
MESHLESS METHOD BASED ON RADIAL BASIS FUNCTIONS FOR GENERAL ROSENAU KdV-RLW EQUATION

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ABSTRACT

In the present study, the meshfree RBFs collocation method is used to find the approximate solutions of the general Rosenau KdV-RLW equation. Firstly, Crank Nicolson method and forward finite difference approximation method is applied for solving the equation. One of the linearization techniques, which is called Rubin graves linearization, is applied for the approximate solution of the equation. Secondly, the numerical values of physical invariants of the motions for the equation are evaluated for studying known fundamental physical conservative properties. Also, to examine the accuracy of the used numerical method L_2 and L_∞ norms are evaluated. The stability analysis for the numerical technique is tested. It is seen that the method is unconditionally stable. At the end of this paper, obtained results show the validity and applicability of the numerical method.

Keywords: Radial basis functions, General rosenau KdV-RLW equation, Meshless Method

1. INTRODUCTION

In the present study, we used the meshless method which based on radial basis functions (RBFs) to get the numerical solution of the general Rosenau KdV-RLW equation. This mentioned equation has the following form:

$$u_t - \gamma^{RLW} u_{xxt} + u_{xxxxt} + \beta^{KdV} u_{xxx} + u_x + \alpha(u^p)_x = 0 \quad (1)$$

$\alpha > 0$, $p \geq 2$, γ^{RLW} , β^{KdV} are real constants.

For this equation following initial condition is taken

$$u(x, 0) = u_0(x), \quad x_l < x < x_r \quad (2)$$

and the boundary condition is used as

$$u(a, t) = u(b, t) = 0, \quad 0 < t < T \quad (3)$$

where value of $u_0(x)$ is known. Equation (1) is a combination of general Rosenau-KdV and general Rosenau-RLW equation. By taking $\beta^{KdV} = 0$ in equation (1), general Rosenau-RLW equation is found as follows:

$$u_t - \gamma^{RLW} u_{xxt} + u_{xxxxt} + u_x + \alpha(u^p)_x = 0, \quad p \geq 2. \quad (4)$$

In the equation (4), for $p = 2$ and 3 usual Rosenau-RLW equation and modified Rosenau-RLW equation is occurred, respectively. Also, in equation (4) by taking $p \geq 4$, the general Rosenau-RLW equation is obtained.

In the literature, various methods are used to solve the equation (4) numerically. Mittal et al. [1] have proposed quintic B-splines collocation method, Zuo et al. [2] studied a new conservative difference

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scheme for finding numerical solutions. On the other hand, if $\gamma^{RLW} = 0$ in equation (1), we have the following general Rosenau-KdV equation:

$$u_t + u_{xxxxt} + \beta^{KdV} u_{xxx} + u_x + \alpha(u^p)_x = 0, \quad p \geq 2 \quad (5)$$

Esfahani [5] studied solitary wave solution. Some numerical techniques have been used for the solution of the general Rosenau-KdV equation [6-7]. Up to now, the general Rosenau KdV-RLW equation has not been solved by using RBFs collocation method. That's why, in this study, we construct RBFs collocation method to obtain the numerical results for the general Rosenau KdV-RLW equation.

1.1. A Brief Review for RBFs

RBFs are much more preferred basis functions in numerical methods to evaluate the approximate solutions of differential equations, in recent years. A RBF defined as follows:

$$\begin{aligned} \phi: R^+ &\rightarrow R, \\ \phi(r_j) &= \phi(\|x - x_j\|) \end{aligned} \quad (6)$$

in which $\| \cdot \|$ represents the Euclidean norm. They are meshless. Therefore, the main advantage of radial basis functions requires neither domain nor surface discretization. Note that, RBFs are positive definite functions and usually symmetric. They can be used as basis function in the scattered data interpolation. They are also divided into Global RBFs and compactly supported RBFs. There are a lot of RBFs in the literature. Most popular RBFs are Gaussian (G), Multiquadric (MQ), Inverse Multiquadric (IMQ), Thin Plate spline (TPS). Some of them include shape parameter. If a RBF has a shape parameter, it is called as infinitely smooth. If it does not include a shape parameter, it is called as piecewise smooth. Actually, the shape parameter plays a critical role in calculations and accuracy of the solution changes as depend on value of shape parameter. Therefore, choosing the value of shape parameter is very important issue and a lot of studies has been done to obtain optimal numerical value for the shape parameter by many authors.

