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Curvatures of Implicit Hypersurfaces in Euclidean 4-space

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ABSTRACT: In this paper, we study the Gaussian and the mean curvatures of a hypersurface in Euclidean 4-space. We obtain the formulas of these curvatures by using the Riemannian connection of Euclidean 4-space. As an application of the obtained formulas, we give the Gaussian and mean curvatures of an implicit hypersurface. Also, the Gaussian curvatures of some quadric hypersurfaces are given.

Keywords: Gaussian curvature, hypersurface, mean curvature

4-boyutlu Öklid Uzayında Kapalı Hiperyüzeylerin Eğrilikleri

ÖZET: Bu çalışmada, 4-boyutlu Öklid uzayında bir hiperyüzeyin Gauss ve ortalama eğrilikleri çalışılmıştır. 4-boyutlu Öklid uzayında Riemann konneksiyonu kullanılarak bu eğriliklerin formülleri elde edilmiştir. Elde edilen formüllerin bir uygulaması olarak bir kapalı hiperyüzeyin Gauss ve ortalama eğrilikleri verilmiştir. Ayrıca, bazı kuadrik hiperyüzeylerin Gauss eğrilikleri elde edilmiştir.

Anahtar Kelimeler: Gauss eğriliği, hiperyüzey, ortalama eğrilik

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INTRODUCTION

The Gaussian and the mean curvatures are the most important curvatures of a surface in Euclidean 3-space. The formulas for these curvatures are well-known not only for parametric surfaces but also for implicit surfaces (see e.g. (Gray et al., 2006)). These curvatures are defined by the shape operator of the surface and they are independent of the chosen basis of the tangent space. In this paper, we study these curvatures for implicit hypersurfaces in Euclidean 4-space.

MATERIALS AND METHODS

Definition 1. The ternary product of the vectors $\mathbf{a} = \sum_{i=1}^{4} a_i \mathbf{e_i}$, $\mathbf{b} = \sum_{i=1}^{4} b_i \mathbf{e_i}$, and $\mathbf{c} = \sum_{i=1}^{4} c_i \mathbf{e_i}$

is defined by, (Hollasch, 1991; Williams and Stein, 1964),

$$\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is the standard basis of $\mathbb{R}^4$.

The ternary product has the following properties (Williams and Stein, 1964):

$$\begin{aligned} &1)\ \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = -\mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c} = \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{a}, \\ &2)\ \langle \mathbf{a}, \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} \rangle = \det\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}, \\ &3)\ \langle \mathbf{a}, \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d} \rangle = -\langle \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}, \mathbf{d} \rangle, \\ &4)\ (\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} \otimes \mathbf{d} = \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{d} + \mathbf{b} \otimes \mathbf{c} \otimes \mathbf{d}. \end{aligned} \tag{1}$$

Definition 2. Let M be a regular hypersurface in Euclidean 4-space E^4, $\mathbf{D}$ be the Riemannian connection of E^4, $\mathbf{U}$ be the unit normal vector field of M, and $P \in M$ be a point. Then, for a tangent vector $\mathbf{v}_P$ to M at P the shape operator of M is defined by $S(v_P) = -D_{v_P} U$ (Lee, 1997).

Definition 3. Let M be a regular hypersurface in Euclidean 4-space E^4. The Gaussian curvature K and the mean curvature H of M at a point $P \in M$ are defined by $K(P) = det(S_P)$ and $H(P) = \frac{1}{3} trace(S_P)$ where S_P is the matrix of the shape operator S of M (Lee, 1997).

RESULTS AND DISCUSSION

Theorem 1. Let P be a point on a regular hypersurface $M \subset E^4$, and let $\mathbf{v}_p$, $\mathbf{w}_p$, $\mathbf{r}_p$ be linearly independent tangent vectors to M at P. Then the Gaussian and the mean curvatures of M at P satisfy the following equations:

$$S(\mathbf{v}_P)\otimes S(\mathbf{w}_P)\otimes S(\mathbf{r}_P)=K(P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P, \tag{2}$$

$$S(\mathbf{v}_P)\otimes\mathbf{w}_P\otimes\mathbf{r}_P+\mathbf{v}_P\otimes S(\mathbf{w}_P)\otimes\mathbf{r}_P+\mathbf{v}_P\otimes\mathbf{w}_P\otimes S(\mathbf{r}_P)=3H(P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P. \tag{3}$$

Proof. Since $\mathbf{v}_P$, $\mathbf{w}_P$, and $\mathbf{r}_P$ are linearly independent, they constitute a basis of the tangent space $T_M(P)$. Then we can write

$$\begin{cases} S(\mathbf{v}_p)=a_1\mathbf{v}_p+b_1\mathbf{w}_p+c_1\mathbf{r}_p, \\ S(\mathbf{w}_p)=a_2\mathbf{v}_p+b_2\mathbf{w}_p+c_2\mathbf{r}_p, \\ S(\mathbf{r}_p)=a_3\mathbf{v}_p+b_3\mathbf{w}_p+c_3\mathbf{r}_p. \end{cases} \tag{4}$$

where $a_i, b_i, c_i \in \mathbb{R}, 1 \leq i \leq 3$.

