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GAUSS AND CODAZZI-MAINARDI FORMULAE

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Abstract

In this paper we have defined ε_i sign functions using the vector fields X_u, X_v, n_u and n_v which have taken derivatives with (u,v) parameters of tangent vector X of any surface in Lorentz space and we obtain Gauss and Codazzi-Mainardi Gauss formulae of the surface.

Preliminaries

It is well known that in a Lorentzian Manifold we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. In this article, however, we have examined some characteristics belonging to the surface by making some special choices on tangent space along the coordinate curves of the surface. Let IR^3 be endowed with the pseudoscalar product of X and Y is defined by

$$\langle X,Y\rangle = x_1y_1 + x_2y_2 - x_3y_3 \qquad X = (x_1,x_2,x_3),\ Y = (y_1,y_2,y_3)$$

$(IR^3,\langle,\rangle)$ is called 3-dimensional Lorentzian space denoted by L^3 [1]. The Lorentzian vector product is defined by

$$X \times Y = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

A vector X in L^3 is called a space-like, light-like, time-like vector if $\langle X,X\rangle > 0$, $\langle X,X\rangle = 0$ or $\langle X,X\rangle < 0$ accordingly. For $X \in L^3$, the norm of X defined by

$$\|X\| = \sqrt{|\langle X,X\rangle|}$$

and X is called a unit vector if $\|X\| = 1$ [2].

1. INTRODUCTION

Definition 1.1. A symmetric bilinear form b on vector space V is

i) positive [negative] definite provided $v \neq 0$ implies $b(v,v) > 0$ [< 0]

ii) positive [negative] semi-definite provided $v \geq 0$ [$v \leq 0$] for all $v \in V$

iii) non-degenerate provided $b(v,w) = 0$ for all $w \in V$ implies $v = 0$ [1].

Definition 1.2. A scalar product g on a vector space V is a non-degenerate symmetric bilinear form on V [1].

Definition 1.3. The index ν of symmetric bilinear form b on V is the largest integer that is the dimension of a subspace $W \subset V$ on which $g|_W$ is negative definite[1].

Lemma 1.4. A scalar product space $V \neq 0$ has an orthonormal basis for V, $\varepsilon_i = \langle e_i, e_i \rangle$. Then each $v \in V$ has a unique expression [1],

$$v = \sum_{i=1}^{n} \varepsilon_i \langle e_i, e_i \rangle e_i$$

Lemma 1.5. For any orthonormal basis $\{e_1, ..., e_n\}$ for V, the number of negative signs in the signature $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ is the index ν of V [1].

Definition 1.6. A metric tensor g on a smooth manifold M is a symmetric nondegenerate (0, 2) tensor field on M of constant index [1].

Definition 1.7. A semi-Riemannian manifold is a smooth manifold furnished with a metric tensor g.

Definition 1.8. A semi-Riemannian submanifold M with (n−1)–dimensional of a semi-Riemannian manifold M with n-dimensional is called semi-Riemannian hypersurface of M [1].

2. GAUSS FORMULAE

Let $X = X(u,v)$ be a surface in Lorentz space and $n(u,v)$ be unit normal vector field of the surface. ε_1, ε_2, ε_3 ε_4, ε_5 be sign functions of the vectors n_u, n_v, X_u, X_v, n, respectively. Then we can write the following equations.

$$\langle n_u, n_u \rangle = \varepsilon_1 \|n_u\|^2, \ \langle n_v, n_v \rangle = \varepsilon_2 \|n_v\|^2, \langle X_u, X_u \rangle = \varepsilon_3 \|X_u\|^2$$
$$\langle X_v, X_v \rangle = \varepsilon_4 \|X_v\|^2, \ \langle n, n \rangle = \varepsilon_5$$

$$\langle n_v, X_u \rangle = \frac{M}{\sqrt{\varepsilon_2 \varepsilon_3}}, \ \langle n_v, X_v \rangle = \frac{N}{\sqrt{\varepsilon_2 \varepsilon_4}}, \ \langle n_u, X_v \rangle = \frac{M}{\sqrt{\varepsilon_1 \varepsilon_4}}, \ \langle n_u, X_u \rangle = \frac{L}{\sqrt{\varepsilon_1 \varepsilon_3}}$$
$$\langle X_u, X_v \rangle = \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}, \ \langle X_u, X_u \rangle = \varepsilon_3 E, \ \langle X_v, X_v \rangle = \varepsilon_4 G$$

The vectors fields X_{uu}, X_{uv}, X_{vv} can be written as linear combinations of X_u, X_v, n as follows.

$$X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + \alpha_1 n$$

$$X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + \alpha_2 n$$

$$X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + \alpha_3 n$$

where the coefficients Γ_{ij}^{k} are Christoffel symbols. These equations are called Gauss formulae of the surface *X(u,v)*. We get $\alpha_1 = \varepsilon_5 L$, $\alpha_2 = \varepsilon_5 M$ and $\alpha_3 = \varepsilon_5 N$ by using inner production with the normal vector n of the Gauss formulae.

