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RESEARCH ARTICLE

LIMITS IN THE CATEGORY OF 2-GENERALIZED CROSSED MODULES

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Abstract

In a previous study, limits were calculated within the category of 2-crossed modules of groups over a fixed group R, where the group actions involved were specifically taken to be conjugation actions. While this framework provides a rich algebraic structure for constructing certain homotopical and categorical structures, the limitation of conjugation-based actions restricts the flexibility of this approach. In this paper, we extend this framework by introducing the notion of 2-generalized crossed modules (2GCM), which generalize the structure of 2-crossed modules by allowing more flexible and arbitrary group actions, rather than restricting them to conjugation actions. Furthermore, we prove that the category of 2-generalized crossed modules is finitely complete, meaning that it possesses all finite limits, such as products and equalizers. This property is important for higher-level categorical analysis and supports the application of 2-generalized crossed modules in both theoretical and applied contexts, particularly in higher-dimensional algebra and homotopy theory.

Keywords

2-Generalized crossed module,
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1. INTRODUCTION

A generalized crossed module $\omega: F \to K$ is a morphism of groups, together with the actions of F on F and of K on K arbitrary instead of conjugation actions. Yavari and Salemkar defined the generalized crossed modules in [1]. They get the generalized crossed modules category, **GCM**, and investigated the relation between epimorphisms and surjective morphisms in **GCM**.

A 2-crossed module of groups $F \overset{\alpha_2}{\to} K \overset{\alpha_1}{\to} H$ is a complex of groups, satisfying certain conditions along with the actions of H on K and F, and a mapping $\{-,-\}: K \times K \to F$, which is often called the Peiffer lifting of 2-crossed module, such that the action of H on H is conjugation, α_1 and α_2 are H -equivariant. 2-crossed modules were first introduced by Conduché in [2] as models for connected homotopy 3-types. The commutative algebra version was also given by Arvasi in [3]. Later, the concept of a 2-crossed module is extended to various algebraic structures; see [4,5-6] for further details.

Brown and Sivera [7], calculated algebraic (co)limits in the homotopy theory with using (co)fibred categories. They proved that, the inclusion map $i_J: X_J \to X$, where $\Psi: X \to B$ is a fibration and $J \in B$, preserves colimits of connected diagram. In [8], they described colimits in the categories of crossed modules (over groupoids) and modules (over groupoids), which are derived by reducing the colimits in the category of crossed complexes. In many of the work, it is shown that a homotopically defined functor

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$$\Pi: \text{(topological data)} \rightarrow \text{(algebraic data)}$$

in a fibred category preserves certain colimits [7,8,9-10]. In the literature, (co)limits of crossed modules were studied over time for various algebraic structures, as seen in [11,12,13,14-15]. The 2-crossed module version is studied in [16-17].

In this paper, we introduce the concept of 2-generalized crossed modules (2GCM), where group actions are arbitrary rather than restricted to conjugation actions. Furthermore, we prove that the category of 2GCM is finitely complete. Put differently, we construct the product, pullback and equalizer objects within the category of 2GCM.

2. GENERALIZED CROSSED MODULES

We recall the definition of a generalized crossed module from [1].

Definition 2.1 A generalized crossed module (F, K, ω), consists of a group morphism $\omega: F \rightarrow K$, together with the following properties:

i) an action of F on F, denoted by $f \,\square_F\, f'$, for every $f, f' \in F$,

ii) an action of K on K , denoted by $k \,\square_K\, k'$, for every $k, k' \in K$,

iii) an action of K on F, denoted by k_f, for every $k \in K, f \in F$,

satisfying the conditions:

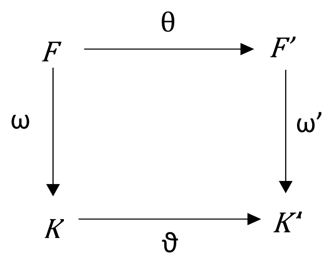
GCM1) $\omega(k_f) = k \,\square_K\, \omega(f)$

GCM2) ${}^{\omega(f)}f' = f \,\square_F\, f'$

for all $f, f' \in F$ and $k \in K$. If ω only satisfies condition **GCM1**, we get a pre-generalized crossed module.

Remark 2.2 Throughout this paper, an action of F on F is denoted by $\cdot$ instead of $\square_F$ for any group F.

A morphism $(\theta, \vartheta): (F, K, \omega) \rightarrow (F', K', \omega')$ of generalized crossed modules consists of group morphisms $\theta: F \rightarrow F'$ and $\vartheta: K \rightarrow K'$ such that the following diagram is commutative.



i.e. $\vartheta\omega = \omega'\theta$ and

$$\theta({}^{k}f) = {}^{\vartheta(k)}\theta(f)$$

for all $f \in F$ and $k \in K$. Thus we obtain the category of generalized crossed modules, denoted by **GCM**.