The radial basis functions interpolation was first used by Kansa [8,9]. It is obtained directly collocating RBFs for solving ordinary differential equations and especially partial differential equations. Also, in this method there is no need to evaluate any integral because of the using of collocation technique. The existence, uniqueness and convergence of RBFs approximation was investigated by [10,11]. Also, some numerical and theoretical studies were presented in references [12-15].

2. RBFs COLLOCATION METHOD FOR SOLVING GENERAL ROSENAU KdV-RLW EQUATION

In this section of the study, RBFs collocation method will be used to find the approximate solution the initial-boundary value problem of the equation (1) with given conditions (2) and (3). Now, we shall obtain a difference equation for the equation (1) by using Crank-Nicolson technique for the function $u(x, t)$ and forward difference approximation for the function $u_t(x, t)$. Therefore by using these approaches the equation (1) is written as follows:

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} - \gamma \frac{u_{xx}^{n+1} - u_{xx}^n}{\Delta t} + \frac{u_{xxxx}^{n+1} - u_{xxxx}^n}{\Delta t} + \beta \frac{u_{xxx}^{n+1} + u_{xxx}^n}{2} + \\ \frac{u_x^{n+1} + u_x^n}{2} + \alpha \cdot p \frac{(u^{p-1} u_x)^{n+1} + (u^{p-1} u_x)^n}{2} = 0, \end{aligned} \quad (7)$$

where γ is the constant for RLW and β is the constant for KdV. The nonlinear term $(u^{p-1}u_x)^{n+1}$ in equation (7) is linearized by using Rubin-Graves linearization technique [16] as follows

$$(u^{p-1}u_x)^{n+1} = (u^{p-1})^n(u_x)^{n+1} + (p-1)(u^{p-2})^n u_x^n u^{n+1} - (p-1)(u^{p-1})^n u_x^n. \quad (8)$$

Equation (7) can be rewritten by substituting linearized difference equation (8) into (7), we get

$$\begin{aligned} u^{n+1} - \gamma u_{xx}^{n+1} + u_{xxxx}^{n+1} + \frac{\beta \Delta t}{2} u_{xxx}^{n+1} + \frac{\Delta t}{2} u_x^{n+1} + \\ \frac{\alpha p \Delta t}{2} [(u^{p-1})^n (u_x)^{n+1} + (p-1)(u^{p-2})^n u_x^n u^{n+1}] = u^n - \gamma u_{xx}^n + u_{xxxx}^n - \frac{\beta \Delta t}{2} u_{xxx}^n - \\ \frac{\Delta t}{2} u_x^n + \frac{\alpha p \Delta t}{2} (p-2)(u^{p-1})^n u_x^n. \end{aligned} \quad (9)$$

The truncation error of the scheme in the given equation (9) is obtained. Taylor's series expansion of above expression about (x_i, t_n) gives

$$\begin{aligned} T_i^{n+1} = \frac{\Delta t (\Delta x)^2}{3} \left(\frac{3\beta}{4} u_{5x} + u_{3x} \right) + \frac{(\Delta t)^2}{2} \frac{\partial}{\partial t} (\beta u_{3x} + u_x) \\ + \frac{(\Delta t)^3}{6} \frac{\partial^2}{\partial t^2} (u_t + \beta u_{3x} + 3u_x) \end{aligned}$$

Hence truncation error of the scheme is of order two in space and order one in time. Since truncation error approaches zeros at $\Delta t \rightarrow 0$ and $\Delta x \rightarrow 0$, the difference scheme is consistent.

