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TIMOSHENKO KOLONU OLARAKMODELLENEN TEK KATLI ÇERÇEVELERİN ZEMİN FLEKSİBİLİTESİ DİKKATE ALINARAK SERBEST TİTREŞİMİ

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ÖZET

Çerçevelerin dinamik analizinde sıkça kullanılan kabullerden birisi hareket denklemini kat hizasında topaklanmış kütleye göre yazmak ve kolonları rijitlik elemanı olarak modellemektir. Ancak gerçekte, kolonlar yayılı kütleye ve rijitliğe sahiptir ve çerçevelerin zeminle bağlantı noktaları, pratikte, dinamik yük sırasında zeminin elastik davranışına bağlı olarak bir miktar dönebilmekte ve ötelenebilmektedir. Bu durumda, çerçevelerin mesnetlerinde dönmeye ve ötelenmeye karşı elastik yaylar kullanılarak, elastik mesnet davranışı modellenebilir. Bu çalışmada, elastik mesnetli Timoshenko kolonu olarak modellenen tek katlı çerçevelerin serbest titreşimi kolonların dönme ataleti de dikkate alınarak incelenmiş ve farklı yay sabitleri için doğal frekanslar elde edilmiştir.

Anahtar Kelimeler: *Elastik mesnet, Serbest titreşim, Tek katlı çerçeve, Timoshenko kolonu*

FREE VIBRATION OF SINGLE STOREY FRAMES MODELED AS TIMOSHENKO COLUMN INCLUDING SOIL FLEXIBILITY

ABSTRACT

One of the assumptions mostly used in dynamic analysis of frames is writing the equation of motion according to concentrated mass at the storey height and modeling the columns as stiffness element. The other one is that the model of the frame is fixed supported. However, columns, in fact, have distributed mass and stiffness; and in practice, column bases of frames may usually rotate and translate a little due to elastic behavior of soil during dynamic loading. In this case, elastic support behavior can be modeled using elastic springs against translation and rotation at the column bases of frames. In this study, free vibration of single storey frames modeled as elastically supported Timoshenko column is studied including rotatory inertia of the columns and natural frequencies are obtained for different spring coefficients.

Keywords: *Elastic Support, Free Vibration, Single Storey Frame, Timoshenko Column*

1. INTRODUCTION

In free vibration analysis of SDOF frames, it is generally assumed that distributed mass of columns is negligible and supports are fully rigid. These assumptions make the dynamic analysis of mathematical calculation model easy.

Michaltsos and Ermopoulos studied free and forced vibration of the model in this study neglecting shear deformation and rotatory inertia [1]. Glabisz studied vibration and stability of elastically supported continuous bars subjected to static loading [2]. Güler searched the effects of soil flexibility on free vibration of tower-like structures using Euler model [3]. Demirdağ obtained inelastic response spectrum of elastically supported frames modeled as a Timoshenko column [4].

Dynamic analysis of framed systems modeled as discrete parameter in which deformations and distributed mass of the columns are neglected is also frequently studied by many researchers [5; 6; 7].

Behavior of the column bases of frames is more appropriate to elastic support model. Dynamic mathematical model of elastically supported single storey frame is presented in Fig. 1. Floor mass of the frame is concentrated at the top of elastic column in the model, and base of the column is supported by elastic springs against rotation and translation modeling the elastic support behavior.

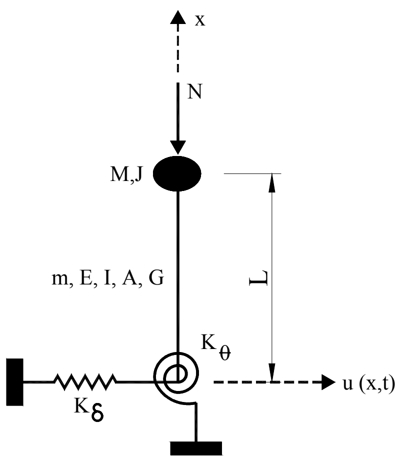

Figure-1: Timoshenko column model of elastically supported SDOF frame.

Following assumptions are made in this study.