Thus, the matrix at P of the shape operator is given by $S_P=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

Using Eq. (4), we obtain

$$\begin{aligned} S(\mathbf{v}_P)\otimes S(\mathbf{w}_P)\otimes S(\mathbf{r}_P) &= (a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-a_1b_3c_2-a_2b_1c_3-a_3b_2c_1).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P \\ &= \det(S_P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P \\ &= K(P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P \end{aligned}$$

and

$$\begin{aligned} S(\mathbf{v}_P)\otimes\mathbf{w}_P\otimes\mathbf{r}_P+\mathbf{v}_P\otimes S(\mathbf{w}_P)\otimes\mathbf{r}_P+\mathbf{v}_P\otimes\mathbf{w}_P\otimes S(\mathbf{r}_P) &= (a_1+b_2+c_3).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P \\ &= \operatorname{trace}(S_P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P \\ &= 3H(P).\mathbf{v}_P\otimes\mathbf{w}_P\otimes\mathbf{r}_P. \end{aligned}$$

Theorem 2. Let $\mathbf{Z}$ be a nonvanishing normal vector field on a regular hypersurface $M \subset E^4$. Let $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{R}$ be the tangent vector fields of M satisfying $\mathbf{V} \otimes \mathbf{W} \otimes \mathbf{R} = \mathbf{Z}$. Then the Gaussian and the mean curvatures of M can be given by

$$K = \frac{1}{\|\mathbf{Z}\|^5} \det\{\mathbf{D_V Z},\ \mathbf{D_W Z},\ \mathbf{D_R Z}, \mathbf{Z}\} \tag{5}$$

and

$$H = \frac{1}{3\|\mathbf{Z}\|^3}\left(\det\{\mathbf{Z}, \mathbf{V}, \mathbf{R}, \mathbf{D_W Z}\} - \det\{\mathbf{Z}, \mathbf{W}, \mathbf{R}, \mathbf{D_V Z}\} - \det\{\mathbf{Z}, \mathbf{V}, \mathbf{W}, \mathbf{D_R Z}\}\right), \tag{6}$$

respectively.

Proof. Let us denote the unit normal vector field of the hypersurface with $\mathbf{U}$. Then we may write $\mathbf{U} = \frac{\mathbf{Z}}{\|\mathbf{Z}\|}$. Then we have $S(\mathbf{V}) = -\mathbf{D_V U} = -\mathbf{V}\left[\frac{1}{\|\mathbf{Z}\|}\right]\mathbf{Z} - \frac{1}{\|\mathbf{Z}\|}\mathbf{D_V Z}$. Therefore, we obtain

$$S(\mathbf{V}) \otimes S(\mathbf{W}) \otimes S(\mathbf{R}) = \frac{-1}{\|\mathbf{Z}\|^3} \mathbf{D_V Z} \otimes \mathbf{D_W Z} \otimes \mathbf{D_R Z} + \Omega,$$

where

$$\Omega = \frac{\mathbf{W}\left[\frac{1}{\|\mathbf{Z}\|}\right]}{\|\mathbf{Z}\|^2}\mathbf{D_V Z} \otimes \mathbf{D_R Z} \otimes \mathbf{Z} - \frac{\mathbf{V}\left[\frac{1}{\|\mathbf{Z}\|}\right]}{\|\mathbf{Z}\|^2}\mathbf{D_W Z} \otimes \mathbf{D_R Z} \otimes \mathbf{Z} - \frac{\mathbf{R}\left[\frac{1}{\|\mathbf{Z}\|}\right]}{\|\mathbf{Z}\|^2}\mathbf{D_V Z} \otimes \mathbf{D_W Z} \otimes \mathbf{Z}$$

is a tangent vector field of M. Then, by using Eq. (2), we have

$$K.\mathbf{V} \otimes \mathbf{W} \otimes \mathbf{R} = K.\mathbf{Z} = \frac{-1}{\|\mathbf{Z}\|^3} \mathbf{D_V Z} \otimes \mathbf{D_W Z} \otimes \mathbf{D_R Z} + \Omega. \tag{7}$$

