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Research Article / Review Article

A Paradigm on the Qualitative Behavior of Dynamical Systems Inspired by Circuit Theory

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Abstract: In this paper, we consider the qualitative analysis of a liquid mechanical tank system with an electrical model. In the prototype phase, such models are more flexible like the construction process of the first nuclear reactors. The mathematical model of this dynamic system is nonlinear and time-varying. Here, physical principles and engineering specifications will be used to find unique results without any mathematical approximation. The energy function of the system is constructed with intuitive physical principles. The system also will be discussed with and without feedback control laws. Global asymptotic controllability of the equilibrium point of the system will be determined. The literature presents us, the level control works with a few multi-tanks up to six. We generalize those with $n-$tanks from a different theoretical perspective. The readymade system and candidate Lyapunov function will not be used here; the study will be conducted by constructing them. The effectiveness of the control mechanism will be determined by both theoretical analysis and simulation. According to the proposed algorithm, the measurement of liquid levels in tanks can be made in volts anywhere in the system, collectively or individually. The algorithm is clear, not large time-consuming and the solution cost is not expensive. Some simulations are also presented that validate our theoretical predictions.

Keywords: Liquid level control, Lyapunov, Passivity, PD control, Stability

Devre Teorisinden Esinlenerek Dinamik Sistemlerin Niteliksel Davranışı Üzerine Bir Paradigma

Öz: Bu makalede, elektriksel bir modele sahip sıvı mekanik tank sisteminin nitel analizini ele alıyoruz. Prototip aşamasında, bu tür modeller ilk nükleer reaktörlerin inşa süreci gibi daha esnektir. Bu dinamik sistemin matematiksel modeli doğrusal olmayan ve zamanla değişendir. Burada, herhangi bir matematiksel yaklaşım olmaksızın benzersiz sonuçlar bulmak için fiziksel ilkeler ve mühendislik özellikleri kullanılacaktır. Sistemin enerji fonksiyonu sezgisel fiziksel ilkelerle oluşturulmuştur. Sistem ayrıca geri bildirim kontrol yasalarıyla ve onlarsız olarak tartışılacaktır. Sistemin denge noktasının küresel asimptotik kontrol edilebilirliği belirlenecektir. Literatür bize seviye kontrolünün altıya kadar birkaç çoklu tankla çalıştığını göstermektedir. Bunları tanklarla farklı bir teorik bakış açısıyla $n-$ tank olarak genelleştiriyoruz. Hazır sistem ve aday Lyapunov fonksiyonu burada kullanılmayacak; çalışma bunları inşa ederek yürütülecektir. Kontrol mekanizmasının etkinliği hem teorik analiz hem de simülasyonla belirlenecektir. Önerilen algoritmaya göre, tanklardaki sıvı seviyelerinin ölçümü sistemin herhangi bir yerinde, topluca veya ayrı ayrı volt cinsinden yapılabilir. Algoritma açıktır, çok zaman alıcı değildir ve çözüm maliyeti pahalı değildir. Ayrıca teorik tahminlerimizi doğrulayan bazı simülasyonlar da sunulmuştur.

Anahtar Kelimeler: Kararlılık, Lyapunov, Pasiflik, PD kontrol, Sıvı seviye kontrolü