Some examples of generalized crossed modules are given below:

Example 2.3 If $\omega: F \rightarrow K$ is any crossed module, then it is also generalized crossed module.

Example 2.4 Let $\omega: F \rightarrow K$ be a group morphism. If all actions are trivial, then ω becomes a generalized crossed module.

Example 2.5 Let F and K be two groups. If the action of F on F is trivial and the actions of K on K and K on F are arbitrary, then the trivial morphism $1: F \rightarrow K$ is a generalized crossed module.

Example 2.6 For any group F and any action of on itself, then (F, F, id_F) is a generalized crossed module.

3. 2-GENERALIZED CROSSED MODULES

Definition 3.1 A 2-generalized crossed module $(F, K, H, \rho_2, \rho_1)$ is a normal complex of groups

$$F \overset{\rho_2}{\to} K \overset{\rho_1}{\to} H$$

i) an action of H on F denoted by $h \triangleright f$ for all $h \in H$, $f \in F$,
ii) an action of H on K denoted by $h \triangleright k$ for all $h \in H$, $k \in K$,
iii) an action of K on K and H on H denoted by $k \cdot k'$ and $h \cdot h'$ for all $k, k' \in K$ and $h, h' \in H$ and a mapping $\{-,-\}: K \times K \to F$ which is called Peiffer lifting, such that ρ_2 and ρ_1 are H-equivarient, satisfying the following conditions:
2GCM1) $\rho_2\{k_0, k_1\} = (k_0 \cdot k_1)(\rho_1(k_0) \triangleright k_1^{-1})$
2GCM2) $\{\rho_2(f_0), \rho_2(f_1)\} = (f_0 \cdot f_1)f_1^{-1}$
2GCM3)

$$\text{i) } \{k_0, k_1k_2\} = {}^{(k_0 \cdot k_1)}\{k_0, k_2\}\{k_0, k_1\}$$
$$\text{ii) } \{k_0k_1, k_2\} = \{k_0, k_1k_2\}(\rho_1(k_0) \triangleright \{k_1, k_2\})$$

2GCM4)

$$\text{i) } \{\rho_2(f), k\} = f({}^{k}f^{-1})$$
$$\text{ii) } \{k, \rho_2(f)\} = ({}^{k}f)(\rho_1(k) \triangleright f^{-1})$$

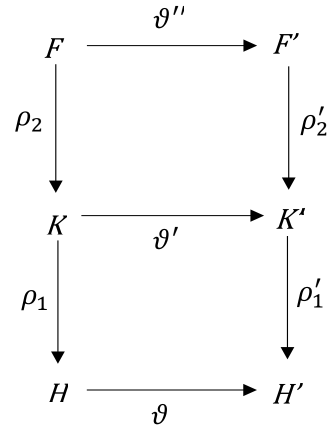
or
2GCM4*) $\{\rho_2(f), k\}\{k, \rho_2(f)\} = f(\rho_1(k) \triangleright f^{-1})$
2GCM5) $h \triangleright \{k_0, k_1\} = \{h \triangleright k_0, h \triangleright k_1\}$
for all $h \in H$, $k, k_0, k_1, k_2 \in K$ and $f, f_0, f_1 \in F$.

There is an action of K on F, defined as

$$k_f = \{k, \rho_2(f)\}(\rho_1(k) \triangleright f)$$

for all $k \in K$ and $f \in F$.

Let $(F, K, H, \rho_2, \rho_1)$ and $(F', K', H', \rho_2', \rho_1')$ be two 2-generalized crossed module. A morphism of 2-generalized crossed modules is defined by the following diagram:



where the diagram is commutative, i.e. $\vartheta\rho_1 = \rho_1'\vartheta'$ and $\vartheta'\rho_2 = \rho_2'\vartheta''$. Furthermore the following equations are satisfied:

$$\vartheta'(h \triangleright k) = \vartheta(h) \triangleright \vartheta'(k)$$
$$\vartheta''(h \triangleright f) = \vartheta(h) \triangleright \vartheta''(f)$$
$$\{-,-\}\vartheta' \times \vartheta' = \vartheta''\{-,-\}$$

for all $h \in H$, $k \in K$ and $f \in F$.

Thus, we get the category of 2-generalized crossed module, which is denoted by **2GCM**.

