Gauss and Codazzi-Mainardi Formulae

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Abstract

In this paper we have defined $\varepsilon$, sign functions using the vector fields $X_u$, $X_v$, $n_u$ and $n_v$ which have taken derivatives with $(u,v)$ parameters of tangent vector $X$ of any surface in Lorentz space and we obtain Gauss and Codazzi-Mainardi Gauss formulae of the surface.

Preliminaries

It is well known that in a Lorentzian Manifold we can find three types of submanifolds: Space-like (or Riemannian), time-like (Lorentzian) and light-like (degenerate or null), depending on the induced metric in the tangent vector space. Lorentz surfaces has been examined in numerous articles and books. In this article, however, we have examined some characteristics belonging to the surface by making some special choices on tangent space along the coordinate curves of the surface. Let $\mathbb{R}^3$ be endowed with the pseudoscalar product of $X$ and $Y$ is defined by

$$
\langle X, Y \rangle = x_i y_i + x_j y_j - x_k y_k \quad X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3)
$$

$$(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$$

is called 3-dimensional Lorentzian space denoted by $L^3$ [1]. The Lorentzian vector product is defined by

$$
X \times Y = \begin{vmatrix}
  e_1 & e_2 & -e_3 \\
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3
\end{vmatrix}
$$

A vector $X$ in $L^3$ is called a space-like, light-like, time-like vector if $\langle X, X \rangle > 0$, $\langle X, X \rangle = 0$ or $\langle X, X \rangle < 0$ accordingly. For $X \in L^3$, the norm of $X$ defined by

$$
\|X\| = \sqrt{\langle X, X \rangle}
$$

and $X$ is called a unit vector if $\|X\| = 1$ [2].

1. INTRODUCTION

Definition 1.1. A symmetric bilinear form $b$ on vector space $V$ is

i) positive [negative] definite provided $v \neq 0$ implies $b(v, v) > 0 [< 0]$

ii) positive [negative] semi-definite provided $v \geq 0 [v \leq 0]$ for all $v \in V$

iii) non-degenerate provided $b(v, w) = 0$ for all $w \in V$ implies $v = 0$ [1].
Definition 1.2. A scalar product $g$ on a vector space $V$ is a non-degenerate symmetric bilinear form on $V$.

Definition 1.3. The index $v$ of symmetric bilinear form $b$ on $V$ is the largest integer that is the dimension of a subspace $W \subset V$ on which $g^{\mid W}$ is negative definite.

Lemma 1.4. A scalar product space $V \neq 0$ has an orthonormal basis for $V$, $e_i = \langle e_i, e_i \rangle$. Then each $v \in V$ has a unique expression $v = \sum_{i=1}^{n} e_i \langle e_i, e_i \rangle e_i$

Lemma 1.5. For any orthonormal basis $\{e_1, ..., e_n\}$ for $V$, the number of negative signs in the signature $(e_1, e_2, ..., e_n)$ is the index $v$ of $V$.

Definition 1.6. A metric tensor $g$ on a smooth manifold $M$ is a symmetric nondegenerate $(0,2)$ tensor field on $M$ of constant index.

Definition 1.7. A semi-Riemannian manifold is a smooth manifold furnished with a metric tensor $g$.

Definition 1.8. A semi-Riemannian submanifold $M$ with $(n-1)$-dimensional of a semi-Riemannian manifold $M$ with $n$-dimensional is called semi-Riemannian hypersurface of $M$.

2. GAUSS FORMULAE

Let $X = X(u,v)$ be a surface in Lorentz space and $n(u,v)$ be unit normal vector field of the surface. $e_1, e_2, e_3, e_4, e_5$ be sign functions of the vectors $n_u, n_v, X_u, X_v, n$, respectively. Then we can write the following equations.

\[
\langle n_u, n_u \rangle = e_1 \|n_u\|^2, \quad \langle n_v, n_v \rangle = e_2 \|n_v\|^2, \quad \langle X_u, X_u \rangle = e_3 \|X_u\|^2, \\
\langle X_v, X_v \rangle = e_4 \|X_v\|^2, \quad \langle n, n \rangle = e_5
\]

\[
\langle n_v, X_u \rangle = \frac{M}{\sqrt{e_2 e_5}}, \quad \langle n_v, X_v \rangle = \frac{N}{\sqrt{e_2 e_4}}, \quad \langle n_u, X_v \rangle = \frac{M}{\sqrt{e_1 e_4}}, \quad \langle n_u, X_u \rangle = \frac{L}{\sqrt{e_1 e_3}}
\]

\[
\langle X_u, X_v \rangle = \frac{F}{\sqrt{e_3 e_4}}, \quad \langle X_u, n \rangle = e_3 E, \quad \langle X_v, n \rangle = e_4 G
\]

