A NOTE ON PAPPIAN AFINNE PLANES

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Abstract

In (Schmidt and Steinitz, 1996), an affine plane with fixed basis \( \{t_1, t_2, 0\} \) is coordinated. Then, a ternary operation \( T \) on \( R \) which is a set of points on \( I \) which is dependent on the coordinate system \( l_1, l_2, t \) is defined. In addition, two different binary operation denoted by \( +, \cdot \) on \( R \) using ternary operation \( T \). After then, it is showed that \( (R,+,\cdot) \) is a division ring. In this paper, first of all we examined the relation between \( (R, T) \) ternary ring and Desargues postulate in affine plane. After then, we showed that \( (R,+,\cdot) \) is field in case affine plane satisfies Pappus Theorem. This results appeared in the first author's Msc thesis.

Keywords: Affine plane, Desargues Postulate, Pappus Theorem

1. INTRODUCTION

Definition 1.1: [1] An affine space is a quadrupel \( A = (P, L, \|, \sim) \) where \( P \) is a set, \( L \) is a set of nonempty subsets of \( P \), \( \| \) is a binary relation on \( L \) and \( \sim \) is a binary relation on \( P \) such that the following conditions are satisfied.

(A1) Line axiom: For all \( p, q \in P \) with \( p \neq q \) there exists (with respect to set inclusion) a least member of \( L \), denoted by \( pq \), which contains \( p \) and \( q \). Further, for every \( l \in L \) and \( p \in l \) there exists a \( p \in P \setminus p \) with \( l := pq \).

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(A2) **Parallel axiom**: \( \parallel \) is an equivalence relation on \( L \) such that for every pair \( (p, l) \in P \times L \) there exists a unique member \( k \) of \( L \) with \( p \in k \) and \( k \parallel l \); we abbreviate \( \Pi (p|l) := k \). Further; \( k \subseteq l \) implies \( \Pi (p|k) \subseteq \Pi (p|l) \) for all \( p \in P \) and \( k, l \in L \).

(A3) **Triangle axiom**: Whenever \( p, q, r \) are pairwise different elements of \( P \) then \( \Pi (a|pq) = \Pi (b|pq) \) implies \( \Pi (a|pr) \land \Pi (b|qr) \neq \emptyset \) for all \( a, b \in P \).

(A4) **Independence axiom**: The relation \( \sim \) is antireflexive and symmetric such that for all \( p, q, r \in P \) with \( p \sim q \) there exists \( s \in P \) with \( r \sim s \) and \( pq \parallel rs \). Further, \( p \sim q \) and \( (pq) \cap l := \{p\} \) implies \( p' \sim q \) and \( (p'q) \cap l := \{p'\} \) for all \( p, p', q \in P \) and \( l \in L \) with \( p, p' \in l \).

An affine space \( A = (P, L, \parallel, \sim) \) is said to be an affine plane if it contains a 3-element basis, i.e there exist \( 0, p, q \in P \) with \( 0 \sim p \) and \( 0 \sim q \) such that every member \( k \) of \( L \) has a 1-element intersection with \( 0q \) provided \( k \parallel 0p \).

In case of \( A \) is an affine plane, the above axioms coincide the axioms which is known.

Let \( A = (P, L, \parallel, \sim) \) be an affine space.

(i) The elements of \( P \) are called points and the members of \( L \) lines. Lines \( k, l \) with \( k \parallel l \) are parallel; points \( p, q \) with \( p \sim q \) are independent.

(ii) For lines \( k, l \) of \( A \), \( k \subseteq \parallel l \) provided \( \Pi (p|k) \subseteq \Pi (p|l) \) is satisfied for some (and hence for every) \( p \in P \).

If \( k \) and \( l \) intersect in a unique point \( r \), we show \( k \land l := r \), in case \( \Pi (p|k) \land \Pi (p|l) := p \) holds for some (and hence for every) \( p \in P \). This is denoted by \( k \neq l \).
(iii) The point at infinity of \( k \in L \) is defined as \( \Pi(k) := \{ l \in L \mid l \| k \} \); the connecting line of a point \( p \) and a point at infinity \( \Pi(k) \) is given by \( p \lor \Pi(p \| k) \) and it will be reasonable to agree upon \( p \sim \Pi(k) \). The set of all points at infinity of \( A \) shall be denoted by \( P_\infty \), the elements of \( P \cup P_\infty \) are called generalized points.