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1. Introduction

In control theory, Lyapunov stability of nonlinear and time-varying systems or machines is an important area for interested researchers. In this context, we consider the dynamics of the fluid in a multi-tank mechanical system by analogizing it to an electrical model. That's why we built the electrical model of the given mechanical model as shown in Figure 1 to make the qualitative analysis of the system more flexible. Since the performance of mechanical systems can be predicted by means of electrical models. This algorithm is preferable in both design and prototype construction. The electrical models are safe, accurate, inexpensive, and have readily available circuit elements. For example, the first nuclear reactors were modeled by electrical models (analog computers) before the reactors themselves were built (Edwards & Penney, 1989). The liquid level determination and the flow control in tank systems have an important role in many industrial processes such as level control for flotation circuits (Sbarbaro & Ortega, 2005), the design of level controller for multi-thank system (Xiuyun, 2015), optimal control of water levels in tanks during distribution (Sankar et al., 2015), PI control of tank's liquid level (Singh et al., 2014), water level positions (Başçi & Derdiyok, 2016). In the prototype phase, analyzing the liquid level and controlling the liquid flow in tank systems with a mechanical model may be difficult, inconvenient, and inaccurate and can even have dangerous consequences. Nevertheless, with mathematical approximations, many scientists of various disciplines used mass-balance or Bernoulli equations as a mathematical model of the mechanical tank systems (Sbarbaro & Ortega, 2005). Nonlinear predictive control is proposed for the stability study of four tank systems (Raff et al., 2006), SISO and MIMO controllers have been implemented in (Kämpjärvi & Jämsä-Jounela, 2003), the issue of the level control two tank system investigated in (Xu et al., 2020), energy-shaping and integral control have been proposed in (Yu et al., 2013), support vector machine-based control (Iplikci, 2011) and sliding mode control (Biswas et al., 2009) have been studied. For the aforementioned works, the mathematical skeleton is roughly similar. We use the Lyapunov function method (Tunç & Ateş, 2006; Yang et al., 2013; Zhang & Yu, 2013) in the qualitative analysis of our study. The system under consideration is passive. Passivity is a basic feature of dissipative dynamical systems (Willems, 1972; Wang et al. 2017; Wang et al., 2018). Viscoelastic, thermodynamic, and the circuit systems (Figure 2) are typical examples of dissipative systems. Passive systems are internally stable. Energy functions of the dissipative systems are bounded (measurable) and decreasing under operation, while those of non-dissipative systems is constants (Ates, 2021). The upper bound of an energy or Lyapunov function can be determined with Gronwall inequality (Eduardo, 1998). A detailed study on dissipative dynamical systems can be found in (Willems, 1972).

This study aims to encourage the development of stability analysis and also by using control design techniques for complex systems. The main focus of the paper is based on the Lyapunov stability theory with: (i) the construction of the differential system, (ii) the methodology of the construction of the energy function by the physical notion of the system, and (iii) result of the time derivative of this function. These three ingredients furnish the paper and they are unique and not published before. This study demonstrates a clear perspective of mathematical analysis and the paper is also welcome in the area of academics in control engineering, automation, robotics, electrical, and mechanical systems, and neural networks.

The paper is structured as: Section 2 presents the preliminary work that defines the general form of the system and its energy function. Section 3 involves the main results that deal with Lyapunov stability, feedback stabilization, linearization, and some simulations. Section 4 closes the paper with a brief discussion and conclusion.

In the subsequent section, we give some basic statements that will guide us in getting the main results.

2. Preliminary

In this paper, we regard the following type of equations for the qualitative analysis of the system theory

$$\dot{v}(\ell) = f(\ell, v(\ell), \theta(\ell)), \tag{1}$$

where $\ell \in \Re_+$ denotes time, $v \in \Re^n$ is the state vector of the system, $\theta \in \Re^m$ is called the input or the control function, and $f \in C^1[\Re_+ \times \Re^n \times \Re^m, \Re^n]$. In this case $\theta(\ell)=0$, let $f(\ell,0,0)=0$, so that (1) admits the zero solution $v(\ell)=0$. For (1) we have the energy function $\Theta(\ell)=\Theta(\ell,v)\in C^1[\Re_+ \times \Re^n, \Re_+]$ and we calculate the time derivative of $\Theta(\ell)$ along the trajectories of (1). In this case, $\theta(\ell)=0$ the isolated equilibrium point 0 of (1) is globally asymptotically controllable, if there exist a class of C^1 Lyapunov (energy) function Θ. Throughout the paper, the time derivative of Θ (with $\theta(\ell)=0$) will be in the form of circuit theory

$$\dot{\Theta}(\ell) = -RI^2 = -Gv^2$$

In the case $\theta(\ell)\neq 0$, the directional derivative of the Lyapunov function Θ of (1) yields a passivity result. A detailed analysis of passivity and dissipative systems can be found in (Ates, 2021).