1) Axial force acting to system is constant through the column.
2) Elastic support is modeled by elastic springs against rotation and translation.
3) Geometrical nonlinearity is valid.

2. EQUATION OF MOTION

Since bending and shear deformations of the column with distributed mass in Fig. 1 are taken into consideration it is possible to write Eq. (1) for total displacement of the system.

$$u(x,t) = u_b(x,t) + u_s(x,t) \tag{1}$$

where u_b and u_s are displacements due respectively to bending and shear; u is total displacement; x and t are respectively, position and time variables. First order differentiation of shear deformation and second order differentiation of bending deformation with respect to x is written respectively, as follows.

$$\frac{\partial u_s}{\partial x} = \frac{V(x,t)k}{AG} \tag{2}$$

$$\frac{\partial^2 u_b}{\partial x^2} = -\frac{M(x,t)}{EI} \tag{3}$$

where V(x,t) and M(x,t) are shear and moment functions, respectively; EI and AG are respectively, flexural and shear rigidity of the column. Relations concerning equilibrium of forces and moments obtained by using equilibrium of differential segment of column in Fig. 2 are given in Eqs. (4) and (5), respectively, where F_I and M_I are inertia force and rotatory inertia moment respectively, of differential segment; N is constant axial force; k is shear area constant.

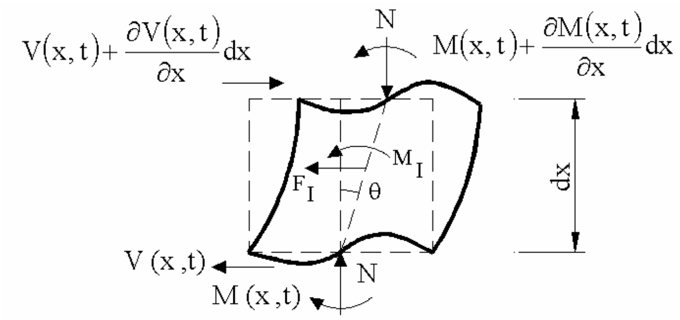

Figure-2: Internal forces and deformations of the differential segment of the column.

$$\frac{\partial V(x,t)}{\partial x} = m\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$\frac{\partial M(x,t)}{\partial x} = V(x,t) + N\frac{\partial u}{\partial x} - m\frac{I}{A}\frac{\partial^3 u_b}{\partial x \partial t^2} \tag{5}$$

where m, A and I are distributed mass, cross-section area and cross-sectional moment of inertia of the column, respectively. Using Eqs. (1), (4) and (5) and making necessary arrangements gives differential equation of motion of system in Fig. 1 as in follows.

$$\frac{\partial^4 u}{\partial x^4}+\frac{N}{EI}\frac{\partial^2 u}{\partial x^2}-\left[\frac{m\cdot k}{AG}+\frac{m\cdot I}{EI\cdot A}\right]\frac{\partial^4 u}{\partial x^2\partial t^2}+\frac{m}{EI}\frac{\partial^2 u}{\partial t^2}+\frac{m^2\cdot I\cdot k}{EI\cdot A\cdot AG}\frac{\partial^4 u}{\partial t^4}=0 \tag{6}$$

As cited in reference [8], contribution of the last term in Eq. (6) to frequencies and mode shapes is very, very small. Therefore, $u_b \cong u$ can be written as taking contribution of rotatory inertia to moment into consideration, and therefore the last term of Eq. (6) can be omitted [8].

Method of separation of variables is applied using transformation given in Eq. (7) for solution of differential Eq. (6).

$$u(x,t)=X(x)\cdot T(t)=X(x)\cdot \sin(\omega t) \tag{7}$$

where X(x) and T(t) are shape and time functions, respectively; ω is natural frequency. Differentiating successively of Eq. (7) with respect to x and t and substituting in Eq. (6) gives

$$X^{IV}+\left[\frac{N}{EI}+\frac{m\cdot k\cdot\omega^2}{AG}+\frac{m\cdot I\cdot\omega^2}{EI\cdot A}\right]X^{II}-\left[\frac{m\cdot\omega^2}{EI}\right]X=0 \tag{8}$$

