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Numerical Solutions of the Gardner Equation via Trigonometric Quintic B-spline Collocation Method

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ABSTRACT

The main purpose of this paper is to get the numerical solutions of the Gardner equation which are widely used in various disciplines. For this purpose, the time integration of the system is achieved by the classical Crank-Nicolson method owing to its large stability region. Space discretization is done by using the trigonometric quintic B-spline functions. Thus the Gardner equation turns into a penta diagonal matrix equation and the Thomas algorithm is applied owing to lower cost of computation when compared Gauss or Gauss-Jordan elimination methods.

Keywords: Gardner Equation, trigonometric quintic B-spline, collocation, wave generation, interaction of two solitary waves.

1. INTRODUCTION

The Gardner equation is a model for the description of weakly nonlinear dispersive waves in situations

$$u_t + \alpha uu_x + \beta u^2 u_x + \gamma u_{xxx} \quad (1.1)$$

where α , β and γ are constant parameters and $u^2 u_x$ is a dissipative term. Nonlinear ion-acoustic waves in plasmas have been studied for a long time. The Gardner equation governing these waves in plasmas with the negative ion concentration close to critical is used [1]. The equation can also describe internal waves with large amplitudes and weakly nonlinear dispersive waves [2]. An

analytical study deals with unsteady wave patterns occurring in the dispersive resolution of the upstream and downstream hydraulic jumps in the transcritical flows governed by the Gardner equation[3]. In [4], construction of conservative finite difference schemes is applied to the Gardner equation. The Restrictive Taylor Approximation is described to obtain numerical solution of Gardner equation in [5]. Extended tanh method was used to construct solitary and soliton solutions of Gardner equations by Bekir [6]. Projective Riccati equations used to generate some hyperbolic type solitary wave [7]. In [8], a new exact traveling wave solutions get by $(G'/G, 1/G)$ expansion approach. The mapping method is employed to carry out the integration of the equation [9]. Lie group and tan-cot methods are also effective

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methods to obtain the solutions of the Gardner equation [10]. A recent study also has dealt with the Gardner equation by solving it numerically using finite element method based on polynomial quintic B-splines[11].

Recently trigonometric B-splines have been adapted to construct numerical techniques for getting solutions of differential equations such as diffusion problems, Fisher equation and Burger equations in [12,13,14]. Quintic trigonometric B-spline functions are newly defined basis function that are different from polynomial [15,16], exponential [17,18] ones but in the family of trigonometric B-splines. Also trigonometric quintic B-spline collocation method was applied to solve coupled Burgers' equation system in [19].

In this paper, the numerical solutions of Gardner equation by the trigonometric quintic B-spline finite element method are searched. Fully-integration of Gardner equation is obtained by using Crank-Nicolson method and trigonometric quintic B-spline collocation method for the time and space discretization respectively. The efficiency of the proposed method together with the trigonometric quintic B-splines is observed on solutions of Gardner equation.

The initial condition

$$u(x, 0) = f(x) \tag{1.2}$$

and the zero Neumann boundary conditions

$$\begin{aligned} u_{xx}(a, t) = 0, u_{xx}(b, t) = 0, \\ u_{xxx}(a, t) = 0, u_{xxx}(b, t) = 0 \end{aligned} \tag{1.3}$$

at both end of the artificial $[a, b]$.

2. TRIGONOMETRIC QUINTIC B-SPLINE COLLOCATION METHOD

Consider a uniform partition of the problem domain $[x_0=a, x_N=b]$, with the grids $x_m, m=0, 1, \dots, N$ and $h=(b-a)/N$. The definition of the trigonometric quintic B-splines requires the support of ghost grids located out of the problem domain. Trigonometric quintic B-splines $T_m(x), m=-2, \dots, N+2$ are defined at the nodes x_m by [20]