3. Main Results

In this study, we tackle the following mechanical scheme.

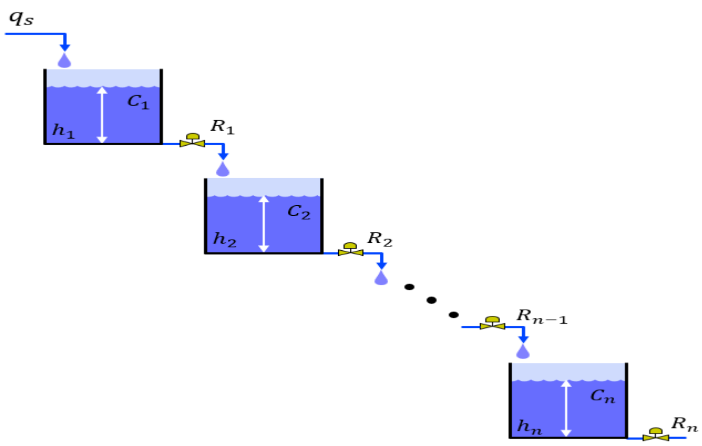

Figure 1. Scheme of interactive $n-$dimensional tank system.

The fluid (q_s), height (h_i), valve (R_i), and translational mass (C_i) of the mechanical system are considered as the current (I_s), voltage (v_i), resistance (R_i), and capacitance (C_i) of the electrical system, $(i=1,\dots,n)$, respectively. By this connection, the electrical scheme of Figure1 (Kämpjärvi et al., 2003) (includes 6 tanks)) is the following:

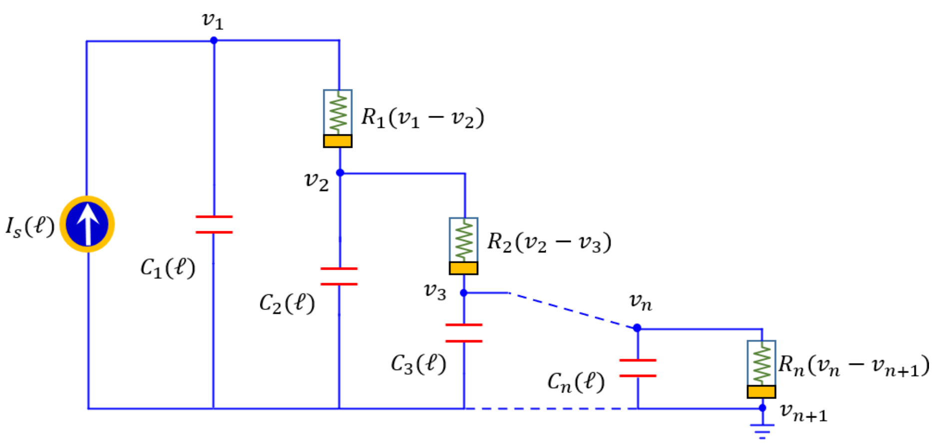

Figure 2. The electrical equivalent circuit of Figure 1.

where $\ell, I_s(\ell)$ denote time and current, $v_i(\ell), c_i(\ell)$ and $R_i(v_i - v_{i+1}) = \dfrac{1}{g_i(v_i - v_{i+1})}$ represent voltage, capacitance, and resistance of the ith section of the above figure. From circuit theory, the relationship between voltage $(v(\ell))$, capacitance $(c(\ell))$, and charge $(q(\ell))$ of the capacitor of an electrical circuit is $q = c\,v.$ Therefore, the current through such a capacitor is

$$\frac{d}{d\ell}q(\ell) = I_c(\ell) = \frac{d}{d\ell}c(\ell)v(\ell) = \dot{c}(\ell)v(\ell) + c(\ell)\dot{v}(\ell) \tag{2}$$

For the $1th$ (first section) of the network we have

$$\dot{v}_1(\ell) = c_1^{-1}(\ell)[-\dot{c}_1(\ell)v_1(\ell) - v_1(\ell)g_1(v_1(\ell)) + I_s(\ell)]$$