Let x_i be $N + 1$ distinct collocation points such that $x_l = x_0 < x_1 \dots < x_N = x_r$ and $h = \frac{x_l - x_r}{N}$. To approximate value of $u(x, t)$ is approached by $u^n = u(x, t^n)$ and it is expressed as

$$u^n = \sum_{j=0}^N \lambda_j^n \phi(r_j) \quad (10)$$

where values of λ_j^n 's will be calculated. Function $\phi(r_j)$ is RBF in the form of Equation (6). The first derivative of approximate solution can be calculated as follows:

$$\frac{d}{dx}(u^n) = \sum_{j=0}^N \lambda_j^n \frac{d}{dx} \phi(r_j). \quad (11)$$

Numerical values of other derivatives in the equation (9) can be evaluated with a similar manner. Substituting the above equations into the equation (9) at the collocation points x_i , following systems of algebraic equations are obtained:

$$\begin{aligned}
 & \sum_{j=0}^N \lambda_j^{n+1} \phi_j(x_i) - \gamma \sum_{j=0}^N \lambda_j^{n+1} \phi_j''(x_i) + \sum_{j=0}^N \lambda_j^{n+1} \phi_j^{iv}(x_i) + \frac{\beta \Delta t}{2} \sum_{j=0}^N \lambda_j^{n+1} \phi_j'''(x_i) + \frac{\Delta t}{2} \sum_{j=0}^N \lambda_j^{n+1} \phi_j'(x_i) \\
 & + \frac{\alpha p \Delta t}{2} \left[\left(\sum_{j=0}^N \lambda_j^n \phi_j(x_i) \right)^{p-1} \sum_{j=0}^N \lambda_j^{n+1} \phi_j'(x_i) \right. \\
 & \left. + (p-1) \left(\sum_{j=0}^N \lambda_j^n \phi_j(x_i) \right)^{p-2} \sum_{j=0}^N \lambda_j^n \phi_j'(x_i) \sum_{j=0}^N \lambda_j^{n+1} \phi_j(x_i) \right] \\
 & = \sum_{j=0}^N \lambda_j^n \phi_j(x_i) - \gamma \sum_{j=0}^N \lambda_j^n \phi_j''(x_i) + \sum_{j=0}^N \lambda_j^n \phi_j^{iv}(x_i) - \frac{\beta \Delta t}{2} \sum_{j=0}^N \lambda_j^n \phi_j'''(x_i) - \frac{\Delta t}{2} \sum_{j=0}^N \lambda_j^n \phi_j'(x_i) \\
 & + \frac{\alpha p (p-2) \Delta t}{2} \left(\sum_{j=0}^N \lambda_j^n \phi_j(x_i) \right)^{p-1} \sum_{j=0}^N \lambda_j^n \phi_j'(x_i), \tag{12}
 \end{aligned}$$

for $i = 1, \dots, N - 1$ and with used boundary conditions

$$\begin{aligned}
 \sum_{j=0}^N \lambda_j^n \phi_j(x_i) &= 0, \quad \text{for } i = 0, \\
 \sum_{j=0}^N \lambda_j^n \phi_j(x_i) &= 0, \quad \text{for } i = N.
 \end{aligned}$$

Therefore a system of linear equations is obtained, where λ_j^{n+1} are unknown parameters and evaluated by using a linear system solver.

3. NUMERICAL RESULTS

For numerical methods, testing the accuracy is an important matter. In where, accuracy will be tested by computing error norms and calculating numerical values of some physical properties for the model equation. The equation (1) has mass and energy conservation. These are evaluated as follows, respectively:

$$Q(t) = \int_{x_l}^{x_r} u(x, t) dx = \int_{x_l}^{x_r} u_0(x, t) dx = Q(0), \tag{13}$$

$$\begin{aligned}
 E(t) &= (\| u \|^2 + \gamma \| u_x \|^2 + \| u_{xx} \|^2) \\
 &= (\| u_0 \|^2 + \gamma \| u_{0x} \|^2 + \| u_{0xx} \|^2) = E(0)
 \end{aligned} \tag{14}$$