$$T_m(x) = \frac{1}{h} \begin{cases} \rho^5(x_{m-2}) & , x_{m-2} \leq x < x_{m-1} \\ -\rho^4(x_{m-2})\rho(x_m) - \rho^3(x_{m-2})\rho(x_{m+1})\rho(x_{m-1}) & , x_{m-1} \leq x < x_m \\ -\rho^2(x_{m-2})\rho(x_{m+2})\rho^2(x_{m-1}) - \rho(x_{m-2})\rho(x_{m+3})\rho^3(x_{m-1}) \\ -\rho(x_{m+4})\rho^4(x_{m-1}) & , x_m \leq x < x_{m+1} \\ \rho^3(x_{m-2})\rho^2(x_{m+1}) + \rho^2(x_{m-2})\rho(x_{m+2})\rho(x_{m-1})\rho(x_{m+1}) \\ +\rho^2(x_{m-2})\rho^2(x_{m+2})\rho(x_m) + \rho(x_{m-2})\rho(x_{m+3})\rho^2(x_{m-1})\rho(x_{m+1}) + \\ \rho(x_{m-2})\rho(x_{m+3})\rho(x_{m-1})\rho(x_{m+2})\rho(x_m) + \rho(x_{m-2})\rho^2(x_{m+3})\rho^2(x_m) \\ +\rho(x_{m+4})\rho^3(x_{m-1})\rho(x_{m+1}) + \rho(x_{m+4})\rho^2(x_{m-1})\rho(x_{m+2})\rho(x_m) \\ +\rho(x_{m+4})\rho(x_{m-1})\rho(x_{m+3})\rho^2(x_m) + \rho^2(x_{m+4})\rho^3(x_m) & , x_{m+1} \leq x < x_{m+2} \\ -\rho^2(x_{m-2})\rho^3(x_{m+2}) - \rho(x_{m-2})\rho(x_{m+3})\rho(x_{m-1})\rho^2(x_{m+2}) \\ -\rho(x_{m-2})\rho^2(x_{m+3})\rho(x_m)\rho(x_{m+2}) - \rho(x_{m-2})\rho^3(x_{m+3})\rho(x_{m+1}) \\ -\rho(x_{m+4})\rho^2(x_{m-1})\rho^2(x_{m+2}) - \rho(x_{m+4})\rho(x_{m-1})\rho(x_{m+3})\rho(x_m)\rho(x_{m+2}) \\ -\rho(x_{m+4})\rho(x_{m-1})\rho^2(x_{m+3})\rho(x_{m+1}) - \rho^2(x_{m+4})\rho^2(x_m)\rho(x_{m+2}) - \\ \rho^2(x_{m+4})\rho(x_m)\rho(x_{m+3})\rho(x_{m+1}) - \rho^3(x_{m+4})\rho^2(x_{m+1}) & , x_{m+2} \leq x < x_{m+3} \\ \rho(x_{m-2})\rho^4(x_{m+3}) + \rho(x_{m+4})\rho(x_{m-1})\rho^3(x_{m+3}) + \rho^2(x_{m+4})\rho(x_m)\rho^2(x_{m+3}) \\ +\rho^3(x_{m+4})\rho(x_{m+1})\rho(x_{m+3}) + \rho^4(x_{m+4})\rho(x_{m+2}) & , x_{m+3} \leq x < x_{m+4} \\ -\rho^5(x_{m+4}) & , x_{m+3} \leq x < x_{m+4} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\rho(x_{m+2}) = \sin((x - x_m)/2),$$

$$\theta = \sin(h/2)\sin(h)\sin(3h/2)\sin(2h)\sin(5h/2),$$

$$m = 0(1)N.$$

Let $U(x, t)$ be approximate solution to $u(x, t)$ defined as

$$U(x, t) = \sum_{m=-2}^{N+2} \delta_m(t) T_m(x)$$

Where δ_m are time dependent parameters that are determined from the collocation points $x_m, m=0, 1, \dots, N$ and the manipulations on initial and boundary data. Trigonometric quintic B-splines and its first four derivatives are continuous on element $[x_{m-2}, x_{m+4}]$. The functional and derivative values of $U(x, t)$ at a grid x_m is described in terms of time dependent parameters δ as