Example 3.2 Any generalized crossed module gives a 2-generalized crossed module. If (K, H, ρ) is a generalized crossed module, then $F \to K \to H$ is a 2-generalized crossed module by taking $F = 1$.
Thus, we have a functor:

$$\Delta: \mathbf{GCM} \to \mathbf{2GCM}$$

defined by $\Delta(K, H, \rho) = (1, K, H, 1, \rho)$.
If $F \overset{\rho_2}{\to} K \overset{\rho_1}{\to} H$ is a 2-generalized crossed module, then $Im\rho_2$ is a normal subgroup of K. Thus, we get an induced generalized crossed module, $\rho': K/Im\rho_2 \to H$. Hence, we get a functor:

$$\Psi: \mathbf{2GCM} \to \mathbf{GCM}.$$

So, Δ is the adjoint functor of Ψ. Therefore, we have the adjunction:

$$Hom_{\mathbf{GCM}}(\Psi(\mathfrak{T}), \mathfrak{X}) \cong Hom_{\mathbf{2GCM}}(\mathfrak{T}, \Delta(\mathfrak{X}))$$

between the category of 2GCM and the category of generalized crossed modules.

4. THE COMPLETENESS of 2GCM

Recall that, the $(\{e\}, \{e\}, id)$ is the terminal object in the category of generalized crossed modules. Moreover, for given two generalized crossed module morphisms $(\theta, \theta'): (K_1, H_1, \alpha) \to (K_3, H_3, \gamma)$ and $(\vartheta, \vartheta'): (K_2, H_2, \beta) \to (K_3, H_3, \gamma)$, we get the generalized crossed module (P_K, P_H, μ), where $P_K = \{(k_1, k_2) | \theta(k_1) = \vartheta(k_2)\}$ and $P_H = \{(h_1, h_2) | \theta'(h_1) = \vartheta'(h_2)\}$, is the pullback of (θ, θ') and (ϑ, ϑ') in the category of generalized crossed modules **GCM**. Therefore, we say that **GCM** is finitely complete, [1].
In this section, we will prove the completeness of the category of 2GCM.

Proposition 4.1 If $(F_1, K_1, H, \rho_2, \rho_1)$ and $(F_2, K_2, H, \sigma_2, \sigma_1)$ are two 2-generalized crossed module, then $(P_F, P_K, H, \mu, \omega)$ is a 2-generalized crossed module, where

$$P_F = \{(f_1, f_2) \mid \rho_2(f_1) = \sigma_2(f_2)\} \subset F_1 \times F_2$$

$$P_K = \{(k_1, k_2) \mid \rho_1(k_1) = \sigma_1(k_2)\} \subset K_1 \times K_2$$

with the peiffer lifting

$$\{-,-\}: P_K \times P_K \to P_F$$

defined by

$$\{(k_1, k_2), (k_1', k_2')\} = (\{k_1, k_2\}, \{k_1', k_2'\})$$

for all $(k_1, k_2), (k_1', k_2') \in P_K$.

Proof Define $\omega: P_K \to H$ by $\omega(k_1, k_2) = \rho_1(k_1) = \sigma_1(k_2)$, $\mu: P_F \to P_K$ by

$$\mu(f_1, f_2) = (\rho_2(f_1), \sigma_2(f_2))$$

for all $(f_1, f_2) \in P_F$ and $(k_1, k_2) \in P_K$. For all $h \in H$ and $(f_1, f_2) \in P_K$;

$$\begin{aligned} \omega({}^{h}(f_1,f_2)) &= \omega({}^{h}f_1, {}^{h}f_2) \\ &= \rho_1({}^{h}f_1) \\ &= h \cdot \rho_1(f_1) \end{aligned}$$

Then (P_K, H, ω), is a pre-generalized crossed module. For all $h \in H$, $(k_0,k_0'),(k_1,k_1'),(k_2,k_2'),(k_1,k_2),(k_1',k_2') \in P_K$ and $(f_1,f_2),(f_1',f_2') \in P_F$;

2GCM1)

$$\begin{aligned} \mu\{(k_1,k_2),(k_1',k_2')\} &= \mu(\{k_1,k_1'\},\{k_2,k_2{}'\}) \\ &= (\rho_2\{k_1,k_1'\},\sigma_2\{k_2,k_2{}'\}) \\ &= ((k_1 \cdot k_1')(\rho_1(k_1) \square (k_1)^{-1},(k_2 \cdot k_2')(\sigma_1(k_2) \square (k_2)^{-1})) \\ &= ((k_1 \cdot k_1'),(k_2 \cdot k_2'))(\rho_1(k_1) \square (k_1)^{-1},\sigma_1(k_2) \square (k_2)^{-1}) \\ &= ((k_1,k_2) \cdot (k_1',k_2{}'))(\rho_1(k_1) \square (k_1)^{-1},\sigma_1(k_2) \square (k_2)^{-1}) \\ &= ((k_1,k_2) \cdot (k_1',k_2{}'))(\omega(k_1,k_2) \square (k_1',k_2')^{-1}) \end{aligned}$$