Secondly, error norms which are a way to determine the error between analytical and numerical solutions, are calculated in this study. The formulas of L_2 and L_∞ norms are as follows:

$$L_2 = \sqrt{h \sum_{j=1}^N |u_j^{exact} - u_j^{numerical}|^2}, \quad L_\infty = \max_{1 \leq j \leq N} |u_j^{exact} - u_j^{numerical}|. \tag{15}$$

3.1. Test Problem I

Our first test problem is the general Rosenau-RLW equation (4) for $\alpha = 1$. We investigate the solitary wave motion for the model equation. The equation (4) has following analytical solution for movement of the solitary wave:

$$u(x, t) = \exp\left(\frac{\ln\left(\frac{(p+3)(3p+1)(p+1)}{2(p^2+3)(p^2+4p+7)}\right)}{p-1}\right) \operatorname{sech}^{\frac{4}{p-1}}\left[\frac{p-1}{\sqrt{4p^2+8p+20}}(x-ct)\right],$$

where $c = \frac{p^4+4p^3+14p^2+20p+25}{p^4+4p^3+10p^2+12p+21}$ is wave velocity.

For $t = 0$, initial condition (2) is found from the exact solution. Also boundary conditions can be evaluated from analytical solution for the chosen solution domain. For this test problem, the solution domain and time interval are chosen as $-60 \leq x \leq 100$ and $0 \leq t \leq 40$, respectively. The results of numerical experiment are indicated for different choices of p , where we take $p = 4, 8$. A comparison error norms with the earlier results [3,4] is presented in Table 1 at time $t = 40$ with $\Delta t = 0.1$ and $h = 0.5$. Table 2 shows the invariant values of approximate solution. From this table, we can say that the changes of mass is nearly unchanged as the time process, while the changes of energy is less than 0.05%. As another result, all the above computations are obtained with Gaussian radial basis function, which is defined as $\phi(r_i) = \exp(-c^2 r_i^2)$, with shape parameter. The optimal shape parameter value is found experimentally. Motion of the single solitary waves is depicted in Figure 1. It is clearly seen that original forms of waves are preserved. For the solitary wave, amplitude's height and location of the peak position at time $t = 40$ are given in Table 3 for $p = 4$ and $p = 8$. As is shown in this table, we can say that the amplitude of wave is larger while the values of p are increasing. The single solitary wave's velocities at time $t = 40$ are found $v_1 = 1.1375, v_2 = 1.05$ for $p = 4, 8$ respectively.

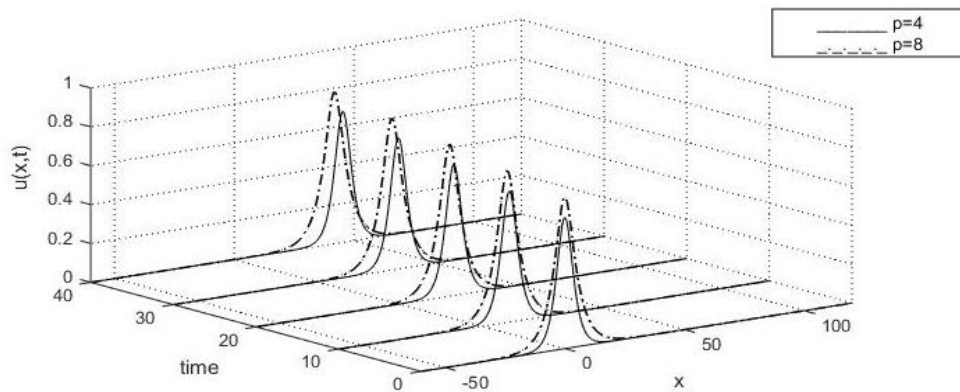


Figure 1. The single solitary wave's motion for $p = 4, 8$.

Table 1. Comparison numerical values of error norms at $t = 40$.

