Çankaya Üniversitesi Fen-Edebiyat Fakültesi,
Journal of Arts and Sciences Sayı: 4 / Aralık 2005

Amply Fws Modules (Aşkın, Sonlu Zayıf Eklenmiş Modüller)

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Abstract

In this work amply finitely weak supplemented module is defined and some properties of it are investigated. Also it is obtained a gathering of variations of supplements in case R is a Dedekind Domain.

Key Words: *Supplemented Module, Dedekind Domain.*

Özet

Bu çalışmada aşkın, sonlu zayıf eklenmiş modül tanımlandı ve bazı özellikleri bulundu. Ayrıca, eklenmiş modüllerin türlü çeşitlemelerinin bir birleşimi bulundu

Anahtar Kelimeler: *Eklenmiş modül, Dedekind Tamlık Bölgesi.*

1. Introduction:

Throughout R will be an associative ring with unity and all modules are unitary left R-modules.

Let M be an R-module, a submodule S of M is called small submodule of M, if $S+N \neq M$ for every proper submodule N of M. If S is a small submodule of M, then we denote it by S<<M.

Let U be a submodule of M and let $U + V = M$ for some submodule V of M. If for every submodule K of V, , then V is called a supplement of U in M.

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An R-module M is called supplemented if every submodule of M has a supplement in M.M is called amply supplemented if whenever $M = X + Y$ where X and Y are submodules of M, then Y contains a supplement of X.

An R-module M is called finitely supplemented (or f-supplemented), if every finitely generated submodule of M has a supplement in M. (See [7], page 349). M is called amply finitely supplemented, if whenever $M = U + V$ where U,V are submodules of M and U is finitely generated then V contains a supplement of U.

Note that some properties of (amply) finitely supplemented modules are given in ([7] 41, 42).

In ([2]) weakly supplemented modules are defined: Let M be an R-module and U be a submodule of M. Then a submodule V of M is called a weak supplement of U in M if $M = U + V$ and $U \cap V << M$. and . M is called weakly supplemented if every submodule of M has weak supplement.

Note that finitely weak supplemented (fws) modules are defined and given some properties in [1]. An R-module M is called finitely weak supplemented if every finitely generated submodule of M has a weak supplement in M.

Let's define amply finitely weak supplemented (amply fws) as expected: Let $M = U + V$ where U is finitely generated submodule of M and V is a submodule of M. If V contains a weak supplement of U in M, then M is called amply fws module.

Let's begin with a very basic result:

Lemma 1.1. Let M be an R-module and U be a submodule of M. A submodule V of M is called a supplement of U in M if and only if $U + V = M$ and $U \cap V << V$.

Proof: See [3] Lemma 4.5.

Proposition 1.2. Amply (weak)supplemented modules are (weakly)supplemented.

Proof: Let M be amply (weak)supplemented and U be a submodule of M. Then $M = U + M$ indeed. Since M is amply (weak)supplemented, M contains a (weak) supplement of U. That means M is (weakly)supplemented.

However, the converse of above proposition is not always true.

Example 1.3. Let R be an incomplete discrete valuation ring with field of fractions Q. Then the R-module M = Q ⊕ Q is supplemented but not amply supplemented. (See [6], page 71.)

Lemma 1.4.

(1.) Let X be a small submodule of M, then any submodule of X is also small in M.

(2.) Let $X \subseteq Y \subseteq M$. If $X<<Y$ then $X<<M$.

(3.) If $f : M \to N$ is a homomorphism and $L<<M$, then $f(L)<<f(M)$.

Proof:

(1.) Let Y be a submodule of X. Suppose that $Y + A = M$ for some submodule A of M. Then since $Y + A \subseteq X + A$, it implies $X + A = M$ too. A contradiction with smallness of X in M.

(2.) Let $X + A = M$ for some submodule A of M. By the modular law, we obtain $X + (Y \cap A) = Y$ and since $X<<Y$, it implies $Y \cap A = Y$.So $Y \subseteq A$ and hence $X \subseteq A$. Therefore $X + A = A = M$.

(3.) See [7]19.3(4)

Corollary 1.5. Supplemented modules are weakly supplemented.

Proof: By Lemma 1.4(2).

Lemma 1.6. Let M be an R-module with submodules U and V, and let V be a weak supplement of U in M. Then for a submodule L of U, $\frac{V+L}{L}$ is a weak supplement of $\frac{U}{L}$ in $\frac{M}{L}$.

Proof: Since V is a weak supplement of U in M, then $M = U + V$ *and* $U \cap V <<M$.

Then $\frac{M}{L} = \frac{U+V}{L} = \frac{U}{L} + \frac{V+L}{L}$. Now with the help of the modular law

$\frac{U}{L} \cap \frac{V+L}{L} = \frac{U \cap (V+L)}{L} = \frac{U \cap V + L}{L}$. Since $U \cap V<<M$, then by Lemma 1.4(3)

$\frac{U \cap V + L}{L} << \frac{M}{L}$.

Lemma 1.7. Let M be an R-module and X<<M. Let $X \subseteq A \subseteq M$ and A be a supplement in M. Then X<<A too.

Proof: Let $X + Y = A$ for some submodule Y of A, then since A is a supplement in M, there is a submodule K in M s.t. A is supplement of K in M. That is, $M = K + A$ and $K \cap A <<A$. So, $M = K+A = K + X + Y$, but since X is small in M, then $M = K + Y$. Appliying the modular law, $A = (A \cap K)+Y$. Hence this implies A=Y.