On the other hand, we take derivatives of following equation with respect to the parameters u and v

$$\langle X_u, X_v \rangle = \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

then we obtain,

$$\langle X_{uu}, X_v \rangle + \langle X_u, X_{uv} \rangle = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} \langle X_{vu}, X_v \rangle + \langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

$$\langle X_{uu}, X_v \rangle + \frac{\varepsilon_3}{2} E_v = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} \frac{\varepsilon_4}{2} G_u + \langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

we multiply Gauss equations by $\vec{X}_u$ and $\vec{X}_v$ we get

$$\langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_4}{2} G_u \langle X_{uu}, X_v \rangle = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_3}{2} E_v$$

We get the following equations using by inner production both X_u and X_v of the Gauss equations.

(2.1) $$\frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_3}{2} E_v = \Gamma_{11}^1 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma_{11}^2 \varepsilon_4 G$$

(2.2) $$\frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_4}{2} G_u = \Gamma_{22}^1 \varepsilon_3 E + \Gamma_{22}^2 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

(2.3) $$\frac{\varepsilon_4}{2} G_v = \Gamma_{22}^1 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma_{22}^2 \varepsilon_4 G$$

(2.4) $$\frac{\varepsilon_3}{2} E_u = \Gamma_{11}^1 \varepsilon_3 E + \Gamma_{11}^2 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

(2.5) $$\frac{\varepsilon_3}{2} E_v = \Gamma_{12}^1 \varepsilon_3 E + \Gamma_{12}^2 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}$$

(2.6) $$\frac{\varepsilon_4}{2} G_u = \Gamma_{12}^1 \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma_{12}^2 \varepsilon_4 G$$

Thus, we can calculate Γ_{11}^1, Γ_{11}^2, Γ_{12}^1, Γ_{12}^2, Γ_{22}^1 and Γ_{22}^2 coefficients. At first, we have to solve (2.1) and (2.4) together, we get

(2.7) $$\Gamma_{11}^{1}=\frac{\varepsilon_3 FE_v}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 GE_u}{2H^2}-\frac{FF_u}{H^2}$$

we solve (2.5) and (2.6) together, we get

(2.8) $$\Gamma_{12}^{1}=\frac{\varepsilon_3\varepsilon_4 GE_v}{2H^2}-\frac{\varepsilon_3 FG_u}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}$$

we solve (2.2) and (2.3) together, we get

(2.9) $$\Gamma_{22}^{1}=\frac{\varepsilon_4}{\sqrt{\varepsilon_3\varepsilon_4}H^2}(GF_v-FG_v)-\frac{GG_u}{2H^2}$$

we solve (2.1) and (2.4) together, we get

(2.10) $$\Gamma_{11}^{2}=\frac{\varepsilon_3(EF_u-\frac{FE_u}{2})}{\sqrt{\varepsilon_3\varepsilon_4}H^2}-\frac{EE_v}{2H^2}$$

we solve (2.5) and (2.6) together, we get

(2.11) $$\Gamma_{12}^{2}=\frac{\varepsilon_3\varepsilon_4 EG_u}{2H^2}-\frac{\varepsilon_3 FE_v}{2\sqrt{\varepsilon_3\varepsilon_4}H}$$

finally we solve (2.2) and (2.3) together, we get

(2.12) $$\Gamma_{22}^{2}=\frac{\varepsilon_4 FG_u}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 EG_v}{2H^2}-\frac{FF_v}{\varepsilon_3\varepsilon_4 H^2}$$

Theorem 2.1: If coordinate lines are normal each other, then F=0 and Gauss formulae are

$$X_{uu}=\frac{E_u}{2E}X_u+\frac{E_v}{2\varepsilon_3\varepsilon_4 G}X_v+\varepsilon_5 Ln$$

$$X_{uv}=\frac{E_v}{2E}X_u+\frac{G_u}{2G}X_v+\varepsilon_5 Mn$$

$$X_{vv}=\frac{-G_u}{2\varepsilon_3\varepsilon_4 E}X_u+\frac{G_v}{2G}X_v+\varepsilon_5 Nn$$