Taking the scalar product of both sides of Eq. (7) with $\mathbf{Z}$ yields

$$K.\langle \mathbf{Z}, \mathbf{Z} \rangle = \frac{-1}{\|\mathbf{Z}\|^3} \langle \mathbf{D_V Z} \otimes \mathbf{D_W Z} \otimes \mathbf{D_R Z}, \mathbf{Z} \rangle.$$

Then, using Eq. (1) we get

$$K = \frac{1}{\|\mathbf{Z}\|^5} \det\{\mathbf{D_V Z},\ \mathbf{D_W Z},\ \mathbf{D_R Z}, \mathbf{Z}\}.$$

Similarly, from Eq. (3) we have

$$3H.\mathbf{V}\otimes\mathbf{W}\otimes\mathbf{R}=S(\mathbf{V})\otimes\mathbf{W}\otimes\mathbf{R}+\mathbf{V}\otimes S(\mathbf{W})\otimes\mathbf{R}+\mathbf{V}\otimes\mathbf{W}\otimes S(\mathbf{R})$$

and

$$3H.\mathbf{Z}=-\frac{1}{\|\mathbf{Z}\|}\left(\mathbf{D_V Z}\otimes\mathbf{W}\otimes\mathbf{R}+\mathbf{V}\otimes\mathbf{D_W Z}\otimes\mathbf{R}+\mathbf{V}\otimes\mathbf{W}\otimes\mathbf{D_R Z}\right)$$

$$-\mathbf{V}\left[\tfrac{1}{\|\mathbf{Z}\|}\right]\mathbf{Z}\otimes\mathbf{W}\otimes\mathbf{R}-\mathbf{W}\left[\tfrac{1}{\|\mathbf{Z}\|}\right]\mathbf{V}\otimes\mathbf{Z}\otimes\mathbf{R}-\mathbf{R}\left[\tfrac{1}{\|\mathbf{Z}\|}\right]\mathbf{V}\otimes\mathbf{W}\otimes\mathbf{Z}. \qquad (8)$$

Taking the scalar product of both sides of Eq. (8) with $\mathbf{Z}$ and using Eq. (1) yield

$$H=\frac{1}{3\|\mathbf{Z}\|^3}\left(\det\{\mathbf{Z},\mathbf{V},\mathbf{R},\mathbf{D_W Z}\}-\det\{\mathbf{Z},\mathbf{W},\mathbf{R},\mathbf{D_V Z}\}-\det\{\mathbf{Z},\mathbf{V},\mathbf{W},\mathbf{D_R Z}\}\right).$$

As an application of Theorem 2, we may give the following:

Example 1. Let M be the hypersurface given by

$$\left\{(x_1,x_2,x_3,x_4)\in E^4 \mid f_1x_1^k+f_2x_2^k+f_3x_3^k+f_4x_4^k=1\right\},$$

where f_1, f_2, f_3, f_4 are constants being not all zero, k is a nonzero real number, and x_1, x_2, x_3, x_4 are the natural coordinate functions of E^4.

Now, by using Theorem 2, let us obtain the Gaussian curvature and the mean curvature of M.

Let $g(x_1,x_2,x_3,x_4)=f_1x_1^k+f_2x_2^k+f_3x_3^k+f_4x_4^k$. Thus, we have $M=\left\{P\in E^4 \mid g(P)=1\right\}$.

Then $\mathbf{Z}=\nabla g$ is a nonvanishing normal vector field of M, i.e. $\mathbf{Z}=k\sum_{i=1}^{4} f_i x_i^{k-1}\frac{\partial}{\partial x_i}$.

Let $\mathbf{V}=\sum_{i=1}^{4} v_i\frac{\partial}{\partial x_i}$, $\mathbf{W}=\sum_{i=1}^{4} w_i\frac{\partial}{\partial x_i}$, and $\mathbf{R}=\sum_{i=1}^{4} r_i\frac{\partial}{\partial x_i}$ be three linearly independent tangent vector fields on M. Since f_i, $1\le i\le 4$, are constants, we have

$$\mathbf{D_V Z}=k\sum_{i=1}^{4}\mathbf{V}\left[f_i x_i^{k-1}\right]\frac{\partial}{\partial x_i}=k(k-1)\sum_{i=1}^{4} f_i v_i x_i^{k-2}\frac{\partial}{\partial x_i},$$

$$\mathbf{D_W Z} = k\sum_{i=1}^{4} \mathbf{W}\left[f_i x_i^{k-1} \right]\tfrac{\partial}{\partial x_i} = k\left(k-1\right)\sum_{i=1}^{4} f_i w_i x_i^{k-2}\tfrac{\partial}{\partial x_i},$$

$$\mathbf{D_R Z} = k\sum_{i=1}^{4} \mathbf{R}\left[f_i x_i^{k-1} \right]\tfrac{\partial}{\partial x_i} = k\left(k-1\right)\sum_{i=1}^{4} f_i r_i x_i^{k-2}\tfrac{\partial}{\partial x_i}.$$