$$U(x_m) = a_1\delta_{m-1} + a_2\delta_m + a_3\delta_{m+1} + a_2\delta_{m+2} + a_1\delta_{m+3},$$

$$U'(x_m) = b_1\delta_{m-1} + b_2\delta_m - b_2\delta_{m+2} - b_1\delta_{m+3},$$

$$U''(x_m) = c_1\delta_{m-1} + c_2\delta_m + c_3\delta_{m+1} + c_2\delta_{m+2} + c_1\delta_{m+3},$$

$$U'''(x_m) = d_1\delta_{m-1} + d_2\delta_m - d_2\delta_{m+2} - d_1\delta_{m+3},$$

$$U''''(x_m) = e_1\delta_{m-1} + e_2\delta_m + e_3\delta_{m+1} + e_2\delta_{m+2} + e_1\delta_{m+3}$$

$$(2.1)$$

The coefficients of the time dependent parameters in (2.1) take the forms

$$a_1 = \sin^5(h/2)/\theta$$

$$a_2 = 2\sin^5(h/2)\cos(h/2) (16\cos^2(h/2)-3)/\theta$$

$$a_3=2(1+48\cos^4(h/2)-16\cos^2(h/2))\sin^5(h/2)/\theta$$

$$b_1=(-5/2)\sin^4(h/2)\cos(h/2)/\theta,$$

$$b_2=-5\sin^4(h/2)\cos^2(h/2)(8\cos^2(h/2)-3)/\theta,$$

$$c_1=(5/4)\sin^3(h/2)(5\cos^2(h/2)-1)/\theta,$$

$$c_2=(5/2)\sin^3(h/2)\cos(h/2)$$

$$(-15\cos^2(h/2)+3+16\cos^4(h/2))/\theta,$$

$$c_3=(-5/2)\sin^3(h/2)(16\cos^6(h/2)-5\cos^2(h/2)+1)/\theta,$$

$$d_1=(-5/8)\sin^2(h/2)\cos(h/2)(25\cos^2(h/2)-13)/\theta,$$

$$d_2=(-5/4)\sin^2(h/2)\cos^2(h/2)(8\cos^4(h/2)-35\cos^2(h/2)+15)/\theta,$$

$$e_1=(5/16)(125\cos^4(h/2)-114\cos^2(h/2)+13)\sin(h/2)/\theta,$$

$$e_2=(-5/8)\sin(h/2)\cos(h/2)(176\cos^6(h/2)-137\cos^4(h/2)-6\cos^2(h/2)+15)/\theta,$$

$$e_3=(5/8)(92\cos^6(h/2)-117\cos^4(h/2)+62\cos^2(h/2)-13)(-1+4\cos^2(h/2))\sin(h/2)/\theta$$

where $\theta=\sin(h/2)\sin(h)\sin(3h/2)\sin(2h)\sin(5h/2)$.

The Crank-Nicolson and the classical forward finite difference discretization converts the equation (1) to

$$\frac{U^{n+1}-U^n}{\Delta t} + \alpha \frac{(UU_x)^{n+1} + (UU_x)^n}{2} + \beta \frac{(U^2U_x)^{n+1} + (U^2U_x)^n}{2} + \gamma \frac{U_{xxx}^{n+1} + U_{xxx}^n}{2} = 0 \quad (2.2)$$

where $U^{n+1}=U(x, (n+1)\Delta t)$ represent the solution at the (n+1)th time level. Here $t^{n+1}=t^n+\Delta t$, Δt is the time step, superscripts denote n th time level, $t^n+n\Delta t$.