3. CODAZZİ-MİNARDİ FORMULAE

If M C^3- manifold then its replacement vector has to satisfy the following equations at point $P(u,v)$.

$$(X_{uu})_v=(X_{uv})_v \ , \ (X_{vv})_u=(X_{uv})_v$$

thus by using Gauss equations we get

(3.1) $$\left(\Gamma_{11}^{1}X_u+\Gamma_{11}^{2}X_v+\varepsilon_5 Ln\right)_v=\left(\Gamma_{12}^{1}X_u+\Gamma_{12}^{2}X_v+\varepsilon_5 M\,n\right)_v$$

(3.2) $$\left(\Gamma_{22}^{1}X_u+\Gamma_{22}^{2}X_v+\varepsilon_5 N\,n\right)_u=\left(\Gamma_{12}^{1}X_u+\Gamma_{12}^{2}X_v+\varepsilon_5 M\,n\right)_v$$

rewrite the coefficients Γ_{ij}^{k} which we obtained for $F\neq 0$ and we get

$$\Gamma_{11}^{1}=\frac{\varepsilon_5 FE_v}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 GE_u}{2H^2}-\frac{FF_u}{H^2}\ ,\ \Gamma_{12}^{1}=-\frac{\varepsilon_4 FG_u}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 GE_v}{2H^2}$$

$$\Gamma_{12}^{2}=-\frac{\varepsilon_4 FE_v}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 G_u E}{2H^2}\ ,\qquad \Gamma_{22}^{1}=\frac{\varepsilon_4(GF_v-FG_v)}{\sqrt{\varepsilon_3\varepsilon_4}H^2}-\frac{GG_u}{2H^2}$$

$$\Gamma_{11}^{2}=\frac{\varepsilon_3(EF_u-\frac{FE_u}{2})}{\sqrt{\varepsilon_3\varepsilon_4}H^2}-\frac{EE_v}{2H^2}\ ,\quad \Gamma_{22}^{2}=\frac{\varepsilon_4 FG_u}{2\sqrt{\varepsilon_3\varepsilon_4}H^2}+\frac{\varepsilon_3\varepsilon_4 EG_v}{2H^2}-\frac{FF_v}{\varepsilon_3\varepsilon_4 H^2}$$

By taking partial derivatives of (3.1) and using the vector X_{uu}, X_{uv}, X_{vv}, n_u and n_v then we get,

(3.3) $$A_1\vec{X}_u+A_2\vec{X}_v+A_3\vec{n}=\vec{0}$$

where the coefficients A_1, A_2, A_3 are as following,

$$A_1=\left(\Gamma_{11}^{1}\right)_v-\left(\Gamma_{12}^{1}\right)_u+\Gamma_{11}^{2}\Gamma_{22}^{1}-\Gamma_{12}^{2}\Gamma_{12}^{1}+\varepsilon_5 L\bar{a}_{21}-\varepsilon_5 M\,\bar{a}_{11}$$

$$A_2=\left(\Gamma_{11}^{1}\right)_v-\left(\Gamma_{12}^{2}\right)_u+\Gamma_{11}^{1}\Gamma_{12}^{2}-\Gamma_{12}^{2}\Gamma_{12}^{2}+\Gamma_{11}^{2}\Gamma_{12}^{2}-\Gamma_{12}^{1}\Gamma_{11}^{2}+\varepsilon_5 L\bar{a}_{22}-\varepsilon_5 M\,\bar{a}_{12}$$

$$A_3=\varepsilon_5 L_v-\varepsilon_5 M_u+\varepsilon_5 M(\Gamma_{11}^{1}-\Gamma_{12}^{2})+\varepsilon_5 N\Gamma_{11}^{2}-\varepsilon_5 L\Gamma_{12}^{1}$$

where $\bar{a}_{11}$, $\bar{a}_{12}$, $\bar{a}_{21}$ and $\bar{a}_{22}$ are the components of Weingarten matrix. Similarly, for (3.2) we get,

(3.4) $$B_1\vec{X}_u+B_2\vec{X}_v+B_3\vec{n}=\vec{0}$$

where the coefficients B_1, B_2, B_3 are as following,

$$B_1=\left(\Gamma_{22}^{1}\right)_u-\left(\Gamma_{12}^{1}\right)_v+\Gamma_{22}^{1}\Gamma_{11}^{1}+\Gamma_{22}^{2}\Gamma_{12}^{1}-\Gamma_{12}^{1}\Gamma_{12}^{1}-\Gamma_{12}^{2}\Gamma_{22}^{1}+\varepsilon_5 N\,\bar{a}_{11}-\varepsilon_5 M\,\bar{a}_{21}$$