Therefore, $\det\{\mathbf{D_V Z}, \mathbf{D_W Z}, \mathbf{D_R Z}, \mathbf{Z}\}$ is equal to

$$\begin{vmatrix} k(k-1) f_1 v_1 x_1^{k-2} & k(k-1) f_2 v_2 x_2^{k-2} & k(k-1) f_3 v_3 x_3^{k-2} & k(k-1) f_4 v_4 x_4^{k-2} \\ k(k-1) f_1 w_1 x_1^{k-2} & k(k-1) f_2 w_2 x_2^{k-2} & k(k-1) f_3 w_3 x_3^{k-2} & k(k-1) f_4 w_4 x_4^{k-2} \\ k(k-1) f_1 r_1 x_1^{k-2} & k(k-1) f_2 r_2 x_2^{k-2} & k(k-1) f_3 r_3 x_3^{k-2} & k(k-1) f_4 r_4 x_4^{k-2} \\ k f_1 x_1^{k-1} & k f_2 x_2^{k-1} & k f_3 x_3^{k-1} & k f_4 x_4^{k-1} \end{vmatrix}$$

$$= k^4\left(k-1\right)^3 f_1 f_2 f_3 f_4 \left(x_1 x_2 x_3 x_4\right)^{k-2}\det\{\mathbf{V},\mathbf{W},\mathbf{R},\mathbf{X}\},$$

where $\mathbf{X} = \sum_{i=1}^{4} x_i \tfrac{\partial}{\partial x_i}$.

We choose $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{R}$ such that $\mathbf{V}\otimes\mathbf{W}\otimes\mathbf{R} = \mathbf{Z}$. Using Eq. (5), we obtain

$$K = -\frac{1}{\|\mathbf{Z}\|^5} k^4 (k-1)^3 f_1 f_2 f_3 f_4 \left(x_1 x_2 x_3 x_4\right)^{k-2}\langle \mathbf{Z},\mathbf{X}\rangle.$$

Since $\sum_{i=1}^{4} f_i x_i^k = 1,$ we get

$$K = -\frac{1}{\left(\sum_{i=1}^{4} f_i^2 x_i^{2k-2}\right)^{\frac{5}{2}}}\left(k-1\right)^3 f_1 f_2 f_3 f_4 \left(x_1 x_2 x_3 x_4\right)^{k-2}.$$

Now let us compute the mean curvature of M. We have

$$\det\{\mathbf{Z},\mathbf{V},\mathbf{R},\mathbf{D_W Z}\} = \begin{vmatrix} k f_1 x_1^{k-1} & k f_2 x_2^{k-1} & k f_3 x_3^{k-1} & k f_4 x_4^{k-1} \\ v_1 & v_2 & v_3 & v_4 \\ r_1 & r_2 & r_3 & r_4 \\ k(k-1) f_1 w_1 x_1^{k-2} & k(k-1) f_2 w_2 x_2^{k-2} & k(k-1) f_3 w_3 x_3^{k-2} & k(k-1) f_4 w_4 x_4^{k} \end{vmatrix}$$

$$
\begin{aligned}
&= k^2(k-1)\Big\{ f_1 f_2 (x_1 x_2)^{k-2}(x_1 w_2 - w_1 x_2)(v_3 r_4 - r_3 v_4) + f_1 f_3 (x_1 x_3)^{k-2}(x_1 w_3 - w_1 x_3)(r_2 v_4 - v_2 r_4) \\
&\quad + f_1 f_4 (x_1 x_4)^{k-2}(x_1 w_4 - w_1 x_4)(v_2 r_3 - r_2 v_3) + f_2 f_4 (x_2 x_4)^{k-2}(x_2 w_4 - w_2 x_4)(r_1 v_3 - v_1 r_3) \\
&\quad + f_2 f_3 (x_2 x_3)^{k-2}(x_2 w_3 - w_2 x_3)(v_1 r_4 - r_1 v_4) + f_3 f_4 (x_3 x_4)^{k-2}(x_3 w_4 - w_3 x_4)(v_1 r_2 - r_1 v_2)\Big\}.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\det\{\mathbf{Z},\mathbf{W},\mathbf{R},\mathbf{D_V Z}\} &= k^2(k-1)\Big\{ f_1 f_2 (x_1 x_2)^{k-2}(x_1 v_2 - v_1 x_2)(w_3 r_4 - r_3 w_4) \\
&\quad + f_1 f_3 (x_1 x_3)^{k-2}(x_1 v_3 - v_1 x_3)(r_2 w_4 - w_2 r_4) + f_1 f_4 (x_1 x_4)^{k-2}(x_1 v_4 - v_1 x_4)(w_2 r_3 - r_2 w_3) \\
&\quad + f_2 f_4 (x_2 x_4)^{k-2}(x_2 v_4 - v_2 x_4)(r_1 w_3 - w_1 r_3) + f_2 f_3 (x_2 x_3)^{k-2}(x_2 v_3 - v_2 x_3)(w_1 r_4 - r_1 w_4) \\
&\quad + f_3 f_4 (x_3 x_4)^{k-2}(x_3 v_4 - v_3 x_4)(w_1 r_2 - r_1 w_2)\Big\}
\end{aligned}
$$