3. SLOPE, MOMENT and SHEAR FUNCTIONS

Second order differentiation of displacement function of bending effect with respect to x is written using Eq. (1) as follows.

$$\frac{\partial^2 u_b}{\partial x^2}=\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u_s}{\partial x^2} \tag{9}$$

Substituting first order differentiation of Eq. (2) with respect to x and Eq.(3) into Eq. (9) gives

$$\frac{M(x,t)}{EI}=-\frac{\partial^2 u}{\partial x^2}+\frac{\partial V(x,t)}{\partial x}\frac{k}{AG} \tag{10}$$

Substituting second order differentiation of Eq. (7) with respect to t into Eq. (4) gives

$$\frac{\partial V(x,t)}{\partial x}=-m\omega^2 u(x,t) \tag{11}$$

Moment function is obtained as follows if Eq. (11) is substituted into Eq. (10) and contribution of axial force and rotatory inertia to moment is included.

$$M(x,t) = -EI\frac{\partial^2 u}{\partial x^2} - EI\frac{m\cdot k\cdot \omega^2}{AG}u(x,t) - N\cdot u(x,t) + \frac{m\cdot I}{A}\frac{\partial^2 u}{\partial t^2} \qquad (12)$$

Substituting the second order differentiation of Eq. (7) with respect to t into Eq. (12) gives moment function as follows [4].

$$M(x,t) = -EI\frac{\partial^2 u}{\partial x^2} - EI\frac{m\cdot k\cdot \omega^2}{AG}u(x,t) - N\cdot u(x,t) - \frac{m\cdot I\cdot \omega^2}{A}u(x,t) \qquad (13)$$

Shear function is obtained by differentiating Eq. (13) with respect to x as

$$V(x,t) = \frac{\partial M(x,t)}{\partial x} = -EI\frac{\partial^3 u}{\partial x^3} - EI\frac{m\cdot k\cdot \omega^2}{AG}\frac{\partial u}{\partial x} - N\frac{\partial u}{\partial x} - \frac{m\cdot I\cdot \omega^2}{A}\frac{\partial u}{\partial x} \qquad (14)$$

Slope function θ(x,t) is written using Eq. (1) and Eq. (2) as follows.

$$\theta(x,t) = \frac{\partial u_b(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x} - \frac{\partial u_s(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial x} - \frac{V(x,t)k}{AG} \qquad (15)$$

Thus, slope function of a distributed mass system subjected to axial compression force with bending, shear and rotatory inertia effects is given with respect to total displacement as follows.

$$\theta(x,t) = \frac{k}{AG}\left\{EI\frac{\partial^3 u}{\partial x^3} + \left[EI\frac{m\cdot k\cdot \omega^2}{AG} + \frac{AG}{k} + \frac{m\cdot I\cdot \omega^2}{A} + N\right]\frac{\partial u}{\partial x}\right\} \qquad (16)$$

4. DIMENSIONLESS ANALYSIS

Taking z=x/L, dimensionless differential equation of motion is obtained as

$$\frac{\partial^4 \bar{u}}{\partial z^4} + \frac{N\cdot L^2}{EI}\frac{\partial^2 \bar{u}}{\partial z^2} - \left[\frac{m\cdot k\cdot L^2}{AG} + \frac{m\cdot I\cdot L^2}{EI\cdot A}\right]\frac{\partial^4 \bar{u}}{\partial z^2 \partial t^2} + \frac{m\cdot L^4}{EI}\frac{\partial^2 \bar{u}}{\partial t^2} = 0 \qquad (17)$$

Substituting the successive differentiations of dimensionless displacement function in Eq. (18) obtained by using method of separation of variables into Eq. (17) gives Eq (19) for general solution of dimensionless equation of motion.

$$\bar{u}(z,t) = Z(z)\cdot T(t) \qquad (18)$$

$$Z^{IV}T + \frac{N \cdot L^2}{EI} Z^{II}T - \left[\frac{m \cdot k \cdot L^2}{AG} + \frac{m \cdot I \cdot L^2}{EI \cdot A}\right] Z^{II}\ddot{T} + \frac{m \cdot L^4}{EI} Z\ddot{T} = 0 \tag{19}$$

