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A NOTE ON PAPPIAN AFINNE PLANES

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Abstract

In (Schmidt and Steinitz,-1996); an affine plane with fixed basis $\{t_1, t_2, 0\}$ *is coordinated. Then, a ternary operation* T *on* R *which is a set of points on* l *which is dependent on the coordinate system* l_1, l_2, t *is defined. In addition, two different binary operation denoted by* $+, \bullet$ *on* R *using ternary operation* T. *After then, it is showed that* $(R, +, \bullet)$ *is a division ring. In this paper, first of all we examined the relation between* (R, T) *ternary ring and Desargues postulate in afine plane. After then, we showed that* $(R, +, \bullet)$ *is field in case affine plane satisfies Pappus Theorem. This results appeared in the first author's Msc thesis.*

Keywords : Afine plane, Desargues Postulat, Pappus Theorem

1. INTRODUCTION

Definition 1.1: [1] An affine space is a quadrupel $A = (P, L, \|, \sim)$ where P is a set, L is a set of nonempty subsets of P, $\|$ is a binary relation on L and $\sim$ is a binary relation on P such that the following conditions are satisfied.

(A1) Line axiom : For all $p, q \in P$ with $p \neq q$ there exists (with respect to set inclusion) a least member of L, denoted by pq, which contains p anb q. Further, for every $l \in L$ and p∈ l there exists a $p \in P \setminus p$ with $l := pq$.

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(A2) Parallel axiom : $\|$ is an equivalence relation on L such that for every pair $(p,l)\in P\times L$ there exists a unique member k of L with $p\in k$ and $k\parallel l$; we abbreviate $\Pi(p|l):=k$. Further; $k\subseteq l$ implies $\Pi(p|k)\subseteq\Pi(p|l)$ for all $p\in P$ and $k,l\in L$.

(A3) Triangle axiom : Whenever p,q,r are pairwise different elements of P then $\Pi(a|pq)=\Pi(b|pq)$ implies $\Pi(a|pr)\wedge\Pi(b|qr)\neq\varnothing$ for all $a,b\in P$.

(A4) Independence axiom : The relation ~ is antireflexive and symmetric such that for all $p,q,r\in P$ with $p\sim q$ there exists $s\in P$ with $r\sim s$ and $pq\parallel rs$. Further, $p\sim q$ and $(pq)\cap l:=\{p\}$ implies $p'\sim q$ and $(p'q)\cap l:=\{p'\}$ for all $p,p',q\in P$ and $l\in L$ with $p,p'\in l$.

An affine space $A=(P,L,\parallel,\sim)$ is said to be an affine plane if it contains a 3-element basis, i.e there exist $0,p,q\in P$ with $0\sim p$ and $0\sim q$ such that every member k of L has a 1-element intersection with $0q$ provided $k\parallel 0p$.

In case of A is an afine plane, the above axioms coincide the axioms which is known.

Let $A=(P,L,\parallel,\sim)$ be an affine space.

(i) The elements of P are called points and the members of L lines. Lines k,l with $k\parallel l$ are parallel; points p,q with $p\sim q$ are independent.

(ii) For lines k,l of A, $k\subseteq\parallel l$ provided $\Pi(p|k)\subseteq\Pi(p|l)$ is satisfied for some (and hence for every) $p\in P$.

If k and l intersect in a unique point r, we show $k\wedge l:=r$, in case $\Pi(p|k)\wedge\Pi(p|l):=p$ holds for some (and hence for every) $p\in P$. This is denoted by $k\#l$.

(iii) The point at infinity of $k \in L$ is defined as $\Pi(k) := \{l \in L | l \| k\}$; the connecting line of a point p and a point at infinity $\Pi(k)$ is given by $p \vee \Pi(p|k)$ and it will be reasonable to agree upon $p \sim \Pi(k)$. The set of all points at infinity of A shall be denoted by P_∞, the elements of $P \cup P_\infty$ are called generalized points.

Definition 1.2 : [1] Let A be an affine space.

(i) Let $a_0, a_1, ..., a_n$ points and let z be a generalized point. We say that an $n-$ tuple $b_0, b_1, ..., b_n$ of points is centrally perspective to $(a_0, a_1, ..., a_n)$ via z briefly $(b_0, b_1, ..., b_n)$ is CP_z to $(a_0, a_1, ..., a_n)$ if $b_i \in a_i z$ and $b_i b_{i+1} \subseteq \| a_i a_{i+1}$ for all $i = 0,1,....., n$ (where $a_{n+1} = a_0, b_{n+1} = b_0$).