Thus, the complete differential system is:

$$\begin{cases} \dot{v}_1 = c_1^{-1}[-\dot{c}_1 v_1 - g_1(v_1 - v_2)(v_1 - v_2) + I_s(\ell)] \\ \dot{v}_i = c_i^{-1}[-\dot{c}_i v_i + g_{i-1}(v_{i-1} - v_i)(v_{i-1} - v_i) - g_i(v_i - v_{i+1})(v_i - v_{i+1})] \end{cases} \tag{3}$$

for $i \geq 2$, $c_i \neq 0$, and $g_i(0) = 0$. An explicit representation of this situation for $n = 3$ is given as:

$$\begin{cases} \dot{v}_1 = c_1^{-1}[-\dot{c}_1 v_1 - g_1(v_1 - v_2)(v_1 - v_2) + I_s(t)] \\ \dot{v}_2 = c_2^{-1}[-\dot{c}_2 v_2 + g_1(v_1 - v_2)(v_1 - v_2) - g_2(v_2 - v_3)(v_2 - v_3)] \\ \dot{v}_3 = c_3^{-1}[-\dot{c}_3 v_3 + g_2(v_2 - v_3)(v_2 - v_3) - g_3(v_3)v_3)] \end{cases} \tag{4}$$

Now, from the above network with the power ($P_{c_i} = v_{c_i} I_{c_i}$) energy (Θ) relationship of the circuit theory we can construct the energy function $\Theta_n(\ell)$ for as:

$$\Theta_n(\ell) = \Theta_n(\ell, v_i(\ell)) = \sum_{i=1}^{n} \int_0^{\ell} \{v_i(\eta)\frac{d}{d\eta}[c_i(\eta)v_i(\mu)]\}d\eta,\ \ell > 0. \tag{5}$$

Let define

$$\inf_{\ell \geq 0} c_i(\ell) = C_i^- \text{ and } \sup_{\ell \geq 0} c_i(\ell) = C_i^+$$

where C_i^- and C_i^+ are positive constants, then it follows that

$$0 \leq \frac{1}{2}C_1^- v_1^2 \leq \frac{1}{2}\sum_{i=1}^{n} C_i^- v_i^2 \leq \Theta_n(\ell) \leq \frac{1}{2}\sum_{i=1}^{n} C_i^+ v_i^2 < \infty. \tag{6}$$

(6) implies that $\Theta_n(\ell)$ is an energy function. Now, we can state the main results:

Theorem 1 The equilibrium state $(v_i)_{i=1}^n = (0,...,0)$ of the system (3) is globally asymptotically controllable if the conditions

(i) $\quad v_{i+1} = 0,\ v_{i+1} \in \Re$,

(ii) $\quad g_i(v_i - v_{i+1}) = \dfrac{1}{R_i(v_i - v_{i+1})} > 0 \quad$ *for* $v_i \neq 0$ *and*

(iii) $\quad g_i(0)=0$,

hold for all i.

Proof. $\dot{v}_i$ is the derivative of the state variable which is constrained by $v_{i+1}=0$ for all i. The time derivative of energy function (5) along the trajectories of the system (3) is

$$\dot{\Theta}_n(\ell)=-\sum_{i=1}^{n-1}[g_i(v_i-v_{i+1})(v_i-v_{i+1})^2+g_n(v_n)v_n^2]+I_s(\ell)v_1(\ell) \tag{7}$$

In the case of the classical Lyapunov method, we set $I_s(\ell)=0$. Therefore, we have Thus, $(v_1,\ldots,v_n)=(0,\ldots,0)$ is the equilibrium state of (3), $\dot{\Theta}_n(\ell)<0$ over $\hat{A}_+ \,'\, \hat{A}^n$, $\Theta_n(\infty)=0$, and $\sum_{i=1}^{n}\Theta_n(\ell,v_i(\ell))\to\infty$ as $\sum_{i=1}^{n}\|v_i\|\to\infty$. Hence, all the motions of (3) are measurable (bounded) as the system depicted in Figure 2. The set Λ where $\dot{\Theta}_n(t)=0$, is $\{0\}$. This implies that $\{0\}$ is the only invariant subset of Λ, and the isolated equilibrium points of (3) are globally asymptotically controllable. Thus, the proof is completed.