The vectors fields $X_{uu}, X_{uv}, X_{vv}$ can be written as linear combinations of $X_u, X_v, n$ as follows.

\[
X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + \alpha_1 n \\
X_{uv} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + \alpha_2 n \\
X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + \alpha_3 n
\]
where the coefficients \( \Gamma^k_j \) are Christoffel symbols. These equations are called Gauss formulae of the surface \( X(u, v) \). We get \( \alpha_1 = \varepsilon_1 L \), \( \alpha_2 = \varepsilon_2 M \) and \( \alpha_3 = \varepsilon_3 N \) by using inner production with the normal vector \( n \) of the Gauss formulae.

On the other hand, we take derivatives of following equation with respect to the parameters \( u \) and \( v \)

\[
\langle X_u, X_v \rangle = \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

then we obtain,

\[
\langle X_{uu}, X_v \rangle + \langle X_u, X_{vv} \rangle = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} \langle X_{uu}, X_v \rangle + \langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

\[
\langle X_{uu}, X_v \rangle + \frac{\varepsilon_3}{2} E_v = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} \frac{\varepsilon_4}{2} G_u + \langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

we multiply Gauss equations by \( \tilde{X}_u \) and \( \tilde{X}_v \) we get

\[
\langle X_u, X_{vv} \rangle = \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_4}{2} G_u \langle X_{uu}, X_v \rangle = \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_3}{2} E_v
\]

We get the following equations using by inner production both \( X_u \) and \( X_v \) of the Gauss equations.

\[
(2.1) \quad \frac{F_u}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_3}{2} E_v = \Gamma^1_{11} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma^2_{11} \varepsilon_4 G
\]

\[
(2.2) \quad \frac{F_v}{\sqrt{\varepsilon_3 \varepsilon_4}} - \frac{\varepsilon_4}{2} G_u = \Gamma^1_{22} \varepsilon_3 E + \Gamma^2_{22} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

\[
(2.3) \quad \frac{\varepsilon_4}{2} G_v = \Gamma^1_{22} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma^2_{22} \varepsilon_3 G
\]

\[
(2.4) \quad \frac{\varepsilon_3}{2} E_u = \Gamma^1_{11} \varepsilon_3 E + \Gamma^2_{11} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

\[
(2.5) \quad \frac{\varepsilon_3}{2} E_v = \Gamma^1_{12} \varepsilon_3 E + \Gamma^2_{12} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}}
\]

\[
(2.6) \quad \frac{\varepsilon_4}{2} G_u = \Gamma^1_{12} \frac{F}{\sqrt{\varepsilon_3 \varepsilon_4}} + \Gamma^2_{12} \varepsilon_4 G
\]

Thus, we can calculate \( \Gamma^1_{11} \), \( \Gamma^2_{11} \), \( \Gamma^1_{12} \), \( \Gamma^2_{12} \), \( \Gamma^2_{22} \) and \( \Gamma^2_{22} \) coefficients. At first, we have to solve (2.1) and (2.4) together, we get
we solve (2.5) and (2.6) together, we get

\[
\Gamma_{11}^1 = \frac{\varepsilon_3 F_{Eu}}{2\sqrt{\varepsilon_3 \varepsilon_4 H^2}} + \frac{\varepsilon_3 \varepsilon_4 G_{Eu}}{2H^2} - \frac{F_{Eu}}{H^2}
\]

we solve (2.2) and (2.3) together, we get

\[
\Gamma_{12}^1 = \frac{\varepsilon_3 \varepsilon_4 GE_u}{2H^2} - \frac{\varepsilon_3 FG_u}{2\sqrt{\varepsilon_3 \varepsilon_4 H^2}}
\]