A satisfies Desargues' postulate for $(a_0, a_1, ..., a_n)$ via z if all $b_0 \in a_0 z$ there exist $b_0, b_1, ..., b_n$ such that $(b_0, b_1, ..., b_n)$ is CP_z to $(a_0, a_1, ..., a_n)$.

(ii) For any generalized point z, a triple (a_0, a_1, a_2) of points with $a_0 \sim z$ and $a_0 a_1 \# a_1 z$, $a_0 a_2 \# a_0 z$ will be called a z-triangle.

In the following we will need a special version of Desargues' postulate:

(D_3) Whenever z is a generalized point, then Desargues' postulate is satisfied for every z-triangle via z.

Remark 1.1 : Let (a_0, a_1, a_2) be a z-triangle (where z is a generalized point.)

(i) For every $b_0 \in a_0 z$ there exists at most one pair of points b_1, b_2 such that (b_0, b_1, b_2) is CP_z to (a_0, a_1, a_2). If z is a point at infinity and (b_0, b_1, b_2) is CP_z to (a_0, a_1, a_2) then (b_0, b_1, b_2) is also a z-triangle and

(a_0, a_1, a_2) is CP_z to (b_0, b_1, b_2) hence $a_i a_j \| b_i b_j$ for all $i,j \in \{0,1,2\}$ with $i \neq j$.

(ii) For all $b_i \in a_i z$ $(i = 0,1,2)$ with $b_0 b_1 \subseteq \| a_0 a_1$ and $b_0 b_2 \subseteq \| a_0 a_2$ the condition (D_3) implies $b_1 b_2 \subseteq \| a_1 a_2$, i.e (b_0, b_1, b_2) CP_z to (a_0, a_1, a_2).

Now we give the Pappus Theorem in an affine plane.

Pappus Theorem : $[2]$ Let x, y, z and x', y', z' be sets of three distinct collinear points on distinct lines such that no one of these points is on both lines an afine plane A. Then $xy' \subseteq \| x'y$ and $xz' \subseteq \| x'z$ implies $y'z \subseteq \| y'z$.

If A satisfies Pappus Theorem then A is called pappian affine plane. If A satisfies Desargues postulate then, A is called desarguesian affine plane.

Theorem 1.1 : $[2]$ Every pappian affine plane is desarguesian.

In $[1]$, A =(P, L , $\|$, ~) which is an affine plane with fixed basis $0, t_1, t_2$ was coordinatized as following. $l_i := 0t_i$ (where $i = 1,2$) and for all $p, q \in P$ it was abbreviated $(p,q) := \Pi(p|l_2) \wedge \Pi(q|l_1)$ Then $t := (t_1, t_2)$ and $l := 0t$. Therefore; $p_1 := (p,0)$, $p_2 := (0,p)$ and $p_* := (t,p)$; hence $(p,q) := (p_1, p_2)$, $p_* := (t_1, p_2)$ hold for all $p, q \in P$.

l_1, l_2, t forms a coordinate system of A where l_i denotes the i th coordinate line $(i = 1,2)$, 0 is the orijin, and t is the unit point, the i th coordinate of a point p is given by p_i. Furthermore; a ternary operation T is defined on R which is a set of points on l which is dependent on the coordinate system l_1, l_2, t.

$T : (a,b,c) \rightarrow l \wedge \Pi(S(a,b,c)|l_1)$

such that $S(a,b,c) := \Pi(a|l_2) \wedge \Pi(c_2|0b*)$.

Then two different binary operation denoted by $+, \bullet$ be defined on R as follows.

$+ := R \times R;\ (a,b) \rightarrow a + b = T(a,t,b)$

$\bullet := R \times R;\ (a,b) \to a \bullet b = T(a,b,0).$

Theorem 1.2 : $[1]$ If A satisfies D_3 then $(R,+,\bullet)$ is a division ring.

2. MAIN RESULT:

Lemma 2.1 : The following statements are equivalent in an afine plane A.

(i) (R,T) is a linear

(ii) (D_3) holds in A, wherever $z = \Pi(l_2), AA' = l_2$ and $BC \subseteq \| B'C'$.

