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Bir Çizgenin Triyametresi

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ÖZET

Bir $G = (V, E)$ çizgesinin triyametresi $\mathrm{tr}(G)$ ile gösterilir ve her $u, v, w \in V$ için $d(u, v) + d(v, w) + d(w, u)$ toplamının maksimumu olarak tanımlanır. Bu çalışmanın ilk iki bölümünde triyameter tanımı yapılmış ve kısaca bu kavramın tarihsel gelişimine değinilmiştir. Üçüncü bölümde bir çizgenin düğüm sayısı n ve minmum derece δ cinsinden triameter için geliştirilmiş üst sınırlar elde edilmiştir. G, mininmum derecesi $\delta \geq 3$ ve $girth(G) \geq 5$ olan n köşeli bir çizge olmak üzere $tr(G) \leq \frac{3(n-2)}{\delta-1} \leq \frac{6n}{n+1}$ olduğu gösterilmiştir. Ayrıca, G, $girth(G) = 4$ olan n mertebeli bir kübik Cayley çizgesi iken $tr(G) \leq n < \frac{6n}{\delta+1}$ ve G, minimal bağlantılı bir çizge iken $r(G) \leq \frac{6n}{\delta+1}$ eşitsizlikleri elde edilmiştir. Çalışmanın dördüncü bölümünde triyametresi 4, 5 ve $2n-3$'e eşit olan n köşel tüm çizgeler için tam bir kareterizasyon elde edilmiştir. Son olarak çalışmanın altıncı bölümünde gelecek araştırmalar için çeşitli açık problemler ortaya konmuştur.

The Triameter of a Graph

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ABSTRACT

The triameter of a graph $G = (V, E)$is denoted by $\mathrm{tr}(G)$and defined as the maximum value of $d(u, v) + d(v, w) + d(w, u)$over all $u, v, w \in V$. In the first two chapter the concept of triameter is first defined, and its historical development is briefly discussed in the first section. In the third section, improved upper bounds for the triameter of a graph are derived in terms of its order n and δ. If G is a graph with $\delta \geq 3$ and $\mathrm{girth}(G) \geq 5$, then $\mathrm{tr}(G) \leq \frac{3(n-2)}{\delta-1} \leq \frac{6n}{\delta+1}$ was obtained. Furthermore, when G is a cubic Cayley graph of order n with $\mathrm{girth}(G) = 4$, $\mathrm{tr}(G) \leq n < \frac{6n}{\delta+1}$ holds, and for minimally connected graph G, it is shown that $\mathrm{tr}(G) \leq \frac{6n}{\delta+1}$. The fourth section, a characterization is provided for all graphs having triameter 4, 5, or $2n-3$. Finally, in the sixth section, several open problems are proposed for future research.

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INTRODUCTION

The concept of the triameter for connected graphs was originally investigated in [1], where it was identified as a significant distance parameter providing a lower bound for the radio chromatic number. Among various findings, [1] established certain bounds relative to the graph's order and its connected domination number. Furthermore, Das [1] proposed four open problems regarding the triameter; notably, questions 1, 3, and 4 have subsequently been addressed by Hak, Kozerenko, and Oliynyk in [2]. These open inquiries can be restated as follows:

1. The currently known bound for $tr(T)$ is not tight. Is it possible to determine a sharp lower bound for $tr(T)$ given specific values for n (vertex count) and l (leaf count)?
2. Can alternative lower bounds for $tr(G)$ be established for all connected graphs using parameters beyond girth, potentially incorporating the minimum degree $\delta(G)$ or maximum degree $\Delta(G)$?
3. Does every set of three vertices defining the triameter in a tree necessarily include a pair that defines the diameter?
4. Can every diametral pair within a tree be augmented to form a triametral triple?

To comprehend the structural nuances of graphs, researchers heavily rely on distance-related metrics such as radius, average distance, and diameter ([3,4]). Within this scope, the triameter-introduced in [1]-is distinguished by its focus on the aggregate distances between vertex triplets rather than pairs. Mathematically, for a connected graph $G = (V, E)$, the triameter, denoted as $tr(G)$, is defined as the maximum value of the sum $d(u, v) + d(v, w) + d(w, u)$ for any vertices u, v, w belonging to V.

While initially introduced as a distance parameter, the triameter has since revealed intriguing connections to various areas in combinatorics and graph labeling, particularly in establishing lower bounds for different chromatic numbers. Moreover, through its behavior in special classes of graphs-such as Cayley graphs and Hamiltonian graphs-the triameter also intersects with algebraic graph theory, where group-theoretic structures influence graph parameters.