3.1. Associated feedback stabilization

Proportional (P), Proportional-derivative (PD), and proportional-derivative-integral (PDI) define feedback control laws. The PDI feedback control law cannot be applied to the system since no inductor exists in the system (Figure 2). If the interaction between the systems produces some inductance, then it can be applied to the system for $n=1$. The connection between the units of the system forces us to do this. Then, the form of PID feedback control law will be

$$I_s(\ell)=-\pi(\ell)v_1(\ell)-\kappa\dot{v}_1(\ell)-\delta\int_0^{\ell}v_1(\eta)d\eta \tag{8}$$

where κ and δ are positive constants, and $\pi>0$ is a function of time. Then the resulting closed-loop equation is obtained when the value of the control (8) is substituted into (4) for $n=1$, yields

$$(\kappa+\dot{c}_1)\ddot{v}_1+[\pi+c_1-\dot{g}_1(v_1)v_1-g_1(v_1)]\dot{v}_1+\ddot{c}_1v_1=0 \tag{9}$$

In the case of circuit theory, (9) denotes a series LRC circuit equation. The qualitative analysis of (9) can be done using the Lyapunov method, for further consideration see (Ates, 2021). The PD control can be applied to the system for any order n. Now consider system (4) with the feedback control law

$$I_s(\ell)=-\pi(\ell)v_1(\ell)-\kappa\dot{v}_1(\ell) \tag{10}$$

Then, the control Lyapunov function will be

$$\begin{aligned}\Theta_{3c}(\ell)=\Theta_{3c}(\ell,v_i(\ell))=&\int_0^{\ell}v_1(\eta)\frac{d}{d\eta}[(c_1(\eta)+\kappa)v_1(\eta)]d\eta+\\&\sum_{i=2}^{3}\int_0^{\ell}\{v_i(\eta)\frac{d}{d\eta}[c_i(\eta)v_i(\eta)]\}d\eta,\ \ell>0\end{aligned} \tag{11}$$

The time derivative of (11) along system (4) with (10) together yields

$$\dot{\Theta}_{3c}(\ell) = -\pi(\ell)v_1^2 - \sum_{i=1}^{2}[g_i(v_i - v_{i+1})(v_i - v_{i+1})^2 + g_3(v_3)v_3^2] < 0. \tag{12}$$

(12) verified by Lyapunov stability.

3.2. Associated linearization

We approach results regarding a nonlinear system by studying the attitude of a linear one. Now, we have the Jacobian matrix Π_c (of (4)) at the equilibrium point $(0,0,0)$ as:

$$\Pi_c = \begin{bmatrix} -(c_1+\kappa)^{-1}(\dot{c}_1+\pi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let pick $c_1(\ell) = 1 + 0.25\sin\ell$, $\kappa = 4$, $\pi(\ell) = 0.25\cos\ell$.

Then

$$v_{1c}(\ell) = k_0 e^{-\frac{\dot{c}_1+\pi}{c_1+\kappa}\ell} = k_0 e^{-\frac{0.5\cos\ell}{1+0.25\sin\ell+4}\ell}$$

and

$$v_1(\ell) = k_0 e^{-\frac{\dot{c}_1}{c_1+}\ell} = k_0 e^{-\frac{0.25\cos\ell}{1+0.25\sin\ell}\ell}$$

where $\ell \geq 0$, k_0 is an arbitrary constant, and in both case the eigenvalues are negative, that is $v_{1c}, v_1 \to 0$ as $\ell \to \infty$. If it is needed in the following procedure, Π denotes the matrix without control $(\kappa = \pi = 0)$ of the above. It is obvious that the given feedback control makes the convergence rate slow. This claim will be verified below with simulation results.