**Definition 1.2:** \([1]\) Let \( A \) be an affine space.

(i) Let \( a_0, a_1, \ldots, a_n \) points and let \( z \) be a generalized point. We say that an \( n \) - tuple \( b_0, b_1, \ldots, b_n \) of points is centrally perspective to \( (a_0, a_1, \ldots, a_n) \) via \( z \) briefly \( (b_0, b_1, \ldots, b_n) \) is \( CP_z \) to \( (a_0, a_1, \ldots, a_n) \) if \( b_i \in a_i z \) and \( b_ib_{i+1} \subseteq a_ia_{i+1} \) for all \( i = 0, 1, \ldots, n \) (where \( a_{n+1} = a_0, b_{n+1} = b_0 \)).

A satisfies Desargues' postulate for \( (a_0, a_1, \ldots, a_n) \) via \( z \) if all \( b_0 \in a_0 z \) there exist \( b_0, b_1, \ldots, b_n \) such that \( (b_0, b_1, \ldots, b_n) \) is \( CP_z \) to \( (a_0, a_1, \ldots, a_n) \).

(ii) For any generalized point \( z \), a triple \( (a_0, a_1, a_2) \) of points with \( a_0 \sim z \) and \( a_0a_1 \neq a_1z \), \( a_0a_2 \neq a_2z \) will be called a \( z \)-triangle.

In the following we will need a special version of Desargues' postulate:

\((D_3)\) Whenever \( z \) is a generalized point, their Desargues' postulate is satisfied for every \( z \)-triangle via \( z \).

**Remark 1.1:** Let \( (a_0, a_1, a_2) \) be a \( z \)-triangle (where \( z \) is a generalized point.)

(i) For every \( b_0 \in a_0 z \) there exists at most one pair of points \( b_1, b_2 \) such that \( (b_0, b_1, b_2) \) is \( CP_z \) to \( (a_0, a_1, a_2) \). If \( z \) is a point at infinity and \( (b_0, b_1, b_2) \) is \( CP_z \) to \( (a_0, a_1, a_2) \) then \( (b_0, b_1, b_2) \) is also a \( z \)-triangle and
\((a_0, a_1, a_2)\) is \(CP_z\) to \((b_0, b_1, b_2)\) hence \(a_i a_j \parallel b_i b_j\) for all \(i, j \in \{0,1,2\}\) with \(i \neq j\).

\(\text{(ii)}\) For all \(b_i \in a_i z\) \((i = 0,1,2)\) with \(b_0 b_i \subseteq a_0 a_1\) and \(b_0 b_2 \subseteq a_0 a_2\) the condition \((D_3)\) implies \(b_1 b_2 \subseteq a_1 a_2\), i.e. \((b_0, b_1, b_2)\) \(CP_z\) to \((a_0, a_1, a_2)\).

Now we give the Pappus Theorem in an affine plane.

**Pappus Theorem:** \([2]\) Let \(x, y, z\) and \(x', y', z'\) be sets of three distinct collinear points on distinct lines such that no one of these points is on both lines an affine plane \(A\). Then \(xy' \subseteq xx' y\) and \(xz' \subseteq xx' z\) implies \(y'z \subseteq yy' z\).

If \(A\) satisfies Pappus Theorem then \(A\) is called pappian affine plane. If \(A\) satisfies Desargues postulate then, \(A\) is called desarguesian affine plane.

**Theorem 1.1:** \([2]\) Every pappian affine plane is desarguesian.

In \([1]\), \(A = (P, L, \parallel, \sim)\) which is an affine plane with fixed basis \(0, t_1, t_2\) was coordinatized as following. \(l_i := 0t_i\) (where \(i = 1,2\)) and for all \(p, q \in P\) it was abbreviated \((p, q) := \Pi(p|l_2) \wedge \Pi(q|l_1)\). Then \(t := (t_1, t_2)\) and \(l := 0 t\). Therefore; \(p_1 := (p, 0)\), \(p_2 := (0, p)\) and \(p_* := (t, p)\); hence \((p, q) := (p_1, p_2)\), \(p_* := (t_1, p_2)\) hold for all \(p, q \in P\).

\(l_1, l_2, t\) forms a coordinate system of \(A\) where \(l_i\) denotes the \(i\) th coordinate line \((i = 1,2)\), \(0\) is the origin, and \(t\) is the unit point, the \(i\) th coordinate of a point \(p\) is given by \(p_i\). Furthermore; a ternary operation \(T\) is defined on \(R\) which is a set of points on \(l\) which is dependent on the coordinate system \(l_1, l_2, t\).

\[T : (a, b, c) \rightarrow l \wedge \Pi(S(a, b, c)|l_1)\]

such that \(S(a, b, c) := \Pi(a|l_2) \wedge \Pi(c_2|0 b*)\).