we solve (2.1) and (2.4) together, we get

\[
\Gamma_{22}^1 = \frac{\varepsilon_3}{\sqrt{\varepsilon_3 \varepsilon_4 H^2}}(GF_v - FG_u) - \frac{GG_u}{2H^2}
\]

we solve (2.5) and (2.6) together, we get

\[
\Gamma_{12}^2 = \frac{\varepsilon_3 \varepsilon_4 EG_u}{2H^2} - \frac{\varepsilon_3 F_{Ev}}{2\sqrt{\varepsilon_3 \varepsilon_4 H^2}}
\]

we solve (2.2) and (2.3) together, we get

\[
\Gamma_{11}^2 = \frac{\varepsilon_3 \varepsilon_4 E_{Eu}}{2H^2} - \frac{\varepsilon_3 F_{Ev}}{2\sqrt{\varepsilon_3 \varepsilon_4 H^2}}
\]

we solve (2.1) and (2.4) together, we get

\[
\Gamma_{22}^2 = \frac{\varepsilon_3 \varepsilon_4 E_{Ev}}{2\sqrt{\varepsilon_3 \varepsilon_4 H^2}} + \frac{\varepsilon_3 \varepsilon_4 E_{Gv}}{2H^2} - \frac{F_{Ev}}{\varepsilon_3 \varepsilon_4 H^2}
\]

Theorem 2.1: If coordinate lines are normal each other, then \( F = 0 \) and Gauss formulae are

\[
X_{uu} = \frac{E_u}{2E} X_u + \frac{E_v}{2\varepsilon_3 \varepsilon_4 G} X_v + \varepsilon_3 Ln
\]

\[
X_{uv} = \frac{E_v}{2E} X_u + \frac{G_u}{2G} X_v + \varepsilon_3 Mn
\]

\[
X_{vv} = -\frac{G_u}{2\varepsilon_3 \varepsilon_4 E} X_u + \frac{G_v}{2G} X_v + \varepsilon_3 Nn
\]

3. CODAZZI-MINARDI FORMULAE

If \( M \) is a \( C^3 \) manifold then its replacement vector has to satisfy the following equations at point \( P(u, v) \).

\[
(X_{uu})_v = (X_{uv})_v, \quad (X_{uv})_u = (X_{vv})_v
\]

thus by using Gauss equations we get

\[
(\Gamma_{11}^1 X_u + \Gamma_{12}^2 X_v + \varepsilon_3 Ln)_v = (\Gamma_{11}^1 X_u + \Gamma_{12}^2 X_v + \varepsilon_3 Mn)_v
\]

\[
(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + \varepsilon_3 Nn)_v = (\Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + \varepsilon_3 Mn)_v
\]

rewrite the coefficients \( \Gamma_{ij}^k \) which we obtained for \( F \neq 0 \) and we get
By taking partial derivatives of (3.1) and using the vector $X_u$, $X_v$, $n_u$ and $n_v$ then we get,

\[(3.3)\]

where the coefficients $A_1$, $A_2$, $A_3$ are as following,

\[A_1 = (\Gamma_{11}^1)_u - (\Gamma_{12}^1)_u + \Gamma_{11}^2 \Gamma_{22} - \Gamma_{12}^2 \Gamma_{11} + \varepsilon_5 L \bar{a}_{21} - \varepsilon_3 M \bar{a}_{11}\]

\[A_2 = (\Gamma_{11}^1)_v - (\Gamma_{12}^1)_v + \Gamma_{11}^1 \Gamma_{22} - \Gamma_{12}^1 \Gamma_{11} + \varepsilon_5 L \bar{a}_{21} - \varepsilon_3 M \bar{a}_{11}\]

\[A_3 = \varepsilon_5 L v - \varepsilon_3 M u + \varepsilon_3 M (\Gamma_{11}^1 - \Gamma_{12}^2) + \varepsilon_3 N \Gamma_{11}^2 - \varepsilon_5 L \Gamma_{12}^1\]

where $a_{11}$, $a_{12}$, $a_{21}$ and $a_{22}$ are the components of Weingarten matrix. Similarly, for (3.2) we get,

\[(3.4)\]

where the coefficients $B_1$, $B_2$, $B_3$ are as following,

\[B_1 = (\Gamma_{22}^2)_u - (\Gamma_{22}^2)_u + \Gamma_{22}^1 \Gamma_{11} - \Gamma_{22}^2 \Gamma_{22} + \varepsilon_3 L \bar{a}_{21} - \varepsilon_3 M \bar{a}_{21}\]