An R-module M is called π-projective, if whenever $M = A+B$ for some submodules A,B of M, there exists $f \in End(M)$ such that $\mathrm{Im} f \subseteq A$ and $\mathrm{Im}(1-f) \subseteq B$.

The following result is from [7], page 359.

Proposition 1.9. Let $M = U+V$. M is π-projective R-module if and only if the epimorphism $\alpha: U \oplus V \to M$ defined by $\alpha((u,v)) = u + v$ splits.

2. Properties of Amply fws Modules.

Proposition 2.1. Let M be an amply fws module and N be a submodule of M. If N is small or finitely generated submodule of M, then $\frac{M}{N}$ is amply fws too.

Proof: Let M and N be R-modules as mentioned above. Let A be a submodule of M containing N. Suppose that $\frac{A}{N}+\frac{B}{N}=\frac{M}{N}$ for some submodule B of M containing N. Let $\frac{A}{N}$ be finitely generated. Then by [7] 19.6 or 13.9(1), A is finitely generated. Since $A+B = M$, by assumption B contains a weak supplement A_0 of A. That is $A+A_0 = M$ and $A \cap A_0 << M$. Then by Lemma 1.6. $\frac{A_0+N}{N}$ is a weak supplement of $\frac{A}{N}$. Clearly $\frac{A_0+N}{N}$ is a submodule of $\frac{B}{N}$.

Proposition 2.2. Every supplement submodule of an amply fws module is amply fws.

Proof: Let M be an amply fws module. Let V be a supplement in M. Suppose that V_0 is a finitely generated submodule of V. Then since M is amply fws module, $M = V_0 + X$ and $V_0 \cap X << M$ for some submodule X of M. By the modular law, $V = V_0 + (V \cap X)$ and so, $V_0 \cap V \cap X = V_0 \cap X << M$. Since V is a supplement in M then by Lemma 1.7 $V_0 \cap X << V$. Hence result follows.

Corollary 2.3. Every direct summand of amply fws module is fws.

Proof: Clear by Proposition 2.2.

The following is the direct adaptation of Lemma 2.3 of [4] to finitely weak supplemented case.

Lemma 2.4. Let M be a fws R-module. If M is π-projective then M is amply fws.

Proof: Direct adaptation of the proof of Lemma 2.3.

Lemma 2.5. Every projective module is π-projective.

Proof: Let M be a projective module s.t. $M = U + V$. It is enough to show, the epimorphism $\alpha : U \oplus V \to M$ defined in Proposition 1.9 splits. Let's construct a diagram:

$$\begin{array}{ccc} & & M \\ & & \downarrow 1_M \\ U \oplus V & \xrightarrow{\alpha} & M \end{array}$$

where 1_M represents the identity map. Since M is projective, there exists a map $\beta: M \to U \oplus V$ s.t. $\alpha_0 \beta = 1_M$. That is the epimorphism $\alpha: U \oplus V \to M$ splits. Hence by the Proposition 1.9., M is π-projective.

Theorem 2.6. Let R be a Dedekind Domain, then the following are equivalent:

1. R is weakly supplemented.
2. R is amply weak supplemented.
3. R is amply fws.
4. R is fws.
5. Every finitely generated R-module is weakly supplemented.
6. Every finitely generated torsion-free R-module is amply weakly supplemented.
7. Every finitely generated torsion-free R-module is fws.
8. Every finitely generated torsion-free R-module is amply fws.

Proof: (1)⇒(2) Since our rings are rings with identity, then R is finitely generated. Clearly R is torsion-free when considered as an R-module. Hence by [5] Corollary 11.107 R is projective. Result follows by Lemma 2.4.

(2) ⇒ (3) Clear.

(3) ⇒ (4) By Proposition 1.2.

(4) ⇒ (5) Let M be a finitely generated R-module, then by [2] Proposition 2.2 and Corollary 2.6. M is weakly supplemented.

(5 ⇒ (6) Let M be a finitely generated torsion-free R-module, then by [5] Corollary 11.107, M is projective by (5) it is also weakly supplemented. Hence by [4] Lemma 2.3., M is amply weakly supplemented.

(6) ⇒ (1) R is finitely generated and torsion-free and hence it is amply weakly supplemented and so weakly supplemented.

(5) ⇒ (7) Clear.

(7) ⇒ (8) Just like proof of (5) ⇒ (6).

(8) ⇒ (1) By assumption R is amply fws and so fws. But since R is a Dedekind domain, it is Noetherian and hence every ideal of R is finitely generated too. Therefore R is fws if and only if weakly supplemented.

REFERENCES

[1] Alizade, R. , Büyükaşık E. (2001). Cofinitely Weak Supplemented Modules Comm. Algebra 31(11):5377-5390.

[2] Lomp C. (1999). On semilocal modules and rings. Comm. Algebra 27(4): 1921-1935.

[3] Mohamed S.H., Müller B.J. (1990). Continuous and Discrete Modules. London Math. Soc. LNS 147, Cambridge University Press.

[4] Nebiyev C. International Journal of Computational Cognition, Vol 3, No:1, March 2005.

[5] Rotman J.J. (2002). Advanced Modern Algebra. Pearson Education.

[6] Smith P.F. (2000) Finitely generated supplemented modules are amply supplemented. The Arabian Journal for Science and Engineering, 25(2c), 69-79.

[7] Wisbauer, R.: Foundations of Module and Ring Theory (Gordon and Breach, 1992.)