In his seminal work, Das presented several bounds on the triameter in terms of basic graph invariants and posed a set of open problems, many of which were subsequently addressed by (Hak, Kozerenko, and Oliynyk, [2]). These developments underscore the triameter's potential as a rich subject of algebraic and combinatorial investigation. In particular, bounding triameter using parameters like graph order, minimum degree, or domination number remains an active area of research.

In this paper, we contribute to this growing body of knowledge by establishing new bounds and structural characterizations related to the triameter. Our main results include:

- A refined upper bound on triameter in terms of the order and minimum degree of the graph.
- A complete characterization of all graphs with triameter 4, 5, and $2n - 3$, where n is the number of vertices.
- A classification of graphs G for which $tr(G) = tr(G^c)$, and a proof that this equality fails for certain triameter values.
- A discussion of the effect of edge and vertex deletions on triameter, along with related open problems and conjectures.

PRELIMINARIES

In this section, we outline the fundamental concepts and notations of graph theory employed throughout this work. For any graph-theoretical terms not explicitly defined here, the reader is directed to [5] for a comprehensive overview.

A graph G is defined by the pair (V, E), comprising a vertex set $V(G)$ and an edge set $E(G)$. Within such a graph, a *cycle* refers to a closed path that traverses from a starting point back to itself without repeating any vertices. Regarding connectivity, G is classified as *connected* if a path exists between every pair of distinct vertices. Conversely, if every pair of distinct vertices is directly joined by an edge, the graph is termed *complete* and is symbolized as K_n for a graph with n vertices. Furthermore, a graph is deemed *planar* if its representation on a 2D plane involves no edge crossings other than at the endpoints.

An acyclic connected graph is known as a *tree*; essentially, this implies that a unique path links any two vertices. A spanning tree for a graph with n vertices is defined as a subgraph containing $n - 1$ edges that maintains connectivity. The notation $u \sim v$ indicates adjacency between vertices u and v. The *diameter* of G, denoted by $diam(G)$, represents the supremum of the shortest path distances $d(a, b)$ between vertex pairs; this value is considered infinite (∞) for disconnected graphs. The length of the shortest cycle within G constitutes its girth, $girth(G)$, which is set to ∞ if the graph is acyclic. Vertices separated by a distance equal to the diameter are termed antipodal, and they are connected by a diametrical path.

A vertex possessing a degree of exactly one is called a *pendant vertex*, and its incident edge is a pendant edge. The $eccentricity$ $e(v)$ corresponds to the maximum distance from a specific vertex v to any other node in G. Consequently, the graph's *radius*, $rad(G)$, is the minimum value among these eccentricities. In terms of Ramsey theory, the *Ramsey number* $r = R(m, n)$ is defined as the smallest integer r ensuring that every simple undirected graph of that order contains either a clique of size m or an independent set of size n.

Let $\mathbb{G}$ represent a group and S be an inverse-closed subset of $\mathbb{G}$ (where $x \in S$ implies $x^{-1} \in S$) excluding the identity element. The *Cayley graph* of $\mathbb{G}$ relative to S is constructed with the vertex set $\mathbb{G}$, where distinct elements x, y are adjacent if $xy^{-1} \in S$. A set of edges sharing no common vertices constitutes a matching. The *complement* graph G^c shares the vertex set of G but connects two vertices x, y if and only if they are non-adjacent in G. A cubic graph is characterized by a regular degree of three for all vertices. The *bull graph* is a specific planar structure with 5 vertices and 5 edges, resembling a triangle with two disjoint pendant edges. Finally, a graph is said to be *Hamiltonian* if it contains a cycle that visits every vertex exactly once.

UPPER BOUNDS RELATING TO GRAPH ORDER AND MINIMUM DEGREE

This section focuses on establishing refined upper bounds for the triameter of a graph, specifically utilizing the graph's order and its minimum degree as key parameters. Previously, Erdös et al. [6] determined an upper limit for the diameter of a graph based on these exact characteristics.

Theorem 3.1 [6] Consider a connected graph G possessing n vertices and a minimum degree $\delta \geq 2$. The diameter satisfies the inequality: $diam(G) \leq 3n/(\delta + 1)$.