Proof : $(i) \Rightarrow (ii)$: Let (R,T) is a linear. Therefore; $T(a,b,c) = ab + c$ for all $a,b,c \in R$. Thus $S(a,b,c)$ and $S(ab,t,c)$ are collinear. ABC is a $\Pi(l_2)-$ triangle for $A = (0,c) = c_2$, $B = S(ab,t,c)$ and $C = S(a,b,c)$. Let $AA' = l_2, BC \subseteq \| B'C'$ and $A'B'C'$ be a $\Pi(l_2)-$triangle for $A' = (0,b) = b_2$, $B' = \Pi(ab|l_2) \wedge \Pi(b_2|0t_*) = S(ab,t,b) = \Pi(b_2|c_2S(ab,t,c))$ and

$$B' = \Pi(a|l_2) \wedge \Pi(b_2|0b_*) = S(a,b,b) = \Pi(b_2|c_2S(a,b,c)).$$

Thus; ABC and $A'B'C'$ are $\Pi(l_2)-$triangle. By the remark 1.1 (i), $ABC\ CP_{\Pi(l_2)}\ A'B'C'$.

From the choose of vertex points of this triangles, $c_2S(ab,t,c) \subseteq \| b_2S(ab,t,b)$ and $c_2S(a,b,c) \subseteq \| b_2S(a,b,b)$. Since (R,T) is a linear, $T(ab,t,b) = T(a,b,c)$ and $T(ab,t,b) = T(a,b,b)$. Thus $S(ab,t,b)$ and $S(a,b,b)$ are collinear and $S(ab,t,b)S(a,b,b) \subseteq \| S(ab,t,c)S(a,b,c)$.

Hence; (ii) is satisfies.

$(ii) \Rightarrow (i)$: Let A be a given affine plane with fixed basis $\{0,t_1,t_2\}$ and $\{b_2, S(ab,t,b), S(a,b,b)\}$ be a $\Pi(l_2)-$triangle in A.

$\{c_2, S(ab,t,c), S(a,b,c)\}$ is a $\Pi(l_2)-$triangle for $c_2 \circ \Pi(b_2|l_2), S(ab,t,c) \circ \Pi(S(ab,t,b)|l_2)$ and $S(a,b,b) \circ \Pi(S(a,b,c)|l_2)$. By the remark 1.1 (i)

$$\{b_2, S(ab,t,b), S(a,b,b)\} CP_{\Pi(l_2)} \{c_2, S(ab,t,c), S(a,b,c)\}$$

Since A satisfies (D_3), $b_2S(ab,t,b) \subseteq \| c_2S(ab,t,c)$ and $b_2S(a,b,b) \subseteq \| b_2S(a,b,c)$ implies $S(ab,t,b)S(a,b,b) \subseteq \| S(ab,t,c)S(a,b,c)$.. Thus $S(ab,t,c)$ and $S(a,b,c)$ are collinear. Therefore;

$$\Pi(S(ab,t,c)|l_1) = \Pi(S(a,b,c)|l_1)$$

$$l \wedge \Pi(S(ab,t,c)|l_1) = l \wedge \Pi(S(a,b,c)|l_1)$$

Since T is a ternary operation on R, $T(ab,t,c) = T(a,b,c)$. Also, by the operation "+", $T(ab,t,c) = T(a,b,c)$ implies $ab + c = T(a,b,c)$. Finally, (R,T) ternary ring is a linear.

Lemma 2.2 : The following statements are equivalent:

(i) (R,T) is a linear and $(R,+)$ is a associative.

(ii) A satisfies (D_3) for the every $\Pi(l_2)-$triangles.

Proof $(i) \Rightarrow (ii)$: Since (R,T) is a linear, by the lemma 2.1, A satisfies (D_3) for , $\Pi(l_2)$, $AA' = l_2$ and $BC \subseteq \| B'C' \subseteq \| l$. Also, T is a associative, $T(a,t,b+c) = T(a+b,t,c)$ for all $a,b,c \in R$. Thus, by the operation "+", $S(a,t,b+c)S(a+b,t,c) \subseteq \| l_1$. Since $\Pi(l_2)^{\sim} b_2$, $b_2S(a,t,b) \# (S(a,t,b)|l_2)$ and $(b_2(0+b)) \# \Pi(b_2|l_2), (b_2, S(a,t,b), a+b)$ is a $\Pi(l_2)-$triangle. Also, $(b+c)_2 \circ \Pi(b_2|l_2)$, $S(a,t,b+c) \circ \Pi(S(a,t,b)|l_2)$ and $S(a+b,t,c) \circ \Pi((a+b)_2|l_2)$. In addition; since (R,T) is a linear, $b_2(b+c)_2 = l_2$, $S(a,t,b+c)S(a+b,t,c \subseteq \| S(a,t,b)(a+b))$ and $(b+c)_2 S(a+b,t,c) \subseteq \| b_2t$. Thus;