Method	p	L_2	L_∞
Present	4	3.7458e-03	1.3674e-03
[3]	4	4.4788e-03	1.7112e-03
[4]	4	7.4517e-02	2.7871e-02
Present	8	4.2968e-03	1.5636e-03
[3]	8	4.3184e-03	1.6189e-03
[4]	8	8.0373e-02	2.9533e-02

Table 2. Invariant values of numerical solution

<i>T</i>	<i>p</i>	<i>Q</i>	<i>E</i>
10	4	6.2658062	2.8676831
20	4	6.2658062	2.8676716
30	4	6.2658062	2.8676602
40	4	6.2658062	2.8676488
10	8	9.7420850	4.7351497
20	8	9.7420850	4.7351353
30	8	9.7420850	4.7351209
40	8	9.7420850	4.7351065

Table 3. Amplitudes solitary wave solutions at time $t = 40$.

<i>Method</i>	<i>p</i>	<i>Peak position</i>	<i>Amplitude of wave</i>
u_{exact}	4	45.5	0.6743
Present	4	45.5	0.6740
u_{exact}	8	42	0.7818
Present	8	42	0.7814

3.2. Test Problem II

In this test problem, for $\alpha = 1, p = 3$ and $p = 5$ the equation (5) is considered. Here, the solution interval of the problem is $[-60, 100]$.

For $p = 3$, the analytical solution is given by [3] as follows

$$u(x, t) = \frac{1}{4} \sqrt{-15 + 3\sqrt{41}} \operatorname{sech}^2 \frac{1}{4} \sqrt{\frac{-5 + \sqrt{41}}{4}} \left[x - \frac{1}{10} (5 + \sqrt{41})t \right]$$

and at $t=0$ solution is

$$u(x, 0) = \frac{1}{4} \sqrt{-15 + 3\sqrt{41}} \operatorname{sech}^2 \frac{1}{4} \sqrt{\frac{-5 + \sqrt{41}}{4}} x.$$

Motion of solitary wave for the general Rosenau-KdV equation is illustrated in Figure 2 for different times. The amplitudes of exact and approximate solution are 0.5096, 0.5093 at peak position $x = 46$, respectively.

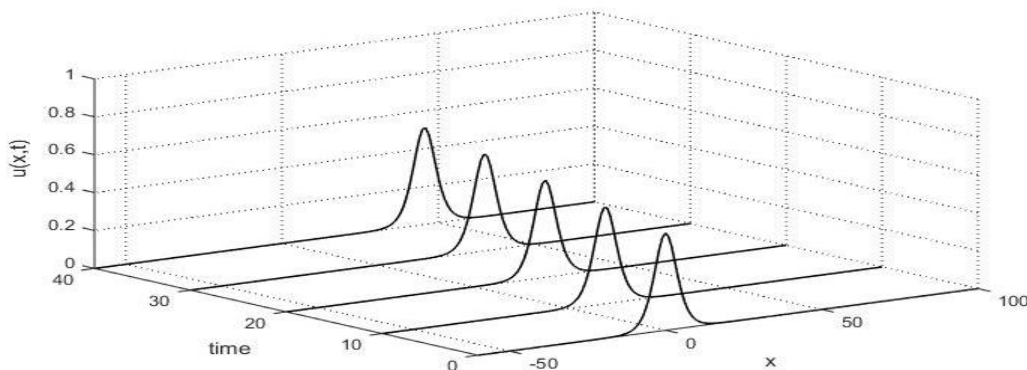


Figure 2. The single solitary wave's motion for $p = 3$.

If we choose the quantity $p = 5$, the solution will be

$$u(x, t) = \frac{4}{15} \sqrt{-5 + \sqrt{34}} \operatorname{sech} \frac{1}{3} \sqrt{-5 + \sqrt{34}} \left[x - \frac{1}{10} (5 + \sqrt{34}) t \right]$$

and for $t = 0$ initial condition will be evaluated as follows:

$$u(x, 0) = \frac{4}{15} \sqrt{-5 + \sqrt{34}} \operatorname{sech} \frac{1}{3} \sqrt{-5 + \sqrt{34}} x.$$

In Figure 3, we can observe the simulation of the single solitary wave at times $t = 0, 10, 20, 30, 40$. When we calculated the height of amplitude for exact and numerical solution, the amplitude of wave has same value for both of them. This value is 0.6828 at position $x = 43$.