and

$$
\begin{aligned}
\det\{\mathbf{Z},\mathbf{V},\mathbf{W},\mathbf{D_R Z}\} &= k^2(k-1)\Big\{ f_1 f_2 (x_1 x_2)^{k-2}(x_1 r_2 - r_1 x_2)(v_3 w_4 - w_3 v_4) \\
&\quad + f_1 f_3 (x_1 x_3)^{k-2}(x_1 r_3 - r_1 x_3)(w_2 v_4 - v_2 w_4) + f_1 f_4 (x_1 x_4)^{k-2}(x_1 r_4 - r_1 x_4)(v_2 w_3 - w_2 v_3) \\
&\quad + f_2 f_4 (x_2 x_4)^{k-2}(x_2 r_4 - r_2 x_4)(w_1 v_3 - v_1 w_3) + f_2 f_3 (x_2 x_3)^{k-2}(x_2 r_3 - r_2 x_3)(v_1 w_4 - w_1 v_4) \\
&\quad + f_3 f_4 (x_3 x_4)^{k-2}(x_3 r_4 - r_3 x_4)(v_1 w_2 - w_1 v_2)\Big\}
\end{aligned}
$$

Therefore, $\det\{\mathbf{Z},\mathbf{V},\mathbf{R},\mathbf{D_W Z}\} - \det\{\mathbf{Z},\mathbf{W},\mathbf{R},\mathbf{D_V Z}\} - \det\{\mathbf{Z},\mathbf{V},\mathbf{W},\mathbf{D_R Z}\}$ is equal to

$$
k^2(k-1)\sum_{1=i<j}^{4} f_i f_j (x_i x_j)^{k-2} \det\{\mathbf{X}_{ij},\mathbf{W},\mathbf{V},\mathbf{R}\}, \tag{9}
$$

where $\mathbf{X}_{ij} = x_i \frac{\partial}{\partial x_i} + x_j \frac{\partial}{\partial x_j}$, $1 = i < j \le 4$. Since $\mathbf{V}\otimes\mathbf{W}\otimes\mathbf{R} = \mathbf{Z}$, substituting $\det\{\mathbf{X}_{ij},\mathbf{W},\mathbf{V},\mathbf{R}\} = -\langle \mathbf{X}_{ij},\mathbf{Z}\rangle$ into Eq. (9), and using Eq. (6), we obtain the mean curvature of M as

$$
H = \frac{-k+1}{3\left(\sum_{i=1}^{4} f_i^2 x_i^{2k-2}\right)^{\frac{3}{2}}} \sum_{1=i<j}^{4} f_i f_j (x_i x_j)^{k-2}\left(f_i x_i^k + f_j x_j^k\right).
$$

Corollary 1. The Gaussian curvatures of the quadric hypersurfaces

$(i)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{w^2}{d^2}=1,$ (Ellipsoid)

$(ii)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{w^2}{d^2}=1,$ (Ellipsoidal hyperboloid with one sheet)

$(iii)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{w^2}{d^2}=1,$ (Troidal hyperboloid)

$(iv)\quad \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{w^2}{d^2}=1,$ (Ellipsoidal hyperboloid with two sheets)

are given in each case by $K=\mp\frac{1}{(abcd)^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}+\frac{w^2}{d^4}\right)^{-\frac{5}{2}}$, where the minus sign appears in (i) and (iii). The lines of curvature on above quadric hypersurfaces are studied in (Sotomayor and Garcia, 2016).

CONCLUSION

In this paper, the Gaussian and the mean curvatures of a hypersurface are studied in 4-dimensional Euclidean space and the formulas for these curvatures of implicit hypersurfaces are obtained.

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