Then two different binary operation denoted by \(+, \ast\) be defined on \(R\) as follows.

\[+ := R \times R; (a, b) \rightarrow a + b = T(a, t, b)\]
Theorem 1.2: \[1\] If \( A \) satisfies \( D_3 \) then \((R, +, \cdot)\) is a division ring.

2. MAIN RESULT:

Lemma 2.1: The following statements are equivalent in an affine plane \( A \).

\( (i) \) \((R, T)\) is a linear

\( (ii) \) \((D_3)\) holds in \( A \), wherever \( z = \Pi(l_2), AA' = l_2 \) and \( BC \subseteq \lVert B'C' \rVert \).

Proof: \((i) \implies (ii)\): Let \((R, T)\) is a linear. Therefore; \( T(a, b, c) = ab + c \) for all \( a, b, c \in R \). Thus \( S(a, b, c) \) and \( S(ab, t, c) \) are collinear. \( ABC \) is a \( \Pi(l_2) \)-triangle for \( A = (0, c) = c_2 \), \( B = S(ab, t, c) \) and \( C = (a, b, c) \). Let \( AA' = l_2, BC \subseteq \lVert B'C' \rVert \) and \( A'B'C' \) be a \( \Pi(l_2) \)-triangle for \( A' = (0, b) = b_2 \), \( B' = \Pi(\lVert ab \rVert l_2) \land \Pi(\lVert b_2 \rVert 0t) = S(ab, t, b) = \Pi(b_2, c_2, S(ab, t, c)) \) and 

\[
B' = \Pi(\lVert a \rVert l_2) \land \Pi(\lVert b_2 \rVert 0b_2) = S(a, b, b) = \Pi(b_2, c_2, S(a, b, c))
\]

Thus; \( ABC \) and \( A'B'C' \) are \( \Pi(l_2) \)-triangle. By the remark 1.1 \((i)\), \( ABC \rightrightarrows_{\Pi(l_2)} A'B'C' \).

From the choose of vertex points of this triangles, \( c_2 S(ab, t, c) \subseteq \lVert b_2 S(ab, t, b) \) and \( c_2 S(a, b, c) \subseteq \lVert b_2 S(a, b, b) \). Since \((R, T)\) is a linear, \( T(ab, t, b) = T(a, b, c) \) and \( T(ab, t, b) = T(a, b, b) \). Thus \( S(ab, t, b) \) and \( S(a, b, b) \) are collinear and \( S(ab, t, b) S(a, b, b) \subseteq S(ab, t, c) S(a, b, c) \).

Hence; \((ii)\) is satisfies.

\((ii) \implies (i)\): Let \( A \) be a given affine plane with fixed basis \( \{0, t_1, t_2\} \) and \( \{b_2, S(ab, t, b), S(a, b, b)\} \) be a \( \Pi(l_2) \)-triangle in \( A \).
\[ \{c_2, S(ab, t, c), S(a, b, c)\} \] is a \( \Pi(l_2) \)-triangle for \( c_2 \circ \Pi(b_2 l_2), S(ab, t, c) \circ \Pi(S(ab, t, b) l_2) \) and \( S(a, b, b) \circ \Pi(S(a, b, c) l_2) \).

By the remark 1.1 (i)

\[ \{b_2, S(ab, t, b), S(a, b, b)\} \subset \Pi(S(ab, t, c), S(a, b, c) \}

Since \( A \) satisfies \((D_3)\), \( b_2 S(ab, t, b) \subset \Pi_2 S(ab, t, c) \) and \( b_2 S(a, b, b) \subset \Pi_2 S(a, b, c) \) implies \( S(ab, t, b)S(ab, t, b) \subset \Pi(S(ab, t, c)S(a, b, c) \). Thus \( S(ab, t, c) \) and \( S(a, b, c) \) are collinear. Therefore;

\[ \Pi(S(ab, t, c) l_2) = \Pi(S(a, b, c) l_1) \]

\[ l \land \Pi(S(ab, t, c) l_2) = l \land \Pi(S(a, b, c) l_1) \]

Since \( T \) is a ternary operation on \( R \), \( T(ab, t, c) = T(a, b, c) \). Also, by the operation "+", \( T(ab, t, c) = T(a, b, c) \) implies \( ab + c = T(a, b, c) \). Finally, \((R, T)\) ternary ring is a linear.

**Lemma 2.2**: The following statements are equivalent:

(i) \((R, T)\) is a linear and \((R, +)\) is a associative.

(ii) \( A \) satisfies \((D_3)\) for the every \( \Pi(l_2) \)-triangles.