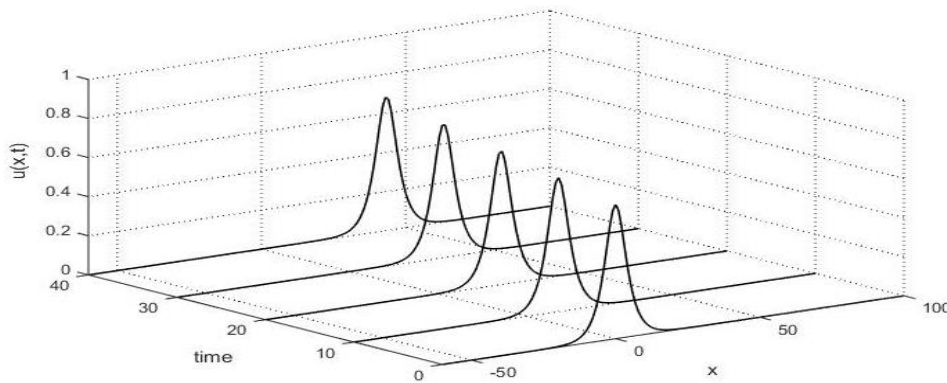


Figure 3. The single solitary wave’s motion for $p = 5$

In Table 4, we present the comparison of error norms at time $t = 40$ with the quantities $\Delta t = 0.1$ and $h = 1$. The numerical results of the physical conservations of the general Rosenau-KdV equation are indicated in Table 5.

Table 4. Comparison error norms of numerical solution at $t = 40$.

Method	p	L_2	L_∞
Present	3	1.7880e-03	6.3620e-04
[6]	3	1.3498e-02	-
[7]	3	-	7.5394e-03
Present	5	3.3217e-03	1.1897e-03
[6]	5	1.7998e-02	-
[7]	5	-	1.2020e-02

Table 5. Invariant values of numerical solution.

T	p	Q	E
10	3	4.8989794	1.682545
20	3	4.8989794	1.682543
30	3	4.8989794	1.682543
40	3	4.8989794	1.682539
10	5	7.0936431	3.206036
20	5	7.0936431	3.205995
30	5	7.0936431	3.205955
40	5	7.0936431	3.205919

3.3. Test Problem III

In this part, we applied the present method to find approximate solution of the general Rosenau KdV-RLW equation. Analytical solution of the equation is given as follows:

$$u(x, t) = A \operatorname{sech}^{\frac{4}{p-1}}[B(x - ct)],$$

where $A = \left[\frac{8(p+1)(p+3)(3p+1)\beta B^4}{\alpha(p-1)^2\{-(p-1)^2\gamma+4(p^2+2p+5)B^2\}} \right]^{\frac{1}{p-1}}$, $c = \frac{\beta(p-1)^2}{-\gamma(p-1)^2+4B^2(p^2+2p+5)}$ and

$$B = \frac{p-1}{p+1} \left[\frac{\sqrt{(p^2+2p+5)^2+16(p+1)^2\beta(\beta+\gamma)} - (p^2+2p+5)}{32\beta} \right]^{\frac{1}{2}}.$$

It is known that this solution function produces a solitary wave. The solution domain is chosen $-40 \leq x \leq 100$ in the time period $0 \leq t \leq 40$. Also, the time and space steps are taken $\Delta t = 0.1$, $h = 0.5$ respectively. Table 6 shows the error norms of numerical solution at different times with $p = 4$. Also, we plotted the numerical solution for $t = 0, 10, 20, 30, 40$ in Figure 4. One can see that the waves move toward right with increase of time. Amplitude of wave computed as 1.0354 at the location of the peak position $x = 59.5$.