2GCM2)

$$\begin{aligned} \{\mu(f_1,f_2),\mu(f_1',f_2')\} &= \{(\rho_2(f_1),\sigma_2(f_2)),(\rho_2(f_1'),\sigma_2(f_2'))\} \\ &= (\{\rho_2(f_1),\rho_2(f_1')\},\{\sigma_2(f_2),\sigma_2(f_2')\}) \\ &= ((f_1 \cdot f_1')(f_1')^{-1},(f_2 \cdot f_2')(f_2')^{-1}) \\ &= ((f_1,f_2) \cdot (f_1',f_2'))(f_1',f_2')^{-1} \end{aligned}$$

2GCM3)

i)

$$\begin{aligned} \{(k_0,k_0'),(k_1,k_1')(k_2,k_2')\} &= \{(k_0,k_0'),(k_1k_2,k_1'k_2')\} \\ &= (\{k_0,k_1k_2\},\{k_0',k_1'k_2'\}) \\ &= ({}^{(k_0\cdot k_1)}\{k_0,k_2\}\{k_0,k_1\},{}^{(k_0'\cdot k_1')}\{k_0',k_2'\}\{k_0',k_1'\}) \\ &= {}^{((k_0\cdot k_1),(k_0'\cdot k_1'))}(\{k_0,k_2\}\{k_0,k_1\},\{k_0',k_2'\}\{k_0',k_1'\}) \\ &= {}^{((k_0\cdot k_1),(k_0'\cdot k_1'))}(\{k_0,k_2\},\{k_0',k_2'\})(\{k_0,k_1\},\{k_0',k_1'\}) \\ &= {}^{((k_0,k_0')\cdot(k_1,k_1'))}\{(k_0,k_0'),(k_2,k_2')\}\{(k_0,k_0'),(k_1,k_1')\} \end{aligned}$$

ii)

$$\begin{aligned} \{(k_0,k_0')(k_1,k_1'),(k_2,k_2')\} &= \{(k_0k_1,k_0'k_1'),(k_2,k_2')\} \\ &= (\{k_0k_1,k_2\},\{k_0'k_1',k_2'\}) \\ &= (\{k_0,k_1k_2\}\rho_1(k_0) \square \{k_1,k_2\},\{k_0',k_1'k_2'\}\sigma_1(k_0') \square \{k_1,k_2'\}) \\ &= (\{k_0,k_1k_2\},\{k_0',k_1'k_2'\})(\rho_1(k_0) \square \{k_1,k_2\},\sigma_1(k_0') \square \{k_1,k_2'\}) \\ &= \{(k_0,k_0'),(k_1,k_1')(k_2,k_2')\}\omega(k_0,k_0') \square \{(k_1,k_1'),(k_2,k_2')\} \end{aligned}$$

2GCM4*)

$$
\begin{aligned}
\{\mu(f_1,f_2),(k_1,k_2)\}\{(k_1,k_2),\mu(f_1,f_2)\} &= \{(\rho_2(f_1),\sigma_2(f_2)),(k_1,k_2)\}\{(k_1,k_2),(\rho_2(f_1),\sigma_2(f_2))\} \\
&= (\{\rho_2(f_1),k_1\},\{\sigma_2(f_2),k_2\})(\{k_1,\rho_2(f_1)\},\{k_2,\sigma_2(f_2)\}) \\
&= (\{\rho_2(f_1),k_1\}\{k_1,\rho_2(f_1)\},\{\sigma_2(f_2),k_2\}\{k_2,\sigma_2(f_2)) \\
&= (f_1(\rho_1(k_1) \triangleright (f_1)^{-1}),f_2(\sigma_1(k_2) \triangleright (f_2)^{-1})) \\
&= (f_1,f_2)(\rho_1(k_1) \triangleright (f_1)^{-1},\sigma_1(k_2) \triangleright (f_2)^{-1}) \\
&= (f_1,f_2)(\omega(k_1,k_2) \triangleright (f_1,f_2)^{-1})
\end{aligned}
$$

2GCM5)

$$
\begin{aligned}
h \triangleright \{(k_1,k_2),(k_1',k_2')\} &= h \triangleright (\{k_1,k_1'\},\{k_2,k_2'\}) \\
&= (h \triangleright \{k_1,k_1'\},h \triangleright \{k_2,k_2'\}) \\
&= (\{h \triangleright k_1,h \triangleright k_1'\},\{h \triangleright k_2,h \triangleright k_2'\}) \\
&= \{(h \triangleright k_1,h \triangleright k_2),(h \triangleright k_1',h \triangleright k_2')\} \\
&= \{h \triangleright (k_1,k_2),h \triangleright (k_1',k_2')\}
\end{aligned}
$$

Then,

$$P_F \overset{\mu}{\rightarrow} P_K \overset{\omega}{\rightarrow} H$$

is a 2-generalized crossed module.