Given the inequality $tr(G) \leq 3 \cdot diam(G)$ a direct consequence is the bound t $tr(G) \leq 9n/(\delta + 1)$. Can we tighten this obvious bound? Based on several observations, we strongly suspect that $tr(G) \leq 6n/(\delta + 1)$. Clearly, this holds for $\delta = 1$ and 2, as $tr(G) \leq 2n - 2$ by [1, Theorem 3.7]. Therefore, we may proceed without loss of generality by assuming $\delta \geq 3$, which implies the existence of a cycle within the graph. Hence the graph contains a cycle. Our initial demonstration verifies this result for graphs exhibiting a girth of at least 5.

Theorem 3.2 Let G be a connected graph on n vertices with $\delta \geq 3$ and $girth(G) \geq 5$. Then

$$\mathrm{tr}(G) \leq \frac{3(n-2)}{\delta - 1} \leq \frac{6n}{\delta + 1}.$$

Proof: Let $P: u = u_0 \sim u_1 \sim u_2 \sim \cdots \sim u_{diam(G)} = v$ be a diametrical path in G. As P is a diametrical path and $girth(G) \geq 5$, u and v, each have at least $\delta - 1$ distinct neighbours and each internal vertex of P has at least $\delta - 2$ distinct neighbours (not lying on P) in G. Thus, counting these distinct neighbours (not on P) and the vertices on the path, we get

$$2(\delta - 1) + (diam(G) - 1)(\delta - 2) + (diam(G) + 1) \leq n, \text{ i.e., } diam(G) + 1 \leq \frac{(n-2)}{\delta - 1}.$$

Now, as $tr(G) \leq 3 \cdot diam(G)$ we have

$$tr(G) < tr(G) + 3 \leq 3 \cdot diam(G) + 3 \leq \frac{3(n-2)}{\delta - 1}.$$

Again as $\delta \geq 3$, we have $(n+2)\delta \geq 3n - 2$, i.e., $(n-2)(\delta + 1) \leq 2n(\delta - 1)$, i.e.,

$$\frac{3(n-2)}{\delta - 1} \leq \frac{6n}{\delta + 1}.$$

To illustrate the aforementioned theorem, the Petersen graph serves as a pertinent model. This well-established cubic graph consists of 10 vertices and possesses a girth of 5, while its triameter is calculated as 6. In this specific instance, the inequality calculation yields 12, which is strictly greater than 6, thereby validating the theorem.

Thus, to prove the conjecture, we need to focus on the case $girth(G) = 3$ or $girth(G) = 4$. A partial answer to the case $girth(G) = 4$ holds in a particular case.

Lemma 3.1 [7] Every cubic Cayley graph characterized by a girth of four possesses a Hamiltonian cycle.

Proposition 3.1 Let G represent a cubic Cayley graph with a girth of 4 and order n. Under these conditions $tr(G) \leq n < \frac{6n}{\delta+1}$.

Proof: It follows from Lemma 3.1 and for an n vertex Hamiltonian graph G, $tr(G) \leq n < \frac{6n}{3+1}$.

The complete bipartite graph $K_{3,3}$, which has a triameter of 3 against an upper bound of 9, stands as a prime example of this scenario. Analogously, this conclusion extends to any Hamiltonian graph where $\delta \leq 5$.

Another partial result can be derived regarding minimally connected graphs. We classify a graph $G = (V, E)$ *as minimally r-connected (or simply minimally connected)* when specific conditions are met: its vertex connectivity is $\kappa(G) = r$ and for any edge $e \in E$ reduces this connectivity to $\kappa(G - e) = r - 1$. Furthermore, the vertex connectivity $\kappa(G)$ is defined as the minimum cardinality of a vertex cut; that is, a subset $S \subseteq V(G)$ whose removal renders G disconnected or reduces it to a single vertex.

Lemma 3.2 [5, Proposition 3.14] Consider a graph G comprising n vertices with a vertex connectivity of κ. The following bound applies: $tr(G) \leq \frac{3(n-2)}{\kappa} + 3$.

Proposition 3.2 Let G be identified as a minimally connected graph with n vertices. Then, $tr(G) \leq \frac{6n}{\delta+1}$.

Proof: It is a standard graph-theoretical fact that for any connected graph G, the vertex connectivity satisfies $\kappa(G) \leq \delta(G)$. As established by Halin in [8], for a minimally connected graph, this equality holds strictly as $\kappa(G) = \delta(G)$. Therefore, by applying Lemma 3.2, we obtain $tr(G) \leq \frac{3(n-2)}{\kappa} + 3 = \frac{3(n-2)}{\delta} + 3 \leq \frac{6n}{\delta+1}$.