Eq. (19) can be rewritten as in the following for T=sinωt≠0.

$$Z^{IV} + \left[\frac{NL^2}{EI} + \frac{m \cdot k \cdot L^2 \cdot \omega^2}{AG} + \frac{m \cdot I \cdot L^2 \cdot \omega^2}{EI \cdot A}\right] Z^{II} - \left[\frac{m \cdot L^4 \cdot \omega^2}{EI}\right] Z = 0 \tag{20}$$

Solution of Eq. (20) is obtained due to n_1 since Δ>0 and $n_2<0$ [4].

for $n_1>o$

$$Z(z) = C_1 \sinh(m_1 z) + C_2 \cosh(m_1 z) + C_3 \sin(m_2 z) + C_4 \cos(m_2 z) \tag{21}$$

for $n_1<o$

$$Z(z) = C_1 \sin(m_1 z) + C_2 \cos(m_1 z) + C_3 \sin(m_2 z) + C_4 \cos(m_2 z) \tag{22}$$

where for $n_1>0$ $m_1 = \sqrt{n_1}$, for $n_1<0$ $m_1 = \sqrt{|n_1|}$; $m_2 = \sqrt{|n_2|}$; $n_1 = \frac{-\alpha_3 + \sqrt{\Delta}}{2}$; $n_2 = \frac{-\alpha_3 - \sqrt{\Delta}}{2}$; $\Delta = \alpha_3^2 - 4\alpha_4$; $\alpha_3 = \overline{N} + \alpha_1 + \alpha_2$; $\overline{N} = \frac{NL^2}{EI}$; $\alpha_4 = -\overline{\omega}^2$; $\alpha_1 = \frac{m \cdot k \cdot L^2 \cdot \omega^2}{AG}$; $\alpha_2 = \frac{m \cdot I \cdot L^2 \cdot \omega^2}{EI \cdot A}$; $\overline{\omega}^2 = \frac{m \cdot L^4 \cdot \omega^2}{EI}$; $C_1 \ldots C_4$=constants.

Thus, dimensionless displacement function is obtained for $n_1>0$ and $n_1<0$, respectively as follows.

$$\overline{u}(z,t) = [C_1 \sinh(m_1 z) + C_2 \cosh(m_1 z) + C_3 \sin(m_2 z) + C_4 \cos(m_2 z)] \sin \omega t \tag{23}$$

$$\overline{u}(z,t) = [C_1 \sin(m_1 z) + C_2 \cos(m_1 z) + C_3 \sin(m_2 z) + C_4 \cos(m_2 z)] \sin \omega t \tag{24}$$

Moment, shear and slope functions in Eqs (13), (14) and (16) are written in terms of the dimensionless displacement function, respectively, as follows [4].

$$\overline{M}(z,t) = -\frac{EI}{L^2}\frac{\partial^2 \overline{u}}{\partial z^2} - EI\frac{m \cdot k \cdot \omega^2}{AG}\overline{u}(z,t) - N \cdot \overline{u}(z,t) - \frac{m \cdot I \cdot \omega^2}{A}\overline{u}(z,t) \tag{25}$$

$$\overline{V}(z,t) = -\frac{EI}{L^3}\frac{\partial^3 \overline{u}}{\partial z^3} - \frac{EI}{L}\frac{m \cdot k \cdot \omega^2}{AG}\frac{\partial \overline{u}}{\partial z} - \frac{N}{L}\frac{\partial \overline{u}}{\partial z} - \frac{m \cdot I \cdot \omega^2}{A \cdot L}\frac{\partial \overline{u}}{\partial z} \tag{26}$$

$$\overline{\theta}(z,t)=\frac{k}{AG}\left\{\frac{EI}{L^3}\frac{\partial^3\overline{u}}{\partial z^3}+\left[\frac{EI}{L}\frac{m\cdot k\cdot\omega^2}{AG}+\frac{AG}{k\cdot L}+\frac{m\cdot I\cdot\omega^2}{A\cdot L}+\frac{N}{L}\right]\frac{\partial\overline{u}}{\partial z}\right\} \tag{27}$$