**Proof** (i) \(\Rightarrow\) (ii): Since \((R, T)\) is a linear, by the lemma 2.1, \( A \) satisfies \((D_3)\) for , \( \Pi(l_2) \). \( AA' = l_2 \) and \( BC \subset \Pi(B'C' \subset l_2) \). Also, \( T \) is a associative, \( T(a, t, b + c) = T(a + b, t, c) \) for all \( a, b, c \in R \). Thus, by the operation "+", \( S(a, t, b + c)S(a + b, t, c) \subset \Pi_2 l_2 \). Since \( \Pi(l_2) \circ b_2, b_2 S(a, t, b) \# (S(a, t, b) l_2) \) and \( (b_2 (0 + b)) \# \Pi(b_2 l_2), (b_2, S(a, t, b), a + b) \) is a \( \Pi(l_2) \)-triangle. Also, \( b_2 (b + c) \circ \Pi(b_2 l_2), S(a, t, b + c) \circ \Pi(S(a, t, b) l_2) \) and \( S(a + b, t, c) \# \Pi((a + b) l_2) \). In addition, since \((R, T)\) is a linear, \( b_2 (b + c) = l_2, S(a, t, b + c)S(a + b, t, c \subset \Pi S(a, t, b)(a + b) \) and \( S(a + b, t, c) \subset \Pi l_2 \). Thus;

\[ (b_2 S(a, t, b), a + b) \subset \Pi_2 (b + c), S(a, t, b + c), S(a + b, t, c) \]
Hence $A$ satisfies $(D_3)$.

\[(ii) \Rightarrow (i):\] We assume that $A$ satisfies $(D_3)$. $(b_2, S(a, t, b), b)$ and $((b + c)_2, S(a, t, b + c), S(b, t, c))$ are $\Pi(l_2)$-triangle. Thus, we obtain

\[(b + c)_2 S(a, t, b + c) \subseteq \|b_2 S(a, t, b)\] and
\[(b + c)_2 S(b, t, c) \subseteq \|b_2 b\]

Since $A$ satisfies $(D_3)$, we obtain the following result.

\[S(a, t, b + c) S(b, t, c) \subseteq \|b S(a, t, b)\] \[(2.1)\]

Now we consider $(S(a, t, b), b, a + b)$ $\Pi(l_2)$-triangle. By (2.1), $S(a, t, b + c) \circ \Pi(S(a, t, b) \|l_2), S(b, t, c) \circ \Pi(b \|l_2)$ and $S(a + b, t, c) \circ \Pi(ab \|l_2).$ Thus; $S(a, t, b + c) S(b, t, c) \subseteq S(a, t, b) S(a + b, t, c) \subseteq \|b(a + b)\.$ Since $A$ satisfies $(D_3)$,

\[S(a, t, b + c) S(a + b, t, c) \subseteq \|S(a, t, b)(a + b)\] \[\|= l_1, \]

and

\[l \wedge \Pi(S(a, t, b + c) \|l_1) = l \wedge \Pi(S(a + b, t, c) \|l_1)\]

$T(a, t, b + c) = T(a + b, t, c)$

$a + (b + c) = (a + b) + c.$

Thus, $(R, T)$ is associative.

Now we show that $(R, T)$ is linear.

\[((b + c)_2, S(a, t, b + c), S(a + b, t, c))\) and $(b_2, S(a, t, b), a + b)$ are $\Pi(l_2)$-triangle. By the lemma 2.1 $b_2 (b + c) := l_2,$

\[(S(a, t, b + c) S(a + b, t, c)) \subseteq \|(S(a, t, b) S(a + b, t, c))\] and $(D_3)$ is satisfies, $(R, T)$ is linear.

**Theorem 2.1**: If $A$ is a Papian plane then $(R, +, \cdot)$ is a field.

**Proof**: Let $A$ is a Papian plane. By the Theorem 1.1, $A$ satisfies $(D_3)$. Also, by the Theorem 1.2 $(R, +, \cdot)$ is a division ring. Since $(R, \cdot)$ is a semigroup,
for every \( a \neq 0 \) there exist an element \( a^{-1} \) of \((R,\cdot)\) such that 
\( a^{-1}a = aa^{-1} = t \). We must show that the operation "\( \cdot \)" has a commutative property in order that \((R,+,\cdot)\) is a field. \( \Pi(a|l_2) \) and \( \Pi(b|l_2) \) are lines in \( A \) such that \( a \neq b \) and \( a, b \in R \). \( x = S(a,a,0), y = S(a,b,0) \) and \( z = S(a,a,b) \) are points on \( \Pi(a|l_2) \). On the other hand \( x' = S(b,a,b), y' = S(b,a,t_2) \) and \( z' = S(b,a,0) \) are points on \( \Pi(b|l_2) \). Also, \( S(a,a,0) \) and \( S(b,a,0) \) are on \( \Pi(0|0a_*) \). \( S(a,a,b) \) and \( S(b,a,b) \) are on \( \Pi(b_2|0a_*) \).

