 cows from the Carnauba farm in Brazil and taken from [15] and it is also analyzed by [16]. The data consists of 107 samples. The data are: 0.4365, 0.426, 0.514, 0.6907, 0.7471, 0.2605, 0.6196, 0.8781, 0.499, 0.6058, 0.6891, 0.577, 0.5394, 0.1479, 0.2356, 0.6012, 0.1525, 0.5483, 0.6927, 0.7261, 0.3323, 0.0671, 0.2361, 0.48, 0.5707, 0.7131, 0.5853, 0.6768, 0.535, 0.4151, 0.6789, 0.4576, 0.3259, 0.2303, 0.7687, 0.4371, 0.3383, 0.6114, 0.348, 0.4564, 0.7804, 0.3406, 0.4823, 0.5912, 0.5744, 0.5481, 0.1131, 0.729, 0.0168, 0.5529, 0.453, 0.3891, 0.4752, 0.3134, 0.3175, 0.1167, 0.675, 0.5113, 0.5447, 0.4143, 0.5627, 0.515, 0.0776, 0.3945, 0.4553, 0.447, 0.5285, 0.5232, 0.6465, 0.065, 0.8492, 0.8147, 0.3627, 0.3906, 0.4438, 0.4612, 0.3188, 0.216, 0.6707, 0.622, 0.5629, 0.4675, 0.6844, 0.3413, 0.4332, 0.0854, 0.3821, 0.4694, 0.3635, 0.4111, 0.5349, 0.3751, 0.1546, 0.4517, 0.2681, 0.4049, 0.5553, 0.5878, 0.4741, 0.3598, 0.7629, 0.5941, 0.6174, 0.686, 0.0609, 0.6488, 0.2747. The results are presented in the Supplementary Material in Table 11. It can be seen from Table 11 that the new proposed UFJ_2 distribution models the real data better than the distributions available in the literature.

The best model is the UFJ_2 distribution, which has lower values for AIC, BIC, CAIC, and HQIC, smaller KS test statistic values, and larger p-values.

5.2 Real Data Example 2

The second dataset represents the Better Life Index, based on self-reported health data from 2015, it is also analyzed by [17], and can be accessed at the following link: https://stats.oecd.org/index.aspx?DataSetCode=BLI2015. The data consists of 34 samples. The data are: 0.85, 0.69, 0.74, 0.89, 0.59, 0.6, 0.72, 0.54, 0.65, 0.67, 0.65, 0.74, 0.57, 0.77, 0.82, 0.8, 0.66, 0.3, 0.35, 0.72, 0.66, 0.76, 0.9, 0.76, 0.58, 0.46, 0.66, 0.65, 0.72, 0.81, 0.81, 0.68, 0.74, 0.88. The results are presented in the Supplementary Material in Table 12. Table 12 shows that the newly proposed UFJ_2 distribution models the second real dataset more effectively than the existing distributions available in the literature, demonstrating its superior fit and applicability. The best model is the UFJ_2 distribution, which has lower values for AIC, BIC, CAIC, and HQIC, smaller KS test statistic values, and larger p-values.

6. Conclusion

In this study, a novel statistical model is defined within the range (0,1). The mathematical properties of the proposed distribution, including moments, skewness, kurtosis, order statistics, stochastic ordering, as well as the Lorenz and Bonferroni curves, are thoroughly investigated. Six estimators are proposed to estimate the unknown parameters of the UFJ distribution, and their performances are evaluated through Monte Carlo simulations. Monte Carlo simulation results show that all estimators perform effectively, especially with large sample sizes. Based on the simulation results, the most efficient estimation methods are identified. Additionally, two numerical examples are provided to demonstrate the practical applicability of the new distribution. Based on the real data analysis results, it is observed that the proposed UFJ distribution best fits the data. This study contributes a new unit distribution, denoted as UFJ, defined on the (0,1) interval to the literature. In this regard, the study holds significant importance.

7. References

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