The four dimensionless boundary conditions of the system in Fig. 1 are

$$\begin{aligned}&\overline{V}(0,t)\frac{L^3}{EI}=\overline{K}_\delta\overline{u}(0,t); \quad \overline{M}(0,t)\frac{L^2}{EI}=-\overline{K}_\theta.L.\overline{\theta}(0,t)\\&\overline{V}(1,t)\frac{L^3}{EI}=\overline{M}\cdot\overline{\omega}^2\overline{u}(1,t); \quad \overline{M}(1,t)\frac{L^2}{EI}=-\overline{J}.L.\overline{\omega}^2\overline{\theta}(1,t)\end{aligned} \tag{28}$$

where $\overline{K}_\delta$ and $\overline{K}_\theta$ are dimensionless parameters of translational and rotational spring coefficients respectively; $\overline{M}$ and $\overline{J}$ are dimensionless parameters of the concentrated mass and its rotational inertia respectively, and are given in Eq. (29).

$$\overline{K}_\delta=\frac{K_\delta L^3}{EI}; \quad \overline{K}_\theta=\frac{K_\theta L}{EI}; \quad \overline{M}=\frac{M}{mL}; \quad \overline{J}=\frac{J}{mL^3} \tag{29}$$

Four linear equations are obtained using boundary conditions in Eq. (28) and are presented in matrix form in the following.

$$\begin{bmatrix}\alpha_{17} & \overline{K}_\delta & \alpha_{18} & \overline{K}_\delta\\ -\alpha_{15} & \alpha_{13} & \alpha_{16} & \alpha_{14}\\ \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{26} & \alpha_{27} & \alpha_{28} & \alpha_{29}\end{bmatrix}\begin{Bmatrix}C_1\\C_2\\C_3\\C_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{30}$$

where $\alpha_5=\frac{AG}{k\cdot EI}$; $\alpha_6=\frac{1}{\alpha_5 L^3}$; $\alpha_7=\alpha_1+\alpha_2+\alpha_5 L^2$; $\alpha_8=\alpha_6\alpha_7$; $\alpha_9=\overline{K}_\theta.\alpha_6 L$;
$\alpha_{10}=\overline{K}_\theta.\alpha_8 L$; $\alpha_{11}=\overline{J}.L.\overline{\omega}^2\alpha_6$; $\alpha_{12}=\overline{J}.L.\overline{\omega}^2\alpha_8$; $\alpha_{13}=m_1^2+\alpha_3$; $\alpha_{14}=\alpha_3-m_2^2$;
$\alpha_{15}=m_1\left(m_1^2\alpha_9+\alpha_{10}\right)$; $\alpha_{16}=m_2\left(m_2^2\alpha_9-\alpha_{10}\right)$; $\alpha_{17}=m_1\alpha_{13}$; $\alpha_{18}=m_2\alpha_{14}$;
$\alpha_{19}=\overline{M}\overline{\omega}^2$; $\alpha_{20}=\alpha_{17}c1+\alpha_{19}s1$; $\alpha_{21}=\alpha_{17}s1+\alpha_{19}c1$; $\alpha_{22}=\alpha_{18}c2+\alpha_{19}s2$;
$\alpha_{23}=-\alpha_{18}s2+\alpha_{19}c2$; c1=cosh($m_1$); s1=sinh($m_1$); c2=cos($m_2$); s1=sin($m_2$);
$\alpha_{24}=m_1\left(m_1^2\alpha_{11}+\alpha_{12}\right)$; $\alpha_{25}=m_2\left(m_2^2\alpha_{11}-\alpha_{12}\right)$; $\alpha_{26}=\alpha_{13}s1-\alpha_{24}c1$;
$\alpha_{27}=\alpha_{13}c1-\alpha_{24}s1$; $\alpha_{28}=\alpha_{14}s2+\alpha_{25}c2$; $\alpha_{29}=\alpha_{14}c2-\alpha_{25}s2$

Determinant of coefficient matrix must be equal to zero for non-trivial solution of Eq. (30). The function obtained by equating determinant to zero is the frequency equation of system in Fig. 1. Roots of this equation are natural frequencies of the system. One of the simple and the widely used methods used for calculation the roots of frequency equation is the secant method [9]. Determinant values are evaluated for a range of ω values in this method.