Remark 1 When it comes to control, the function with index c means with control.

3.3. Simulations

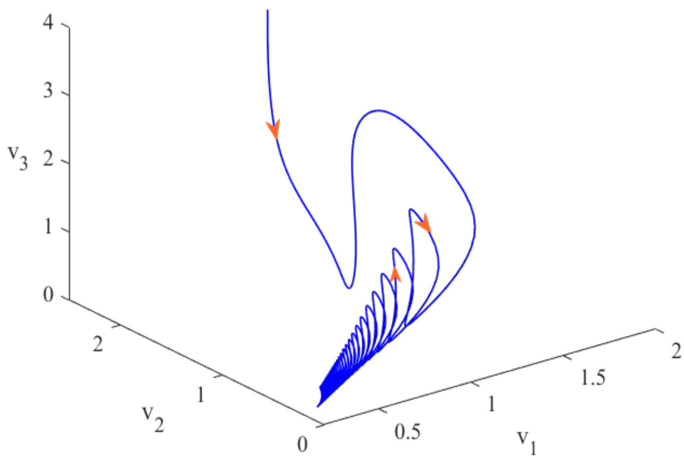

Figure 3. Phase portrait of (4) without control. The axes are given in volt.

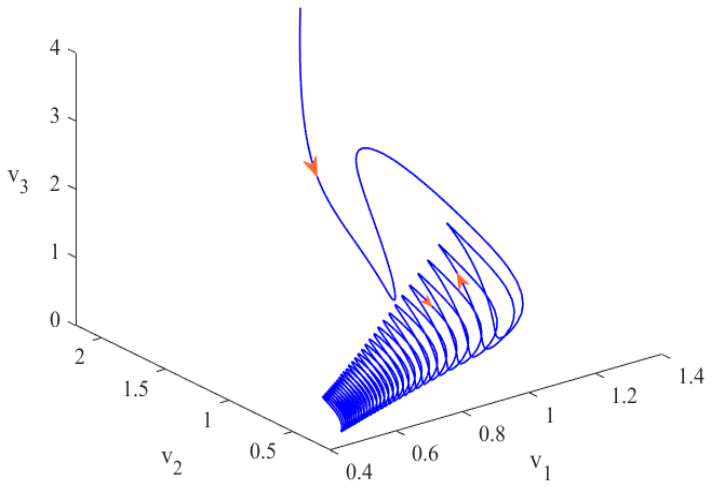

Figure 4. Phase portrait of (4) with control. The axes are given in volt.

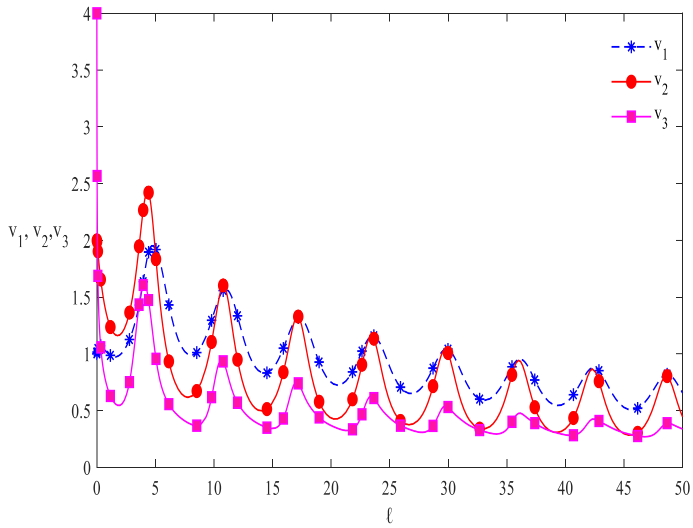

Figure 5. Solution of (4) without control. V_1, V_2, V_3 are given in volt, and ℓ in second.