Since \( \Pi(0|0a_*) \subseteq \Pi(b_2|0a_*) \):
\[
S(a,a,0)S(b,a,0) \subseteq S(a,a,b)S(b,a,b) \ldots (2.2).
\]

We consider, \( \{S(b,a,t),S(a,a,t),S(a,a,0)\} - \) triangle and \( \{S(b,a,b),S(a,a,b),S(a,b,0)\} - \) triangle. It is trivial that,
\[
\{S(b,a,b),S(a,a,b),S(a,b,0)\} \cap \{S(b,a,t),S(a,a,t),S(a,a,0)\} = \emptyset.
\]
From the theorem 1.1 and \( A \) is a pappian plane, \( A \) satisfies \((D_3)\). Thus
\[
S(a,b,0)S(a,a,b) \subseteq S(a,a,0)S(a,a,t)
\]
\[
S(a,a,b)S(b,a,b) \subseteq S(a,a,t)S(b,a,t)
\]
and
\[
S(a,b,0)S(b,a,b) \subseteq S(a,a,0)S(b,a,t) \ldots (2.3).
\]

Since \( A \) is a Pappian plane;
\[
S(a,a,0)S(b,a,0) \subseteq S(a,a,b)S(b,a,b),
\]
\[
S(a,a,0)S(b,a,t) \subseteq S(a,b,0)S(b,a,b)
\]
implies
\[
S(a,b,0)S(b,a,0) \subseteq S(a,a,b)S(b,a,t).
\]
Thus, it is shown that \( S(a,b,0) \) and \( S(b,a,0) \) are collinear. But we must show that \( S(a,b,0)S(b,a,0) \subseteq \Pi_1 \).

Now, we consider \( \{S(b,a,b),S(a,b,0),S(b,a,0)\} - \) triangle and \( \{S(b,a,t),S(a,a,0),S(b,aa)\} - \) triangle. It is trivial that;
\[
\{S(b,a,b),S(a,b,0),S(b,a,0)\} \cap \{S(b,a,t),S(a,a,0),S(b,aa)\} = \emptyset.
\]
Again from the Theorem 1.1 and \( A \) is a pappian plane, \( A \) satisfies \((D_3)\).

Thus
$$S(a,a,0)S(b,a,t) \subseteq S(a,b,0)S(b,a,b),$$
$$S(b,a,t)(b,aa) \subseteq S(b,a,b)S(b,a,0)$$
and
$$S(a,b,0)S(b,a,0) \subseteq S(a,a,0)(b,aa).$$
Since
$$S(a,a,0) = (a,aa); (a,aa)(b,aa) = S(a,a,0)(b,aa) \subseteq \mathbb{I}_1.$$ Thus;
$$S(a,b,0)S(b,a,0) \subseteq S(a,a,0)(b,aa) \subseteq \mathbb{I}_1.$$ From (2.4) and (2.5), we obtain
$$S(a,b,0)S(b,a,0) \subseteq \mathbb{I}_1$$ and
Thus;
$$l \wedge \Pi(S(a,b,0)\mathbb{I}_1) = l \wedge \Pi(S(b,a,0)\mathbb{I}_1)$$
$$T(a,b,0) = T(b,a,0)$$
$$a \cdot b = b \cdot a$$
Thus \((R,+,\cdot)\) is a field.
References


Özet

(Schmidt ve Ralph, 1996) da \( \{0, t_1, t_2 \} \) tabanına bağlı olarak bir afin düzlem koordinatlanmıştır. Daha sonra \( l_1, l_2, l \) koordinat sisteminine bağlı olarak \( l \) doğruusu üzerindeki noktaların kümesi \( R \) olmak üzere \( R \) kümesi üzerinde bir \( T \) üçlü işlem tanımlanarak, \((R, +, \cdot)\) nın bir bölümlü halka olduğu gösterilmiştir. Bu makalede ilk olarak afin düzlemden \((R, T)\) üçlü halkası ile Desargues Postulatu arasındaki ilgi inceledi. Daha sonra, afin düzlemin Pappus Teoremini sağlaması durumunda \((R, +, \cdot)\) nın bir cism olduğu gösterildi. Bu sonuçlar ilk yazarın Master tezinde görülebilir.

Anahtar Kelimeler: Afin düzlem, Dezarg Postulatı, Pappus Teoremi.