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ON FINITE $\{s-1, s\}$-SEMIAFFINE LINEAR SPACES

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Abstract

In this paper, We investigate $\{s-1, s\}$-semiaffine linear spaces with constant point degree. Using only combinatorial techniques we obtaine some results.

1.Introduction

The subject of finite semiaffine linear spaces has been studied and nice combinatorial corollaries ([1], [2], [3], [4], [5], [6]) have been obtained on this subject. In this paper, We investigate $\{s-1, s\}$-semiaffine linear spaces with constant point degree. A finite linear space is a pair $S=(P, L)$ consisting of a finite set P of elements called points and a finite set L of distinguished subsets of points, called lines satisfying the following axioms.

(L1) Any two distinct points of S belong to exactly one line of S.
(L2) Any line of S has at least two points of S.
(L3) There are three points of S not on a common line.

The degree $[p]$ of a point p is the number of lines through p. If $n+1=\max\{[p], p\in P\}$, then n is called the order of the space $S=(P, L)$. We use v and b to denote respectively the number of points and of lines of S.

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The terms i-point and i-line may also be used to refer respectively to a point and a line of degree i.

An affine plane is a linear space A which satisfies the following axiom.

(A)If the point p is not on the line l, then there is a unique line on p missing l.

A projective plane is a linear space satisfying the following axioms.

(P1) Any two distinct lines have a point in common.

(P2) There are four points, no three of which are on a same line.

A linear space with v points in which any line has just two points is a complete graph and is often denoted by K_v.

Let $v \geq 3$ be an i*nteger. A near-pencil on v points is the linear space having one $(v-1)$-line and $v-1$ 2-lines.

Nwankpa-Shrikhandeplane is a linear space on 12 points and 19 lines with constant point size 5, each point being on one 4-line and four 3-lines.

If q consists of a single point $q=\{q\}$, we often write $S-q$ instead of $S-\{q\}$, and we say that S is punctured.

Suppose that we remove a set X of a projective plane P of order n. Then we obtain a linear space $P-X$ having certain parameters (i.e., the number of points, the number of lines , the point-and line-degrees). We call any linear space which has the same parameters as $P-X$ a pseudo-complement of X in P. A pseudo-complement of one line is a linear space with n^2 points, n^2+n lines in which any point has degree $n+1$ and any line has degree n. We know that this is an affine plane, which is a structure embedded into a projective plane of order n. A pseudo-complement of two lines in a projective plane of order n is a linear space having n^2-n points, n^2+n-1 lines in which any point has degree $n+1$ and any line has degree $n-1$ or n.

Let H be a set of non-negative integers. A linear space S is called an H-semiaffine plane if for any non-incident point-line pair (p,l) the number of lines through p disjoint to l belongs to H.

Suppose v,k,μ are integers with $2 \leq k \leq v-2$. A $2-(v,k,\mu)$ blockdesign is an incidence structure with v points in which every line has degree k and any two distinct points are contained in exactly μ lines. The designs

$2-(v,3,1)$ were much studied by J. Steiner, 1796-1863, and we shall refer to them as Steiner triple systems. The notation $S(2,3,v)$ in also used in this case.

Define $S_{s,t}$ to be the unique linear space with $t+3$ points and exactly one line of degree $t-s+2$, while every other line has two points. Then $S_{s,t}$ is $\{s,t\}$-affine of order $t+1$, and with point degree $s+2$ and $t+2$.

Kuiper-Dembowski Theorem: If S is a finite $\{0,1\}$-semiaffine linear space, then it is one of the following:

(a) a near-pencil,
(b) a projective or affine plane,
(c) a punctured projective plane,
(d) an affine plane with one point at infinity.

2. $\{s-1, s\}$ -SEMIAFFINE LINEAR SPACES

We give $\{s-1, s\}$-semiaffine linear spaces with constant point degree. Note that $s=1$ is the Kuiper Dembowski case. We therefore suppose $s \geq 2$.

Clearly, each line has either $n+1-s$ or $n+2-s$ points, and each point is on the same number of $(n+1-s)$- and of $(n+2-s)$-lines.