\[B_2 = (\Gamma_{22}^1)_u - (\Gamma_{22}^1)_u + \Gamma_{22}^1 \Gamma_{11} - \Gamma_{22}^2 \Gamma_{22} + \varepsilon_3 L \bar{a}_{21} - \varepsilon_3 M \bar{a}_{21}\]

\[B_3 = \varepsilon_3 N v - \varepsilon_3 M u + \varepsilon_3 L v + (\Gamma_{22}^1 - \Gamma_{12}^1) \varepsilon_3 M - \varepsilon_3 L \varepsilon_3 N\]

Since the vectors $X_u$, $X_v$ and $n$ are linearly independent in (3.3) and (3.4) then we get

\[A_i = 0, \ B_i = 0, (i = 1, 2, 3) \]

\[(3.5)\]

\[(3.6)\]

The equations (3.5) and (3.6) are called Codazzi-Mainardi formulae of the surface $X(u, v)$.

a) The case $F=0$; We calculate $\Gamma_{ij}^k$ coefficients from Gauss formulae, we get,

\[\Gamma_{11}^1 = \frac{E_x}{2E}, \ \Gamma_{12}^1 = \frac{E_y}{2E}, \ \Gamma_{22}^1 = \frac{-G_x}{2\varepsilon_3 \varepsilon_4 E}, \ \Gamma_{11}^2 = \frac{-E_x}{2\varepsilon_3 \varepsilon_4 E}, \ \Gamma_{12}^2 = \frac{G_x}{2G}, \ \Gamma_{22}^2 = \frac{G_y}{2G}\]

and substitute these values in Codazzi-Mainardi equations, then we get,
Gauss And Codazzi-Mainardi Formulae

N. EKMEKÇİ & Y. TUNCER

\[ L_v + M \frac{E_u}{2E} - N \frac{E_v}{2\varepsilon_3 \varepsilon_4 G} = M_u + L \frac{E_u}{2E} + M \frac{G_u}{2G} \]  

\[ N_u + M \frac{G_v}{2G} - L \frac{G_u}{2\varepsilon_3 \varepsilon_4 E} = M_v + M \frac{E_v}{2E} + N \frac{G_u}{2} \]

b) The case \( F=0 \) and \( M=0 \); In this case (3.7) and (3.8) equations will be as following

\[ L_v - N \frac{E_v}{2\varepsilon_3 \varepsilon_4 G} = L \frac{E_u}{2E} , \quad N_u - L \frac{G_u}{2\varepsilon_3 \varepsilon_4 E} = N \frac{G_u}{2} \]

If the surface is compared with zero length curves-minimal curves then \( E, G \) will be vanish on the surface. And we get following equations by using the equations (2.7), (2.8), (2.9), (2.10), (2.11), (2.12).

\[ I_{11}^1 = -\frac{F F_u}{H^2} = \frac{F F_u}{\varepsilon_3 \varepsilon_4 F^2} = \varepsilon_3 \varepsilon_4 \left( F_u \log|F| \right)_u \]
\[ I_{22}^2 = -\frac{F F_v}{\varepsilon_3 \varepsilon_4 H^2} = -\frac{F F_v}{-F^2} = \frac{F_v}{F} = \left( \log|F| \right)_v \]

and we obtain

\[ I_{11}^1 = I_{12}^1 = I_{22}^2 = \Gamma_{22}^1 = 0 \]

Thus the Gauss formulae are obtained as follows;

\[ X_{uu} = \varepsilon_3 \varepsilon_4 \left( \log|F| \right)_u, \quad X_u + \varepsilon_3 L n, \quad X_{uv} = \varepsilon_3 M n, \quad X_v = \left( \log|F| \right)_v, \quad X_v + \varepsilon_5 N n \]

Furthermore Codazzi-Mainardi formulae will be as follows

\[ L_v - M_v = -\varepsilon_3 \varepsilon_4 M \left( \log|F| \right)_u, \quad N_u - M_v = -M \left( \log|F| \right)_v \]

and then we obtain

\[ \pm \left( \frac{M_u - L_v}{M} \right)_v = \left( \frac{M_v - N_u}{M} \right)_u \]

References