The nonlinear term $(UU_x)^{n+1}$ and $(U^2U_x)^{n+1}$ in Eq. (2.2) is linearized by using the following form [21]

$$(UU_x)^{n+1}=U^{n+1}U_x^n+U^nU_x^{n+1}-U^nU_x^n,$$

$$(U^2U_x)^{n+1}=2U^{n+1}U^nU_x^n+(U^n)^2U_x^{n+1}-2(U^n)^2U_x^n$$

So Eq (2.2) is discretized in time as

$$\frac{U^{n+1}-U^n}{\Delta t} + \alpha \frac{U^{n+1}U_x^n + U^nU_x^{n+1}}{2} + \beta \frac{2U^{n+1}U^nU_x^n + (U^n)^2U_x^{n+1} - (U^n)^2U_x^n}{2} + \gamma \frac{U_{xxx}^{n+1} + U_{xxx}^n}{2} = 0 \quad (2.3)$$

Substitute Eqs. (2.1) into (2.3) and collocate the resulting the equation at the knots x_m , $m=0,1,\dots,N$ yields a linear algebraic system of equations:

$$\begin{aligned} & \left[\left(\frac{2}{\Delta t} + \alpha\lambda + 2\beta\kappa\lambda \right) a_1 + \left(\alpha\kappa + \beta\kappa^2 \right) b_1 + \gamma d_1 \right] \delta_{m-1}^{n+1} \\ & + \left[\left(\frac{2}{\Delta t} + \alpha\lambda + 2\beta\kappa\lambda \right) a_2 + \left(\alpha\kappa + \beta\kappa^2 \right) b_2 + \gamma d_2 \right] \delta_m^{n+1} \\ & + \left[\left(\frac{2}{\Delta t} + \alpha\lambda + 2\beta\kappa\lambda \right) a_3 \right] \delta_{m+1}^{n+1} \\ & + \left[\left(\frac{2}{\Delta t} + \alpha\lambda + 2\beta\kappa\lambda \right) a_2 - \left(\alpha\kappa + \beta\kappa^2 \right) b_2 - \gamma d_2 \right] \delta_{m+2}^{n+1} \\ & + \left[\left(\frac{2}{\Delta t} + \alpha\lambda + 2\beta\kappa\lambda \right) a_1 - \left(\alpha\kappa + \beta\kappa^2 \right) b_1 - \gamma d_1 \right] \delta_{m+3}^{n+1} \\ & = \left[\left(\frac{2}{\Delta t} + \beta\kappa^2\lambda \right) a_1 - \gamma d_1 \right] \delta_{m-1}^n + \left[\left(\frac{2}{\Delta t} + \beta\kappa^2\lambda \right) a_2 - \gamma d_2 \right] \delta_m^n \\ & + \left[\left(\frac{2}{\Delta t} + \beta\kappa^2\lambda \right) a_3 \right] \delta_{m+1}^n + \left[\left(\frac{2}{\Delta t} + \beta\kappa^2\lambda \right) a_2 + \gamma d_2 \right] \delta_{m+2}^n \\ & + \left[\left(\frac{2}{\Delta t} + \beta\kappa^2\lambda \right) a_1 + \gamma d_1 \right] \delta_{m+3}^n \end{aligned} \quad (2.4)$$

where

$$\kappa = a_1\delta_{m-1} + a_2\delta_m + a_3\delta_{m+1} + a_2\delta_{m+2} + a_1\delta_{m+3}$$

$$\lambda = b_1\delta_{m-1} + b_2\delta_m - b_2\delta_{m+2} - b_1\delta_{m+3}$$

The equation (2.4) can be represented the following matrix system;

$$\mathbf{Ax}^{n+1} = \mathbf{Bx} \quad (2.5)$$

where

$$\mathbf{A} = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & & & & & \\ & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \psi_1 & \psi_2 & \psi_3 & \psi_4 & \psi_5 & & \end{bmatrix},$$