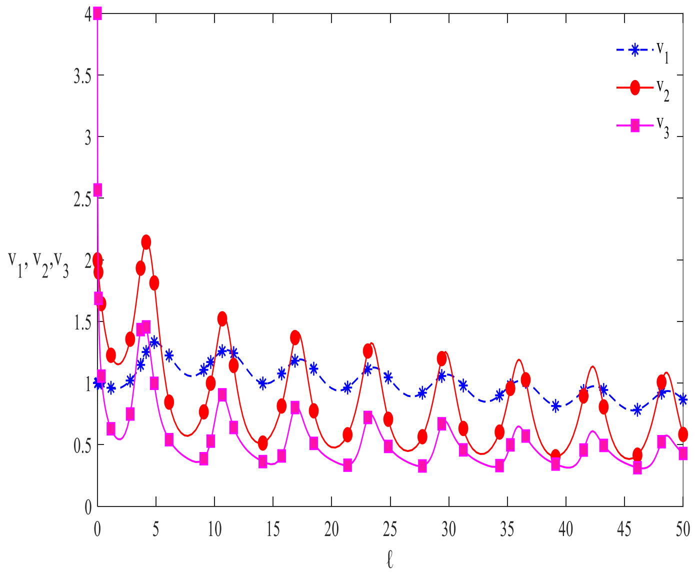

Figure 6. Solution of (4) with control. V_1, V_2, V_3 are given in volt, and ℓ in second.

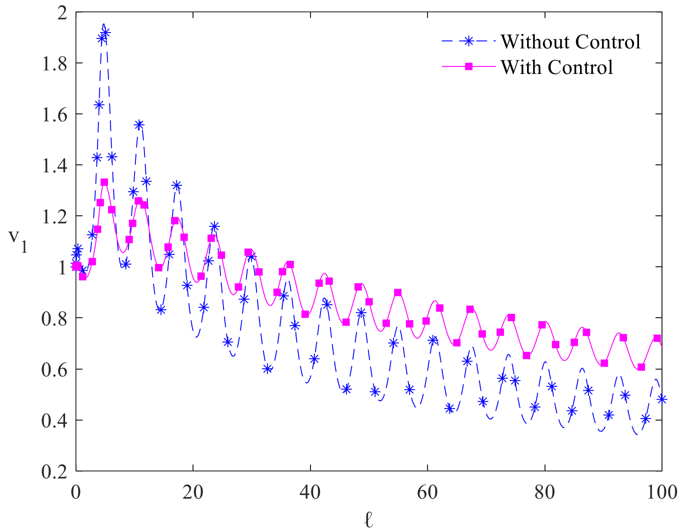

Figure 7. Time series Solution of v_1 in (4) without and with control. V_1 is given in volt, and ℓ in second.

As shown in Figure 3 and Figure 4, the phase platforms have been stabilized. The convergences are decaying exponentially with oscillation. Figure 5 and Figure 6 give measurements of liquid levels of tanks in Volts. The topology of the system shows that feedback controllers only effect v_1 as shown in Figure 5, Figure 6, and Figure 7. The convergence rate is slow when the machine operates with controls since the system operates in parallel. In the series case, the convergence rate is faster.

4. Discussion and Conclusion

In industry, liquid tanks with flotation cells are generally used by mining and chemical engineers in the process of diverse minerals, sediments, inorganic waste constituents, and waste water. In nature, physical phenomena can be represented by differential equations. It is very logical to consider differential equations with the concept of circuit theory and examine electrical models of mechanical systems. Hence, the qualitative performance of mechanical systems can be measured by electrical models. Moreover, electrical systems are more efficient, inexpensive, and safe than mechanical systems. Hence, many technical complications can be avoided. Compared with the relevant references, the designed dynamic systems, the constructed Lyapunov functions, and the time derivative of these functions are unique. Related studies cover a small number of tanks, up to six tanks, but there is no limitation on the number of tanks in this study. Despite this, the proposed algorithm produces systematic results. In this study, the explicit mathematical expressions come from the physical principles of the system and engineering specifications without any approximation. In the future, this work may trigger studies on the stability of dynamic systems, especially Hopfield-type neural networks and complex dynamical systems.

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