$$(b_2S(a,t,b), a+b) CP_{\Pi(l_2)} ((b+c)_2, S(a,t,b+c), S(a+b,t,c)).$$

Hence A satisfies (D_3).

$(ii) \Rightarrow (i)$: We assume that A satisfies (D_3). $(b_2, S(a,t,b), b)$ and $((b+c)_2, S(a,t,b+c), S(b,t,c))$ are $\Pi(l_2)-$triangle. Thus; we obtain

$(b+c)_2 S(a,t,b+c) \subseteq \| b_2 S(a,t,b)$ and

$(b+c)_2 S(b,t,c) \subseteq \| b_2 b$

Since A satisfies (D_3), we obtain following result.

$S(a,t,b+c)S(b,t,c) \subseteq \| bS(a,t,b)$............(2.1)

Now we consider $(S(a,t,b), b, a+b)\ \Pi(l_2)-$triangle. By (2.1), $S(a,t,b+c) \circ \Pi(S(a,t,b)|l_2)$; $S(b,t,c) \circ \Pi(b|l_2)$ and $S(a+b,t,c) \circ \Pi(ab|l_2)$.Thus; $S(a,t,b+c)S(b,t,c) \subseteq \| S(a,t,b)b$ and $S(b,t,c)S(a+b,t,c) \subseteq \| b(a+b)$. Since A satisfies (D_3),

$S(a,t,b+c)S(a+b,t,c) \subseteq \| S(a,t,b)(a+b) \subseteq \| l_1$,

and

$$l \wedge \Pi(S(a,t,b+c)|l_1) = l \wedge \Pi(S(a+b,t,c)|l_1)$$
$$T(a,t,b+c) = T(a+b,t,c)$$
$$a+(b+c) = (a+b)+c .$$

Thus, (R,T) is associative.

Now we show that (R,T) is linear. $((b+c)_2, S(a,t,b+c), S(a+b,t,c))$ and $(b_2, S(a,t,b), a+b)$ are $\Pi(l_2)-$triangle. By the lemma 2.1 $b_2(b+c) := l_2$, $(S(a,t,b+c)S(a+b,t,c)) \subseteq \| (S(a,t,b)t)$ and (D_3) is satisfies, (R,T) is linear.

Theorem 2.1 : If A is a Papian plane then $(R,+,\bullet)$ is a field.

Proof : Let A is a Papian plane. By the Theorem 1.1, A satisfies (D_3). Also, by the Theorem 1.2 $(R,+,\bullet)$ is a division ring. Since $(R,\bullet)$ is a semigroup, for every $a \neq 0$ there exist an element a^{-1} of $(R,\bullet)$ such that $a^{-1}a = aa^{-1} = t$. We must show that the operation "•" has a commutative property in order that $(R,+,\bullet)$ is a field. $\Pi(a|l_2)$ and $\Pi(b|l_2)$ are lines in A such that $a \neq b$ and $a,b \in R$. $x = S(a,a,0), y = S(a,b,0)$ and $z = S(a,a,b)$ are points on $\Pi(a|l_2)$. On the otherhand $x' = S(b,a,b), y' = S(b,a,t_2)$ and $z' = S(b,a,0)$ are points on $\Pi(b|l_2)$. Also, $S(a,a,0)$ and $S(b,a,0)$ are on $\Pi(0|0a_*)$. $S(a,a,b)$ and $S(b,a,b)$ are on $\Pi(b_2|0a_*)$.

Since $\Pi(0|0a_*) \subseteq \| \Pi(b_2|0a_*)$;

$S(a,a,0)S(b,a,0) \subseteq \| S(a,a,b)S(b,a,b)$..........(2.2).