$$B = \begin{bmatrix} \psi_6 & \psi_7 & \psi_8 & \psi_9 & \psi_{10} & & & & & & \\ & \psi_6 & \psi_7 & \psi_8 & \psi_9 & \psi_{10} & & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & \psi_6 & \psi_7 & \psi_8 & \psi_9 & \psi_{10} & & & \end{bmatrix}$$

and

$$\psi_1 = (2/\Delta t + \alpha\lambda + 2\beta\kappa\lambda)a_1 + (\alpha\kappa + \beta\kappa)b_1 + \gamma d_1,$$

$$\psi_2 = (2/\Delta t + \alpha\lambda + 2\beta\kappa\lambda)a_2 + (\alpha\kappa + \beta\kappa^2)b_2 + \gamma d_2$$

$$\psi_3 = (2/\Delta t + \alpha\lambda + 2\beta\kappa\lambda)a_3,$$

$$\psi_4 = (2/\Delta t + \alpha\lambda + 2\beta\kappa\lambda)a_2 - (\alpha\kappa + \beta\kappa^2)b_2 - \gamma d_2$$

$$\psi_5 = (2/\Delta t + \alpha\lambda + 2\beta\kappa\lambda)a_1 - (\alpha\kappa + \beta\kappa^2)b_1 - \gamma d_1,$$

$$\psi_6 = (2/\Delta t + \beta\kappa^2\lambda)a_1 - \gamma d_1$$

$$\psi_7 = (2/\Delta t + \beta\kappa^2\lambda)a_2 - \gamma d_2,$$

$$\psi_8 = (2/\Delta t + \beta\kappa^2\lambda)a_3$$

$$\psi_9 = (2/\Delta t + \beta\kappa^2\lambda)a_2 + \gamma d_2,$$

$$\psi_{10} = (2/\Delta t + \beta\kappa^2\lambda)a_2 + \gamma d_1$$

The boundary conditions $U_{xx}(a,t)=0$, $U_{xx}(b,t)=0$, $U_{xxx}(a,t)=0$ and $U_{xxx}(b,t)=0$

are used to eliminate parameters $\delta_{-1}^{n+1}, \sigma_{-1}^{n+1}, \delta_{-0}^{n+1}, \sigma_{-0}^{n+1}, \delta_{N+2}^{n+1}, \sigma_{N+2}^{n+1}, \delta_{N+3}^{n+1}, \sigma_{N+3}^{n+1}$ from the system (2.5) so that we have a solvable $(2N+2) \times (2N+2)$ 5-banded matrix system. This system is solved with Matlab program using Thomas algorithm.

Time evolution of parameters δ_m^{n+1} and σ_m^{n+1} is computed once the initial parameters δ_m^0 and σ_m^0 are obtained via initial and boundary conditions as below:

$$U_{xx}(a,0) = c_1\delta_{-2}^0 + c_2\delta_{-1}^0 + c_3\delta_0^0 + c_2\delta_1^0 + c_1\delta_2^0 = 0,$$

$$U_{xxx}(a,0) = d_1\delta_{-2}^0 + d_2\delta_{-1}^0 - d_2\delta_1^0 - d_1\delta_2^0 = 0,$$

$$U(x,0) = a_1\delta_{m-1}^0 + a_2\delta_m^0 + a_3\delta_{m+1}^0 + a_2\delta_{m+2}^0 + a_1\delta_{m+3}^0 = U(x_m,0), m = 0(1)N-1,$$

$$U_{xx}(b,0) = c_1\delta_{N-1}^0 + c_2\delta_N^0 + c_3\delta_{N+1}^0 + c_2\delta_{N+2}^0 + c_1\delta_{N+3}^0 = 0,$$

$$U_{xxx}(b,0) = d_1\delta_{N-1}^0 + d_2\delta_N^0 - d_2\delta_{N+2}^0 - d_1\delta_{N+3}^0 = 0.$$

3. NUMERICAL EXAMPLES

In this section, we solve some analytical and non-analytical initial boundary value problems to validate the proposed method and present the

results. The accuracy of suggested method problem is shown by calculating the error norm

$$L_\infty = |u - U|_\infty = \max |u_m^n - U_m^n|$$

where u_m and U_m represent exact and numerical solutions at the n.th time level, respectively.