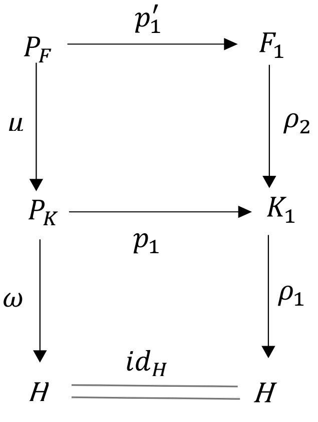
Proposition 4.2 $(F_1,K_1,H,\rho_2,\rho_1)$ and $(F_2,K_2,H,\sigma_2,\sigma_1)$ are two 2-generalized crossed module, then we have natural morphisms of 2GCM;

$$(p_1',p_1,id_H):(P_F,P_K,H,\mu,\omega) \rightarrow (F_1,K_1,H,\rho_2,\rho_1)$$

and

$$(p_2',p_2,id_H):(P_F,P_K,H,\mu,\omega) \rightarrow (F_2,K_2,H,\sigma_2,\sigma_1).$$

Proof Consider the following diagram:



For all $h \in H$, $(k_1,k_2) \in P_K$ and $(f_1,f_2),(f_1',f_2') \in P_F$;

$$
\begin{aligned}
p_1(h \triangleright (k_1,k_2)) &= p_1(h \triangleright k_1,h \triangleright k_2) \\
&= h \triangleright k_1 \\
&= id_H(h) \triangleright k_1 \\
&= id_H(h) \triangleright p_1(k_1,k_2)
\end{aligned}
$$

$$\begin{aligned} p_1'(h \square (f_1, f_2)) &= p_1'(h \square f_1, h \square f_2) \\ &= h \square f_1 \\ &= id_H(h) \square f_1 \\ &= id_H(h) \square p_1'(f_1, f_2) \end{aligned}$$

$$\begin{aligned} \{-,-\}p_1 \times p_1((f_1, f_2), (f_1', f_2')) &= \{-,-\}p_1(f_1, f_2) \times p_1(f_1', f_2') \\ &= \{f_1, f_1'\} \\ &= p_1'\{(f_1, f_2), (f_1', f_2')\} \end{aligned}$$

Moreover, the verification of the following equations is straightforward and follows directly from the definitions:

$$\rho_1 p_1 = id_H\, \omega \quad \text{and} \quad \rho_2 p_1' = p_1\, \mu.$$

Thus, $(p_1', p_1, id_H): (P_F, P_K, H, \mu, \omega) \to (F_1, K_1, H, \rho_2, \rho_1)$ is a morphism of 2GCM. Similarly, it is shown that $(p_2', p_2, id_H): (P_F, P_K, H, \mu, \omega) \to (F_2, K_2, H, \sigma_2, \sigma_1)$ is a morphism of 2GCM.

Theorem 4.3 The category of 2GCM has product object.

Proof: We just need to prove the universal property. Let $(T, R, H, \tau_1, \tau_2)$ be a 2-generalized crossed module with two morphisms of 2GCM;

$$(\alpha_1, \beta_1, id_H): (T, R, H, \tau_1, \tau_2) \to (F_1, K_1, H, \rho_2, \rho_1)$$

$$(\alpha_2, \beta_2, id_H): (T, R, H, \tau_1, \tau_2) \to (F_2, K_2, H, \sigma_2, \sigma_1).$$

Then, there is a unique morphism of 2GCM,

$$(\varphi, \psi, id_H): (T, R, H, \tau_1, \tau_2) \to (P_F, P_K, H, \mu, \omega)$$

such that the diagram commutes:

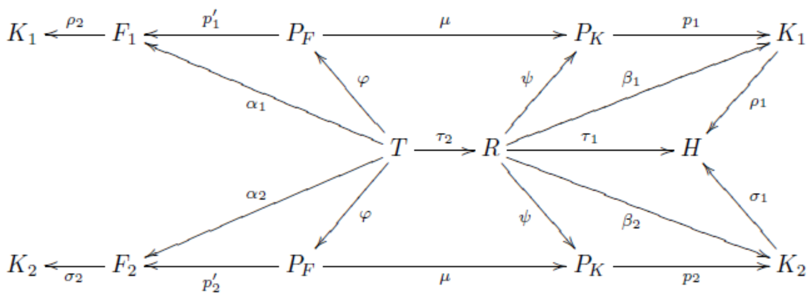
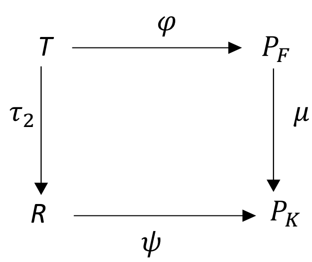
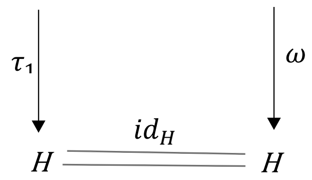


Diagram 1.

Define $\varphi(t) = (\alpha_1(t), \alpha_2(t))$ and $\psi(r) = (\beta_1(r), \beta_2(r))$ for all $t \in T$ and $r \in R$. (φ, ψ, id_H) defines a morphism of 2GCM with the following diagram:





$$\begin{aligned}
\psi(h \square r) &= (\beta_1(h \square r), \beta_2(h \square r)) \\
&= (h \square \beta_1(r), h \square \beta_2(r)) \\
&= id_H(h) \square (\beta_1(r), \beta_2(r)) \\
&= id_H(h) \square \psi(r)
\end{aligned}$$

$$\begin{aligned}
\varphi(h \square t) &= (\alpha_1(h \square t), \alpha_2(h \square t)) \\
&= (h \square \alpha_1(t), h \square \alpha_2(t)) \\
&= id_H(h) \square (\alpha_1(t), \alpha_2(t)) \\
&= id_H(h) \square \varphi(t)
\end{aligned}$$

$$\begin{aligned}
\{-,-\}\psi \times \psi(r, r') &= \{\psi(r), \psi(r')\} \\
&= \{(\beta_1(r), \beta_2(r)), (\beta_1(r'), \beta_2(r'))\} \\
&= (\{\beta_1(r), \beta_1(r')\}, \{\beta_2(r), \beta_2(r')\}) \\
&= (\alpha_1\{r, r'\}, (\alpha_2\{r, r'\}) \\
&= \varphi\{r, r'\} \\
&= \varphi\{-,-\}(r, r')
\end{aligned}$$

for all $t \in T$, $r, r' \in R$ and $h \in H$.

Furthermore, for all $r \in R$ and $t \in T$,

$$p_1\psi(r) = p_1(\beta_1(r), \beta_2(r)) = \beta_1(r)$$

$$p_1'\varphi(t) = p_1'(\alpha_1(t), \alpha_2(t)) = \alpha_1(t).$$

Then, we get $(p_1', p_1, id_H)(\varphi, \psi, id_H) = (\alpha_1, \beta_1, id_H)$, similarly $(p_2', p_2, id_H)(\varphi, \psi, id_H) = (\alpha_2, \beta_2, id_H)$. Thus, the diagram 1 is commutative. Finally, let

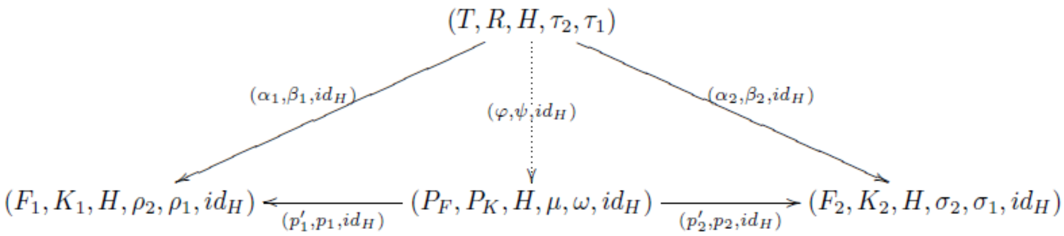
$$(\varphi', \psi', id_H): (T, R, H, \tau_1, \tau_2) \to (P_F, P_K, H, \mu, \omega)$$

be a morphism of 2GCM with the same properties of (φ, ψ, id_H). Define $(f_1, f_2) \in P_F$ by $\varphi'(t) = (f_1, f_2)$ and $(k_1, k_2) \in P_K$ by $\psi'(r) = (k_1, k_2)$.