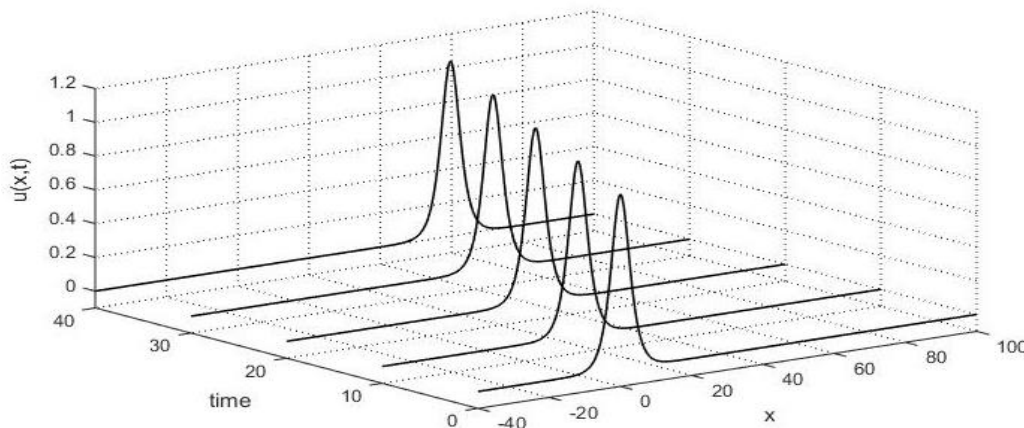


Figure 4. The single solitary wave’s motion for $p = 4$

Table 6. Error norms of numerical solution at different times for $p = 4$.

T	L_2	L_∞
10	1.0340e-02	4.1664e-03
20	2.1139e-02	8.4541e-03
30	3.2570e-02	1.3017e-02
40	4.4636e-02	1.7823e-02

4. STABILITY ANALYSIS OF NUMERICAL SCHEME

In order to examine the stability of the method, Fourier analysis [17] is used. Firstly, we must obtain linearized form of the equation (1), since we will apply the linear stability analysis. Therefore, by choosing the quantity pu^{p-1} as locally constant, we obtain following the linear equation with constant coefficient:

$$u_t - \gamma^{RLW} u_{xxt} + u_{xxxxt} + \beta^{KdV} u_{xxx} + u_x + \alpha v u_x = 0, \quad v \text{ constant.} \quad (16)$$

Fourier method's general principle is replacement of the solution given by RBF collocation method at time t_n by the value $u^n = \xi^n e^{i\theta x}$ where θ is positive constant and $i = \sqrt{-1}$. If this equation is substituted into the linear difference equation, we get

$$\xi = \frac{(1 + \gamma^{RLW}\theta^2 + \theta^4) + i\Delta t\left(\frac{\beta^{KdV}}{2}\theta^3 - \frac{\theta}{2} - \alpha v\theta\right)}{(1 + \gamma^{RLW}\theta^2 + \theta^4) - i\Delta t\left(\frac{\beta^{KdV}}{2}\theta^3 - \frac{\theta}{2} - \alpha v\theta\right)}. \quad (17)$$

It is conclude that the present method is unconditionally stable because of taking the modulus of equation (17) gives $|\xi| = 1$.

4. CONCLUSION

In the earlier similar works the governing equation was solved using finite difference methods based on an average difference scheme and conservative difference scheme. But for the solved equation any meshless method didn't used. As a difference scheme meshless technique based on collocation approximation is used for solving the equation in this current study. Numerical calculations for different cases of the equation is obtained by using RBF collocation method. Gaussian radial basis function is used to find the numerical solution for all test problems. This present method is applied for three test problems of simulation of solitary waves. After investigation of the stability of the method, it is seen that this performed numerical method is unconditional stable. The error norms and invariants are also computed. The numerical algorithm well conserves the properties related to mass and energy. It should be noted that RBFs collocation method is reliable and effective to solve similar type nonlinear problems.

ACKNOWLEDGEMENTS

This study is supported by the Scientific Research Projects of Anadolu University (No: 1504F166).

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