The ω value causing a sign change between the successive determinant values is a root of frequency equation and means a frequency for the system.

5. NUMERICAL ANALYSIS

Physical properties for dynamic calculation model of single-storey elastically supported frame chosen for numerical analysis are given as follows.

Distributed mass of column: m=17.982 kNs^2/m^2; Cross-section area of column: A=2.45 m^2; Cross-sectional moment of inertia of column: I=0.0625 m^4; Flexural stiffness of column: EI=1987500 kNm^2; Shear stiffness of column: AG=31164000 kN; Numerical constant for column with rectangular cross-section: k=1.2; Length of column: L=3 m.

Dimensionless natural frequencies for the first mode of system are calculated by a computer algorithm depending on secant method using values of 0.1-0.25-0.5-0.75-1.0, 0.1-0.5-1.0-5.0-10, 0.1-0.5-1.0-5.0-10, 0.1-1.0-10-100-1000 and 0.1-1.0-10-100-1000 for dimensionless parameters of axial force, concentrated mass, rotational inertia of concentrated mass, translational spring coefficient and rotational spring coefficient, respectively.

The results indicate that increase in the value of spring coefficients increases the frequency values; however, no more increase in frequency values is seen after the dimensionless spring value of 1000. Graphical presentation of the change in normalized frequency values of the first mode due to increase in rotational spring coefficient for different values of $\overline{J}$ is given for $\overline{N} = 0.1$ in Figs. 3, 4 and 5 for $\overline{M}$ values of 0.1, 1 and 10 respectively; for $\overline{N} = 1.0$ in Figs. 6, 7 and 8 for $\overline{M}$ values of 0.1, 1 and 10 respectively. γ in horizontal axis is the ratio of dimensionless rotational spring coefficient to dimensionless translational spring coefficient.

As the values of concentrated mass and its rotational inertia are increased, natural frequency values show a decrease, as expected. The graphs of change in normalized frequency values due to increase in $\overline{M}$ for different γ values are presented for $\overline{N} = 0.1$ in Figs. 9, 10 and 11 for $\overline{J}$ values of 0.1, 1 and 10 respectively; for $\overline{N} = 1.0$ in Figs. 12, 13 and 14 for $\overline{J}$ values of 0.1, 1 and 10 respectively.

6. CONCLUSIONS

In this study, free vibration analysis of elastically supported s-d-o-f frames modeled as in Fig. 1 is investigated. Shear deformation and rotatory inertia for the elastic column modeled as having distributed mass and having a concentrated mass representing floor at the top are taken into consideration. Rotational inertia of the concentrated mass is also included in the dynamic analysis. Elastic connection between elastic column and the soil is modeled by elastic springs against rotation and translation.

Dimensionless frequency values calculated by a computer algorithm depending on secant method show a decrease when shear deformation and rotatory inertia effects are respectively included in transverse vibration due to bending of elastic column having distributed mass. However, this decrease has no practical meaning and can be neglected.

An increase through a certain value is observed in frequency values as both spring coefficient values increase, in other words as the support behavior closes to rigid support the system will have greater period. However, a decrease in frequency values is observed as the values of concentrated mass and its rotational inertia increase. In the case of increasing rotational spring coefficient with constant translational spring coefficient the increase of frequency values is greater than the case of increasing translational spring coefficient with constant rotational spring coefficient. This indicates that the rotational spring cannot be neglected when modeling the column base as an elastic support while the translational spring may not be used in the free vibration analysis.

To the contrary of discrete parameter modeling, continuous system modeling allows one to study the behavior of the system along its whole height, not only the behavior at the storey height.

In practice, column bases of frames may usually rotate and translate a little due to elastic behavior of soil. In this case, elastic support behavior can be modeled using elastic springs against translation and rotation at the column bases of frames; therefore, more realistic free vibration response of frames may be obtained in the case of elastic support.

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Şekillerin devamı gelecek

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