$$\begin{aligned}
&p_1'\varphi'(t) = \alpha_1(t) \Leftrightarrow p_1'(f_1, f_2) = \alpha_1(t) \Leftrightarrow f_1 = \alpha_1(t) \\
&p_2'\varphi'(t) = \alpha_2(t) \Leftrightarrow p_2'(f_1, f_2) = \alpha_2(t) \Leftrightarrow f_2 = \alpha_2(t) \\
&p_1\psi'(r) = \beta_1(r) \Leftrightarrow p_1(k_1, k_2) = \beta_1(r) \Leftrightarrow k_1 = \beta_1(r) \\
&p_2\psi'(r) = \beta_2(r) \Leftrightarrow p_2(k_1, k_2) = \beta_2(r) \Leftrightarrow k_2 = \beta_2(r)
\end{aligned}$$

for all $r \in R$ and $t \in T$.

Thus, this proves that (φ, ψ, id_H) is unique. Consequently, we get the following product diagram of 2GCM:



Theorem 4.4 The category of 2GCM has pullback object.

Proof: If $(\nu_1, \upsilon_1, id_H): (F_1, K_1, H, \rho_2, \rho_1) \to (F_3, K_3, H, \lambda_2, \lambda_1)$ and $(\nu_2, \upsilon_2, id_H): (F_2, K_2, H, \sigma_2, \sigma_1) \to (F_3, K_3, H, \lambda_2, \lambda_1)$ are two morphisms of 2GCM, then we know from Proposition 4.1 that, $(P_F, P_K, H, \eta, \varepsilon)$ is a 2-generalized crossed module, where

$$P_F = \{(f_1, f_2) \mid \nu_1(f_1) = \nu_2(f_2)\} \subset F_1 \times F_2$$

$$P_K = \{(k_1, k_2) \mid \upsilon_1(k_1) = \upsilon_2(k_2)\} \subset K_1 \times K_2$$

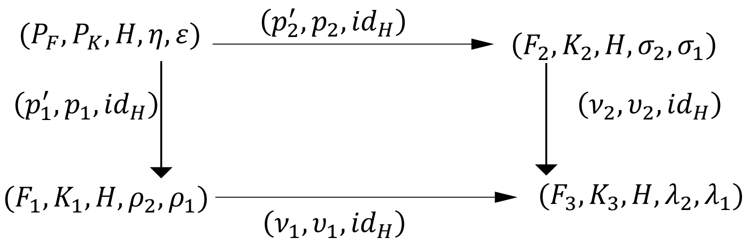
with the peiffer lifting

$$\{-,-\}: P_K \times P_K \to P_F$$

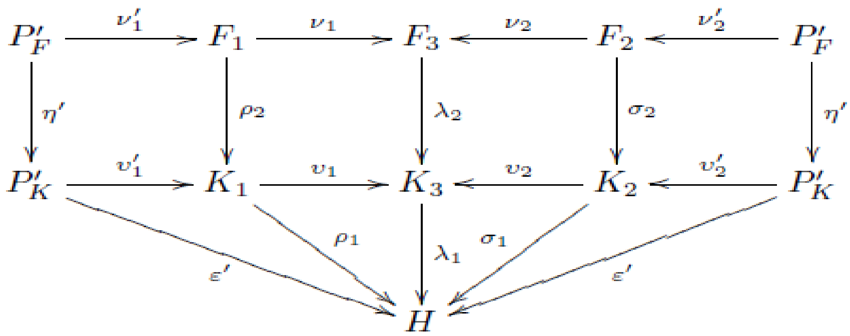
defined by

$$\{(k_1, k_2), (k_1', k_2')\} = (\{k_1, k_1'\}, \{k_2, k_2'\})$$

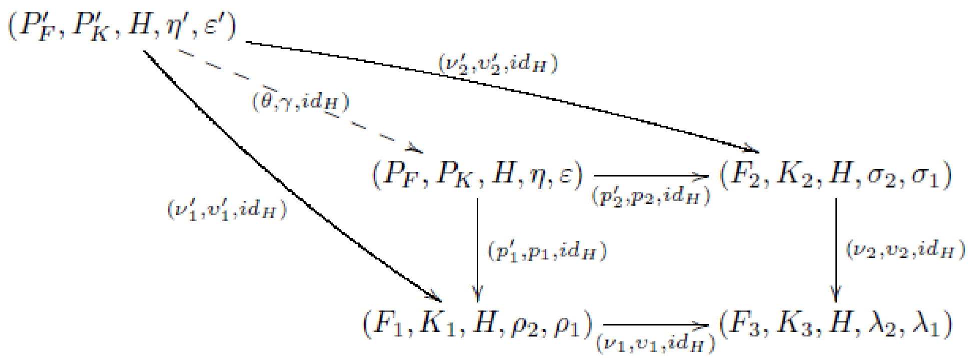
for all $(k_1, k_2), (k_1', k_2') \in P_K$. Also, we have two morphisms of 2GCM, $(p_1', p_1, id_H): (P_F, P_K, H, \eta, \varepsilon) \to (F_1, K_1, H, \rho_2, \rho_1)$ and $(p_2', p_2, id_H): (P_F, P_K, H, \eta, \varepsilon) \to (F_2, K_2, H, \sigma_2, \sigma_1)$ from Proposition 4.2. Thus, we get the following commutative diagram:



Now to check that (p_1', p_1, id_H) and (p_2', p_2, id_H) provide the universal porperty. $(P_F', P_K', H, \eta', \varepsilon')$ be any 2-generalized crossed module and let $(\nu_1', \upsilon_1', id_H): (P_F', P_K', H, \eta', \varepsilon') \to (F_1, K_1, H, \rho_2, \rho_1)$ and $(\nu_2', \upsilon_2', id_H): (P_F', P_K', H, \eta', \varepsilon') \to (F_2, K_2, H, \sigma_2, \sigma_1)$ be two morphisms of 2GCM such that the diagram



commutes, i.e. $(\nu_1, \upsilon_1, id_H)(\nu'_1, \upsilon'_1, id_H) = (\nu_2, \upsilon_2, id_H)(\nu'_2, \upsilon'_2, id_H)$. Then, there is a unique morphism of 2GCM $(\theta, \gamma, id_H): (P'_F, P'_K, H, \eta', \varepsilon') \to (P_F, P_K, H, \eta, \varepsilon)$ such that the following diagram is commutative:



Define $\theta: P'_F \to P_F$ by $\theta(f'_1, f'_2) = (\nu'_1(f'_1), \nu'_2(f'_2))$ for all $(f'_1, f'_2) \in P'_F$, and $\gamma: P'_K \to P_K$, by $\gamma(k'_1, k'_2) = (\upsilon'_1(k'_1), \upsilon'_2(k'_2))$ for all $(k'_1, k'_2) \in P'_K$. We can easily show that (θ, γ, id_H) is a morphism of 2GCM. Furthermore, we get

$$(p'_1, p_1, id_H)(\theta, \gamma, id_H) = (\nu'_1, \upsilon'_1, id_H)$$

$$(p'_2, p_2, id_H)(\theta, \gamma, id_H) = (\nu'_2, \upsilon'_2, id_H)$$

and this prove the commutativity of diagram above. The uniqueness of (θ, γ, id_H) is shown similarly to the previous proof.

Lemma 4.5 As a result, in the category of 2GCM, an equalizer object can be constructed using the product and pullback objects. More precisely, in any category, the equalizer of the parallel morphisms $(\theta, \vartheta): T \to S$, is the pullback of $(1_T, \theta): T \to T \times S$ and $(1_T, \vartheta): T \to T \times S$.

Remark 4.6 The category of 2GCM has a zero object $(\{e\}, \{e\}, \{e\}, id, id)$, where $(\{e\}, \{e\}, id)$ is the zero object in the category of generalized crossed modules.

Theorem 4.7 The category of 2GCM is finitely complete.

Proof: Follows from Theorem 4.3, Theorem 4.4 and Lemma 4.5.

5. CONCLUSIONS

We already know that, we have the adjunction:

$$\mathrm{Hom}_{\mathbf{GCM}}(\Psi(\mathfrak{R}), \mathfrak{C}) \cong \mathrm{Hom}_{2\mathrm{GCM}}(\mathfrak{R}, \Omega(\mathfrak{C})),$$

between the category of 2GCM and the category of generalized crossed modules, where $\mathfrak{C}$ is a generalized crossed module and $\mathfrak{R}$ is a 2-generalized crossed module. As a result of this adjunction, we can say that the functor Ω preserves limits, while Ψ preserves colimits. Consequently, all the constructions discussed in the previous section, which are the certain cases of limits, are preserved under the functor Ω. For instance, let $\mathcal{H}$ and $\mathcal{R}$ be two generalized crossed modules with the same codomain and $\boldsymbol{P}$ be the product of them. By using the adjunction above, we can say that, the product of 2GCM $\Omega(\mathcal{H})$ and $\Omega(\mathcal{R})$ is $\Omega(\boldsymbol{P})$. Moreover, as mentioned above, similar properties can be applied not only to the product object, but also to all of the notions we have defined.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

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