As any tree is minimally connected, any tree is an example to justify the above proposition.

GRAPHS WITH TRIAMETER 4, 5, AND $2n - 3$

It is trivially observed that complete graphs represent the sole class of graphs possessing a triameter of 3. In a parallel vein, previous findings in [1] established the upper bound $tr(G) \leq 2n - 2$ identifying that the only n-vertex graphs with $tr(G) = 2n - 2$ achieving this limit are trees characterized by having either 2 or 3 leaves. The primary objective of this section is to provide a characterization for graphs exhibiting triameters of 4, 5, and $2n - 3$.

Theorem 4.1 Consider an n vertex graph G. The condition $tr(G) = 4$ holds if and only if G is isomorphic to $K_n \setminus M$ where M represents a non-empty matching within K_n.

Proof: Let us begin with the complete graph K_n composed of n vertices. The removal of any single edge e results in $tr(K_n - e) = 4$. Suppose we select a second edge e' from the graph $K_n - e$. Should e and e' share a common vertex (i.e., they are incident), the triameter increases to $tr(K_n \setminus \{e, e'\}) = 5$. Conversely, if the edges are disjoint (non-incident), the triameter remains stable at $tr(K_n \setminus \{e, e'\}) = 4$. By extending this logic, it becomes evident that removing a set of disjoint edges- specifically, a matching M of K_n- results in $tr(K_n \setminus M) = 4$

For the converse argument, assume we have a graph G with n vertices where $tr(G) = 4$. Structurally, G must be derived from K_n through the deletion of a specific set of edges. However, as previously demonstrated, the removal of any pair of incident edges elevates the triameter beyond 4. Consequently, to maintain the triameter at 4, the deleted edges must be mutually non-incident, thereby forming a matching. This concludes the proof.

Theorem 4.2 Let $G = (V, E)$ be a graph on n vertices. The equality $tr(G) = 5$ is valid if and only if the diameter is 2 and the graph can be expressed as $G = K_n \setminus T$. Here $T \subseteq E(K_n)$ represents an edge subset that includes at least two incident edges but is devoid of any triangles.

Proof: Let G be a graph on n vertices with $diam(G) = 2$ and $G = K_n \setminus T$, for some $T \subseteq E(K_n)$ such that T contains at least two incident edges and T does not contain any triangle. Since $diam(G) = 2$, then $tr(G) = 4,5$ or 6. As T contains at least two incident edges, $tr(G) \neq 4$. If $tr(G) = 6$, then there exist three vertices u, v, w such that $d(u, v) = d(v, w) = d(w, u) = 2$. (Note that $d(u, v) + d(v, w) + d(w, u) = 3 + 2 + 1$ is not possible, as $diam(G) = 2$). Thus G contains three vertices which are mutually non-adjacent, i.e., T contains a triangle, a contradiction. Thus $tr(G) = 5$.

Conversely, let $tr(G) = 5$. Then $diam(G) = 2$ and G can be obtained by removing some suitable edges T from K_n. If T does not contain any incident edges, then T is a matching and hence $tr(G) = 4$, a contradiction. If T contains a triangle, then $tr(G) \geq 6$, a contradiction. Hence T contains at least two incident edges and T does not contain any triangle.

Corollary 4.1 Let $G = (V, E)$ be a graph on n vertices such that $tr(G) = 6$ and $diam(G) = 2$. Then $G = K_n \setminus T$, for some $T \subseteq E(K_n)$ such that T contains at least a triangle.

Theorem 4.3 Let $G = (V, E)$ be a graph on $n \geq 3$ vertices. Then $tr(G) = 2n - 3$ if and only if G is either K_3 or isomorphic to a graph formed by joining 3 paths to the three vertices of a K_3.

Proof: Clearly, if G is either K_3 or isomorphic to a graph formed by joining 3 paths to the three vertices of a K_3, then $tr(G) = 2n - 3$.

Conversely, let $tr(G) = 2n - 3$. Then for any spanning tree T of G, we have $tr(T)\geq tr(G) = 2n - 3$. Since T is bipartite, $tr(T)$ can not be odd. Thus $tr(T) = 2n - 2$ for all spanning trees T of G and any spanning tree is either with 2 or 3 leaves. We claim that $\Delta(G) < 4$. Because, for every connected graph G, there exists a spanning tree T of G such that $\Delta(G) = \Delta(T)$ and any tree T has at least $\Delta(T)$ leaves. Thus $\Delta(G) = 2$ or 3.