3.1. Kink type Wave Propagation

Use Kink type wave solution of the Gardner equation is [17]

$$u(x,t) = \frac{1}{10} - \frac{1}{10} \tanh\left(\frac{\sqrt{30}}{60}\left(x - \frac{1}{30}t\right)\right) \quad (3.1)$$

kink type wave travels to the right with the velocity $1/30$, Fig 1. The initial condition required to start the iteration of the time integration is determined by assuming $t=0$ in the analytical solution (3.1). We choose homogeneous Neumann conditions. We compute the numerical solutions using the selected values $\alpha=1$, $\beta=-5$ and $\mu=1$ with different values of time step size Δt . In our first computation, we take $\Delta t=0.1$ and $\Delta t=0.01$ while the number of partition N changes. The corresponding results are presented in Table 1. In our computation, we compute the maximum absolute errors at time level $t=12$ in the finite interval $[-80,80]$. The error distribution is plotted in Fig 2. The error is concentrated about the points where the wave height changes rapidly.

Table 1. Comparison of the max. Error norms at $t=12$

N	$\Delta t=0.1$	$\Delta t=0.01$
200	1.2361×10^{-3}	1.3288×10^{-3}
400	7.2063×10^{-5}	8.3593×10^{-5}
600	1.3221×10^{-5}	1.6551×10^{-5}
800	3.9570×10^{-6}	5.1655×10^{-6}

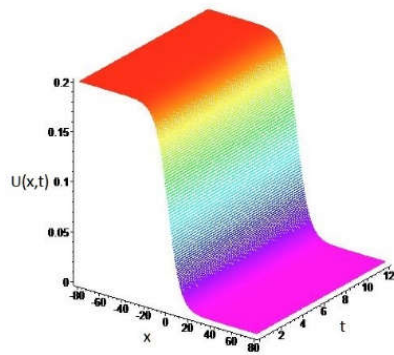


Fig. 1. Motion of the kink type wave at $t=12$, $N=200$

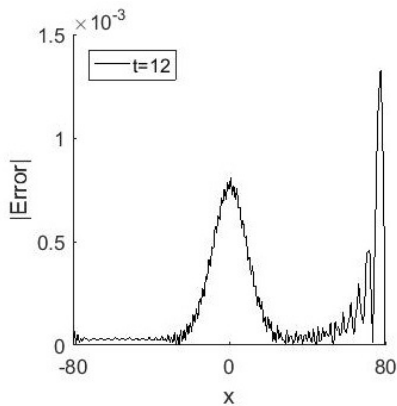


Fig. 2. The maximum error distributions at $t=12$, $N=200$

3.2. Generation of new wave

Generation of wave is studied for the Gardner's equation in the study [4]. In this section, same simulation is carried out for the perturbed Gardner's equation. Thus to the perturbed Gardner equation of the form

$$u_t + \alpha u u_x + \beta u^2 u_x + \mu u_{xxx} = \zeta \tag{3.2}$$

for $\zeta > 0$ can be useful to study the wave generation from an initial positive pulse. The decomposition of the balance among the nonlinear terms and the third order derivative is expected not to keep the shape or velocity as propagating. Thus, the initial condition is generated from the initial condition of the first problem sensitively as

$$u(x,t) = \frac{2}{3} \frac{5}{4 + \sqrt{14} \cosh\left(\frac{x}{3} - \frac{5}{3}\right)} \tag{3.3}$$

by perturbation the initial condition. We choose the parameters $\alpha=10$, $\beta=-3$ and $\mu=1$ in the Gardner equation (3.2). We run the proposed algorithms with the discretization parameters $N=400$ and $\Delta t=0.01$ in the artificial problem interval $[-40,60]$ up to the time $t=15$. Simulation of the wave generation is shown in Figs 3-6. Initial wave is split into three new solitary waves and further one has started to become solitary wave when time reach at $t=15$, seen in Fig. 6.