Let σ be the number of $(n+1-s)$- lines on any point, and let $b' = b_{n+1-s}$ be total number of $(n+1-s)$-lines. We obtained the following equations.

$$v-1=\sigma(n-s)+(n+1-\sigma)(n+1-s) \tag{1}$$

$$b'(n+1-s)=v\sigma=[(n+1)(n+1-s)-\sigma+1]\sigma \tag{2}$$

$$\begin{aligned}(b-b')(n+2-s) &= v(n+1-\sigma) \\ &= [(n+1)(n+1-s)-\sigma+1](n+1-\sigma)\end{aligned} \tag{3}$$

Equations (1,2) and (3) implies the existence if integers x (non-negative) and y such that

$$(n+1-s)x=\sigma(\sigma-1) \tag{4}$$

$$(n+2-s)y=(\sigma+s-2)(\sigma+1-s) \qquad (5)$$

Then (4) and (5) together give

$$y+(n+1-s)y = (n+2-s)y=(\sigma-1+s-1)(\sigma+1-s)$$

$$= (n+1-s)x-(s-1)(s+2) \qquad (6)$$

or

$$(n+1-s)(x-y) = y+(s-1)(s-2) \qquad (7)$$

It follows from equation (7) that $(n+1-s)y+(s-1)(s-2)$.

Proposition 1. We have $y+(s-1)(s-2)\geq 0$. Equality holds if and only if $s=2$ and S is an affine plane or a punctured affine plane.
Proof: Assume $y+(s-1)(s-2)<0$. Then

$$(n+2-s)y<-(n+2-s)(s-1)(s-2).$$

Equations (6) implies

$$(n+1-s)x-(s-1)(s-2)<-(n+2-s)(s-1)(s-2).$$

So

$$(n+1-s)(x+(s-1)(s-2))<0.$$

Since $n+1-s>0$, we get
$0>x+(s-1)(s-2)\geq 0$,

a contradiction..

Suppose, then, that $y+(s-1)(s-2)=0$. From equations (7), we get $x=y\geq 0$; subsequently $x=0=y$ and $s=2$. In view of equation (4) now, $\sigma=0$ or 1. If $\sigma=0$, then S is an affine plane of order n. If $\sigma=1$, equations $(1),(2)$ and (3) imply $v=n^2-1, b'=n+1$ and $b=n^2+n$. Moreover, the $(n+1-s)-$ lines partition the points. Adjoining a point at infinity corresponding to this partition yields an affine plane of order n. Thus, S is a punctured affine plane of order n.

For the remainder of the section, we assume $y+(s-1)(s-2)>0$.

Proposition 2. Either $n \leq s^2 - 1$ or σ satisfies

$$\sigma^2 - \sigma - (s-1)(s-2) - (n+2-s)(n-(s-1)^2) = 0 \qquad (8)$$

In the later case, we get in particular: If $s=2$, then S is the pseudo-complement of two lines in a projective plane of order n; if $s=3$, then $\sigma = n-2$ and S is the pseudo-complement of a triangle in a projective plane of order n if $s \geq 4$, then $n \leq (s^4 - 6s^3 + 13s^2 - 8s - 1)\ /4$.

Proof: Since $y+(s-1)(s-2)>0$, we can use equation (7) to write

$$(n+1-s)z = y+(s-1)(s-2) \geq 2 \qquad (9)$$

Suppose first of all that $z \geq 2$. Since $\sigma \leq n+1$ equationon (5) implies

$$(n+2-s)y \leq (\sigma+s-2)(n+2-s),$$

and hence $y \leq \sigma + s - 2$. Therefore,

$$\begin{aligned} 2(n+1-s) \leq y+(s-1)(s-2) &\leq \sigma+s-2+(s-1)(s-2) \\ &\leq n+1+s-2+(s-1)(s-2) \qquad (10) \end{aligned}$$

from which obtain $n \leq s^2 - 1$..

Now suppose $z=1$, and so $y = n-(s-1)^2$. Substituting in equation (5) gives

$$(n+2-s)(n-(s-1)^2) = \sigma^2 - \sigma - (s-1)(s-2).$$

Solving this quadratic in σ we get as discriminant

$$\Delta = 1+4(n^2 - s^2 n + sn + n + s^3 - 3s^2 + 2s).$$

If $s=2$ this equation reduces to $\Delta = 1+4(n^2-n)(2n-1)^2$. So $\sigma = (1 \pm (2n-1))\ /2$. The non-negative solution is $\sigma = n$. Using equations $(1), (2)$ and (3) we obtain $v = n^2 - n, b' = n^2, b = n^2 + n + 1 - 2$, and so S is the pseudo-complement of two lines in a projective plane of order n.

If $s=3, \Delta=(2n-5)^2$, implying $\sigma=n-2$. Consequently, by equations $(1),(2)$ and $(3), v=(n-1)^2, b'=(n-1)^2$ and $b=(n-1)^2+3(n-1)$. So S is the pseudo-complement of a triangle in a projective plane of order n. Finally, if $s\geq 4$,

$$\Delta<(2n-s^2+s+1)^2. \tag{11}$$

If $2n-s\ \ +s+1\leq 0$, then $n<s^2-1$. On the other hand, if $2n-s^2+s+1>0$, then equation (11) implies $\Delta\leq(2n-s^2+s)^2$, which reduces to $4n\leq s^4-6s^3+13s^2-8s-1$.