$$B_2=\left(\Gamma_{22}^{2}\right)_u-\left(\Gamma_{12}^{2}\right)_v+\Gamma_{22}^{1}\Gamma_{11}^{2}+\Gamma_{22}^{2}\Gamma_{12}^{1}-\Gamma_{12}^{1}\Gamma_{12}^{2}-\Gamma_{12}^{2}\Gamma_{22}^{2}+\varepsilon_5 N\,\bar{a}_{12}-\varepsilon_5 M\,\bar{a}_{22}$$

$$B_3=\varepsilon_5 N_u-\varepsilon_5 M_v+\Gamma_{22}^{1}\varepsilon_5 L+(\Gamma_{22}^{2}-\Gamma_{12}^{1})\varepsilon_5 M-\Gamma_{12}^{2}\varepsilon_5 N$$

Since the vectors X_u, X_v and n are linearly independent in (3.3) and (3.4) then we get

$A_i=0,\ B_i=0,\ (i=1,2,3)$. For $A_3=0$ and $B_3=0$;

(3.5) $$L_v-M_u=M(\Gamma_{12}^{2}-\Gamma_{11}^{1})-N\Gamma_{11}^{2}+L\Gamma_{12}^{1}$$

(3.6) $$N_u-M_v=M(\Gamma_{12}^{1}-\Gamma_{22}^{2})+N\Gamma_{12}^{2}-L\Gamma_{22}^{1}$$

The equations (3.5) and (3.6) are called Codazzi-Mainardi formulae of the surface $X(u,v)$.

a) The case F=0 ; We calculate Γ_{ij}^{k} coefficients from Gauss formulae, we get,

$$\Gamma_{11}^{1}=\frac{E_u}{2E}\ ,\ \Gamma_{12}^{1}=\frac{E_v}{2E}\ ,\ \Gamma_{22}^{1}=\frac{-G_u}{2\varepsilon_3\varepsilon_4 E},\ \Gamma_{11}^{2}=\frac{-E_v}{2\varepsilon_3\varepsilon_4 E},\ \Gamma_{12}^{2}=\frac{G_u}{2G},\ \Gamma_{22}^{2}=\frac{G_v}{2G}$$

and substitute these values in Codazzi-Mainardi equations, then we get, (3.7) $$L_v + M\frac{E_u}{2E} - N\frac{E_v}{2\varepsilon_3\varepsilon_4 G} = M_u + L\frac{E_v}{2E} + M\frac{G_u}{2G}$$

(3.8) $$N_u + M\frac{G_v}{2G} - L\frac{G_u}{2\varepsilon_3\varepsilon_4 E} = M_v + M\frac{E_v}{2E} + N\frac{G_u}{2}$$

b) The case F=0 and M=0 ; In this case (3.7) and (3.8) equations will be as following

$$L_v - N\frac{E_v}{2\varepsilon_3\varepsilon_4 G} = L\frac{E_v}{2E} \;,\; N_u - L\frac{G_u}{2\varepsilon_3\varepsilon_4 E} = N\frac{G_u}{2}$$

If the surface is compared with zero lengt curves-minimal curves then E, G will be vanish on the surface. And we get following equations by using the equations (2.7), (2.8), (2.9), (2.10), (2.11), (2.12).

$$\Gamma_{11}^1 = \frac{-FF_u}{H^2} = \frac{FF_u}{\varepsilon_3\varepsilon_4 F^2} = \varepsilon_3\varepsilon_4\left(\frac{F_u}{F}\right) = \varepsilon_3\varepsilon_4\left(\log|F|\right)_u$$
$$\Gamma_{22}^2 = \frac{-FF_v}{\varepsilon_3\varepsilon_4 H^2} = \frac{-FF_v}{-F^2} = \frac{F_v}{F} = \left(\log|F|\right)_v$$

and we obtain

$$\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0$$

Thus the Gauss formulae are obtained as follows;

$$X_{uu} = \varepsilon_3\varepsilon_4\left(\log|F|\right)_u X_u + \varepsilon_5 Ln \,,\; X_{uv} = \varepsilon_5 Mn \,,\; X_{vv} = \left(\log|F|\right)_v X_v + \varepsilon_5 Nn$$

Furthermore Codazzi-Mainardi formulae will be as follows

$$L_v - M_u = -\varepsilon_3\varepsilon_4 M\left(\log|F|\right)_u,\; N_u - M_v = -M\left(\log|F|\right)_v$$

and then we obtain

$$\mp\left(\frac{M_u - L_v}{M}\right)_v = \left(\frac{M_v - N_u}{M}\right)_u$$

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