If $\Delta(G) = 2$, then any spanning tree of G can not have 3 leaves. Thus P_n is the only spanning tree of G and $tr(P\_n) = 2n - 2 > tr(G)$. Thus G has at least one edge more than P_n. However, joining any edge between the internal vertices of P_n creates a vertex of degree more than 2 in G. Thus, only the leaves of P_n can be joined by an edge. But, in that case $G = C_n$ and $tr(G) = n = 2n - 3$. This implies that $n = 3$, i.e., $G = K_3$.

If $\Delta(G) = 3$, let T be a spanning tree of G and $\Delta(T) = 3$. Then, we have $\gamma_c(G) = \gamma_c(T) = n - 3 = n - \Delta$. By [9], this implies that T has exactly one vertex of degree 3 , say u. Let u_1, u_2, u_3 be the three neighbours of u in T. Now, $tr(T) = 2n - 2 > tr(G)$. Thus G has at least one edge more than T. Clearly, no edge in G, apart from uu_1, uu_2, uu_3, can be incident to u, as that would violate $\Delta(G) = 3$. Joining any two of u_i and u_j by an edge in T yields a graph with triameter $2n - 3$. However adding any other edges in T yields a graph with triameter smaller than $2n - 3$. Hence the theorem holds.

Corollary 4.2 Let G be a graph on n vertices such that $tr(G) = 2n - 3$ and G^c is connected. Then

$$tr(G^c) = \begin{cases} 7, & if\ G \cong Bull\ graph, \\ 6, & otherwise \end{cases}$$

Proof: It follows from the characterization of graphs with $tr(G) = 2n - 3$ in Theorem 4.3.

CHARACTERIZING GRAPHS WITH $tr(G)$= $tr(G^c)$

Lemma 5.1 [1, Lemma 5.2] Let $G = (V; E)$ be a graph such that G and G^c are connected. If $tr(G) > 9$, then $tr(G^c) \leq 6$.

In the light of Lemma 5.1 if for a graph G, $tr(G) = tr(G^c)$ holds, then $tr(G) = tr(G^c) \in \{3,4,5,6,7,8,9\}$. Then the natural question to ask is to characterize those graphs.

Proposition 5.1 Let There does not exist any graph G, such that G and G^c are connected and $tr(G) = tr(G^c) \in \{3,4,9\}$.

Proof: The only graphs G with $tr(G) = 3$ are K_n. As complement of K_n is disconnected, there does not exist any graph G such that $tr(G) = tr(G^c) = 3$.

Let, if possible, G be a graph such that $tr(G) = tr(G^c) = 4$. As $2 \cdot diam(G) \leq tr(G)\leq 3 \cdot diam(G)$, we have $diam(G) = diam(G^c) = 2$. Let u, v, w be the vertices in G for which the triameter is attained. Then, without loss of generality, let $4 = tr(G) = d_G(u, v) + d_G(v, w) + d_G(w, u) = 1 + 1 + 2$. This means $(u, v), (v, w) \in E(G)$, i.e., $(u, v), (v, w) \notin E(G^c)$. Hence $d_{G^c}(u, v) + d_{G^c}(v, w) + d_{G^c}(w, u) = 2 + 2 + 1 = 5 > tr(G^c)$, a contradiction. Thus, there does not exist any graph G such that $tr(G) = tr(G^c) = 4$.

Let, if possible, G be a graph such that $tr(G) = tr(G^c) = 9$. Thus $diam(G), diam(G^c) \in \{3,4\}$. However, as $diam(G^c) > 3$ implies $diam(G) < 3$, we have $diam(G) = diam(G^c) = 3$. By [10, Corollary 2.4 (v)], this implies $rad(G) = rad(G^c) = 2$. As $rad(G) < diam(G)$, this means that G is not self-centered. Then, by [10, Corollary 2.5], G has two adjacent vertices x and y such that $N_G(x) \cup N_G(y) = V(G)$. However, this means for any three vertices u, v, w, $d_G(u, v) + d_G(v, w) + d_G(w, u) \leq$

8, a contradiction. Hence, there does not exist any graph G such that $tr(G) = tr(G^c) = 9$.

Theorem 5.1 If G is a graph such that $tr(G) = tr(G^c) = 5$, then $G \cong C_5$.