We consider, $\{S(b,a,t), S(a,a,t), S(a,a,0)\}-$triangle and $\{S(b,a,b), S(a,a,b), S(a,b,0)\}-$triangle. It is trivial that,

$\{S(b,a,b), S(a,a,b), S(a,b,0)\} CP_{\Pi(l_2)} \{S(b,a,t), S(a,a,t), S(a,a,0)\}$.

From the theorem1.1 and A is a pappian plane, A satisfies (D_3). Thus

$S(a,b,0)S(a,a,b) \subseteq \| S(a,a,0)S(a,a,t)$

$S(a,a,b)S(b,a,b) \subseteq \| S(a,a,t)S(b,a,t)$

and

$S(a,b,0)S(b,a,b) \subseteq \| S(a,a,0)S(b,a,t)$..........(2.3).

Since A is a Pappian plane; $S(a,a,0)S(b,a,0) \subseteq \| S(a,a,b)S(b,a,b)$, $S(a,a,0)S(b,a,t) \subseteq \| S(a,b,0)S(b,a,b)$ implies $S(a,b,0)S(b,a,0) \subseteq \| S(a,a,b)S(b,a,t)$. Thus; it is shown that $S(a,b,0)$ and $S(b,a,0)$ are collinear. But we must show that $S(a,b,0)S(b,a,0) \subseteq \| l_1$.

Now, we consider $\{S(b,a,b), S(a,b,0), S(b,a,0)\}-$triangle and $\{S(b,a,t), S(a,a,0), (b,aa)\}-$ triangle. It is trivial that;

$\{S(b,a,b), S(a,b,0), S(b,a,0)\} CP_{\Pi(l_2)} \{S(b,a,t), S(a,a,0), (b,aa)\}$.

Again from the Theorem 1.1 and A is a pappian plane, A satisfies (D_3). Thus

$S(a,a,0)S(b,a,t) \subseteq \| S(a,b,0)S(b,a,b)$,

$S(b,a,t)(b,aa) \subseteq \| S(b,a,b)S(b,a,0)$

and $S(a,b,0)S(b,a,0) \subseteq \| S(a,a,0)(b,aa)$. Since $S(a,a,0) = (a,aa)$, $(a,aa)(b,aa) = S(a,a,0)(b,aa) \subseteq \| l_1$. Thus;

$S(a,b,0)S(b,a,0) \subseteq \| S(a,a,0)(b,aa)$........(2.4)

$S(a,a,0)(b,aa) \subseteq \| l_1$................(2.5).

From (2.4) and (2.5), we obtain

$S(a,b,0)S(b,a,0) \subseteq \| l_1$ and

Thus;

$l \wedge \Pi(S(a,b,0) \| l_1) = l \wedge \Pi(S(b,a,0) \| l_1)$

$T(a,b,0) = T(b,a,0)$

$a \bullet b = b \bullet a$

Thus $(R,+,\bullet)$ is a field.

References

[1] Anapa, P. The coordinatization of affine planes and its ternary ring, Fen Bilimleri Enstitüsü ,Msc Thesis, 1998.

[2] Dembowski , P. Finite Geometries, Springer-Verlag New York Inc.1968.

[3] Stefan E. Schmidt and Ralph Steinitz . The Coordinatization of Affine Planes by Rings, Geo.Ded. 62.299-317, 1996.

Özet

(Schmidt ve Ralph,-1996) da $\{0,t_1,t_2\}$ *tabanına bağlı olarak bir afin düzlem koordinatlanmıştır. Daha sonra* l_1,l_2,t *koordinat sisitemine bağlı olarak* l *doğrusu üzerindeki noktaların kümesi* R *olmak üzere* R *kümesi üzerinde bir* T *üçlü işlem tanımlanarak,* $(R,+,\bullet)$ *nın bir bölümlü halka olduğu gösterilmiştir. Bu makalede ilk olarak afin düzlemde* (R,T) *üçlü halkası ile Desargues Postulatı arasındaki ilgi incelendi. Daha sonra, afin düzlemin Pappus Teoremini sağlaması durumunda* $(R,+,\bullet)$ *nın bir cisim olduğu gösterildi. Bu sonuçlar ilk yazarın Master tezinde görülebilir.*

***Anahtar Kelimeler** : Afin düzlem, Dezarg Postulatı, Pappus Teoremi.*