<u>Corollary 1.</u> $\{2,3\}-$semiaffine linear space of order $n, n\geq 4$ and $\sigma=n-2$, is the pseudo-complement of a triangle in a projective plane of order n.

<u>Proof:</u> In $\{2,3\}-$semiaffine linear space of order $n, n\geq 4$ and $\sigma=(n-2)$, the number of points

$$\begin{aligned} v &= (n-2)(n-3)+(n+1-n+2)(n-2)+1 \\ &= n^2-2n+1. \end{aligned}$$

In addition, by equations (2) and $(3), b'=n^2-2n+1, b=n^2+n-2$. These parameters are the same parameters as the pseudo-complement of a triangle in a projective plane order n. Therefore $\{2,3\}-$semiaffine linear space of order $n, n\geq 4$ and $\sigma=n-2$, is the pseudo-complement of a triangle in a projective plane of order n.

<u>Corollary 2.</u> $\{1,2\}-$ semiaffine linear space of order $n, n\geq 3$ and $\sigma=n$, is the pseudo-complement of two lines in a projective plane of order n.

<u>Proof:</u> In $\{1,2\}-$ semiaffine linear space of order $n, n\geq 3$ and $\sigma=n$, the number of points

$$\begin{aligned} v &= n(n-2)+(n+1-n)(n-1)+1 \\ &= n^2-n \end{aligned}$$

In addition, by equations (2) and $(3), b'=n^2-2n+1, b=n^2+n-2$. These parameters are the same parameters as the pseudo-complement of two lines in a projective plane of order . Therefore $\{1,2\}-$ semiaffine linear space of order $n, n\geq 3$ and $\sigma=n$ is the pseudo-complement of two lines in a projective plane of order n.

Proposition 3. (a) A $\{1,2\}-$semiaffine linear space of order 3 is $S_{1,2}, K_5$ or can be obtained from an affine plane of order 3 by removing nothing, a single point, or all points of a line along with the line.

(b) A $\{2,3\}-$semiaffine linear space of order n is the pseudo-complement of a triangle, a block design $2-(46,6,1)$ or $S-(2,3,13)$ or K_6.

Proof: (a) Since any point is incident with at most four lines, any line has only to tree points.

From equation $(5): 3y = \sigma(\sigma-1)$. Hence $y \geq 0$. Proposition 1 handled the case $y = 0$, so we assume $y > 0$. If $z \geq 2$, then by equation (9),

$$4 \;=\; 2(n+1-s) \leq y \leq \sigma \leq n+1 = 4.$$

So $\sigma = n+1$ and consequently all lines are $2-$lines. Therefore, S is K_5. If $z = 1$, then by Proposition 2, S is the complement of a line in an affine plane of order 3.

(b) By Proposition 2, we have $n \leq 3^2 - 1 = 8$. In case $n = 8$, it is obtained by equation (10)

$$\begin{aligned} 12 = 2(n+1-s) &\leq y+(s-1)(s-2) \leq \sigma + s - 2 + (s-1)(s-2) \\ &\leq n+1+s-2+(s-1)(s-2) = 12. \end{aligned}$$

Therefore, we have $\sigma = n+1$; so S is a block design in which any line has $n+1-s = 6$ points. Hence $(n+1)(n-2)+1 = 46.$.

In any case, equations (4) and (5) read

$$(n-2)x = \sigma(\sigma-1) \qquad\qquad (n-1)y = (\sigma+1)(\sigma-2).$$

If $4 \leq n \leq 7$, we have only the following possibilities: $n = 4$ and $\sigma = 2$ or $5; n = 5$ and $\sigma = 3$ or $6; n = 6$ and $\sigma = 4; n = 7$ and $\sigma = 5$ If $n = 4$ and $\sigma = 5$, then any line is a $2-$line and $S = K_6$. If $n = 5$ and $\sigma = 6, S$ is an

$S(2,3,13)$. In all other cases, S is the pseudo-complement of a triangle in a projective plane of order n.

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SONLU $\{s-1, s\}$*-YARIAFİN LİNEER UZAYLAR*

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Özet

Bu makalede, sabit nokta dereceli $\{s-1, s\}$*-yarıafin lineer uzayları inceledik. Sadece kombinatoryel özellikleri kullanarak bazı sonuçlar elde ettik.*

Anahtar Kelimeler: *Afin Düzlem H-yarıafin Lineer Uzay, Lineer Uzay, Projektif Düzlem.*