Proof: Let G be a graph on n vertices such that $tr(G) = tr(G^c) = 5$. By Theorem 4.2, we have $G = K_n \setminus T$, for some$T \subseteq E(K_n)$ such that T contain at least two incident edges and T does not contain any triangle. Now, as $G = K_n \setminus T$, we have $G^c = T$ and all the n vertices of G^c are incident to one or more edges in T. Again as $tr(G^c) = 5$, by Theorem 4.2, $K_n \setminus T$ does not contain any triangle. Thus both G and G^c is triangle-free, i.e., G neither contains a clique of size 3 nor an independent set of size 3. Thus $n < R(3,3) = 6$, where $R(a,b)$ denotes the Ramsey number. Hence $n \leq 5$. As G and G^c are both connected, we have $n \geq 4$. However, the only connected graph on 4 vertices whose complement is also connected is P_4 and $tr(P_4) = 6$. Thus the only option left is $n = 5$. An exhaustive search on connected graphs G on 5 vertices such that its complement is connected and $tr(G) = tr(G^c) = 5$ gives $G \cong C_5$.

CONCLUSION AND OPEN PROBLEMS

In this final section, we outline potential avenues for future inquiry and present several open problems derived from our findings.

- **(Characterizing graphs G with $tr(G) = tr(G^c) \in \{6,7,8\}$)**
- It was shown that $tr(G) = tr(G^c) \in \{3,4,9\}$ can not hold and $tr(G) = tr(G^c) = 5$ implies $G \cong G^c \cong C_5$. Thus characterizing the class of graphs G with $tr(G) = tr(G^c) \in \{6,7,8\}$ can be an interesting topic of research. See Figure 1 which shows that each of these classes are non-empty.
- **(Effect of vertex and edge removal on triameter)**

Proposition 6.1 Consider a connected graph G and a vertex subset $A \subseteq V(G)$. Suppose that for every element $a \in A$ be the subgraph $G - a$ remains connected but exhibits a strictly smaller triameter, i.e., $tr(G - a) < tr(G)$. Under these constraints, the cardinality of A satisfies$|A| \leq 3$.

Proof: Let $tr(G) = d(u,v,w)$. Let $x \in V(G) \setminus \{u,v,w\}$ such that $G' = G - x$ is connected. Then $tr(G') \geq d_{G'}(u,v,w) \geq d_G(u,v,w) = tr(G)$. Thus $x \notin A$, i.e., $A \subseteq \{u,v,w\}$, i.e., $|A| \leq 3$.

This can be a starting point for studying triameter vertex-critical graphs.

- (**Conjecture:** $tr(G) \leq \frac{6n}{\delta+1}$) In Theorem 3.2, Proposition 3.1 and Proposition 3.2, it was proved that $tr(G) \leq \frac{6n}{\delta+1}$ for certain families of graphs. However, we strongly suspect that the result is true for any connected graph G and the bound is assymptotically tight. To settle this, one need to prove the result for connected graphs G with girth 3 and 4 and $\delta(G) \geq 3$.

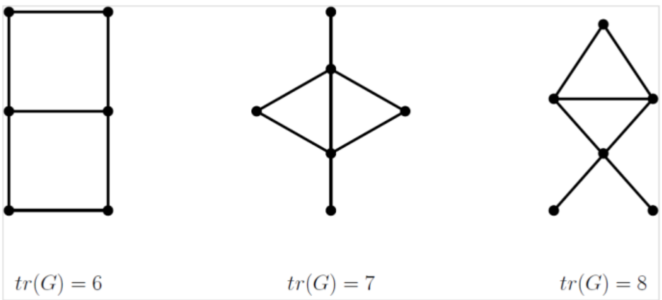

Figure 1
Some graphs with $tr(G) = tr(G^c)$.

Ethical Statement

The present study is an original research article designed and developed by the authors.

Author Contributions

Research Design (CRediT 1) A.D. (%60) – C.A. (%40)
Data Collection (CRediT 2) A.D. (%50) – C.A. (%50)
Research - Data Analysis – Validation (CRediT 3-4-6-11) A.D. (%70) – C.A. (%30)
Writing the Article (CRediT 12-13) A.D. (%40) – C.A. (%60)
Revision and Improvement of the Text (CRediT 14) A.D. (%50) – C.A. (%50)

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Conflict of Interest

The authors have no conflicts of interest to disclose for this study.

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