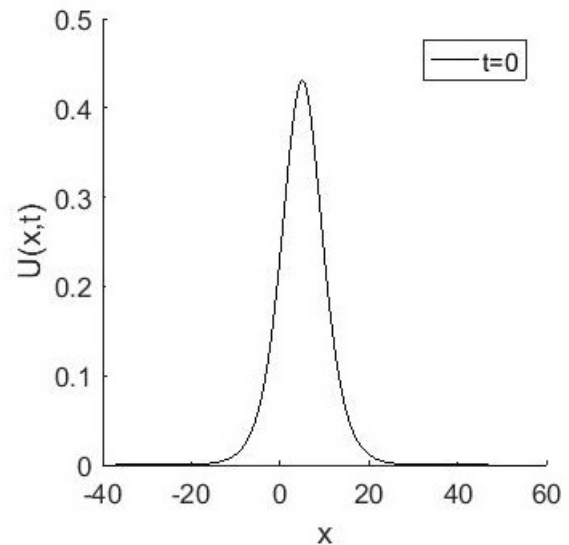


Fig. 3. Wave generation for initial data

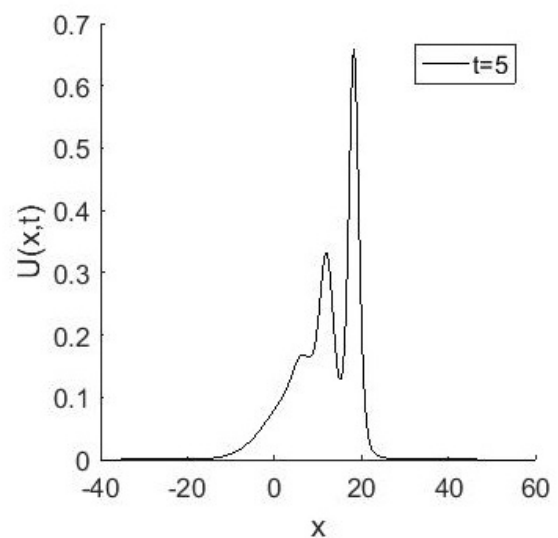


Fig. 4. Wave generation for $t=5$

solitaries propagate in the opposite directions along the horizontal axis as time goes. Continuation of simulation is depicted Fig. 8-11. We assume that $\alpha=6$, $\beta=6$ and $\mu=1$ in the Gardner equation (1.1). The designed routines are run up to the terminating time $t=5$ with the discretization parameters $N=600$ and $\Delta t=0.01$ in the finite problem interval $[-10,20]$.

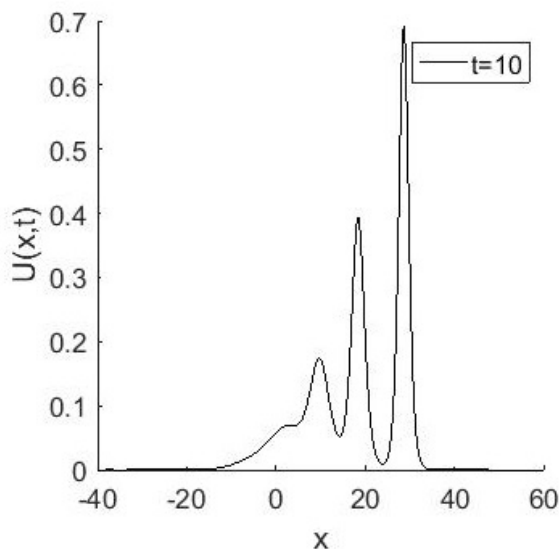


Fig. 5. Wave generation for $t=10$

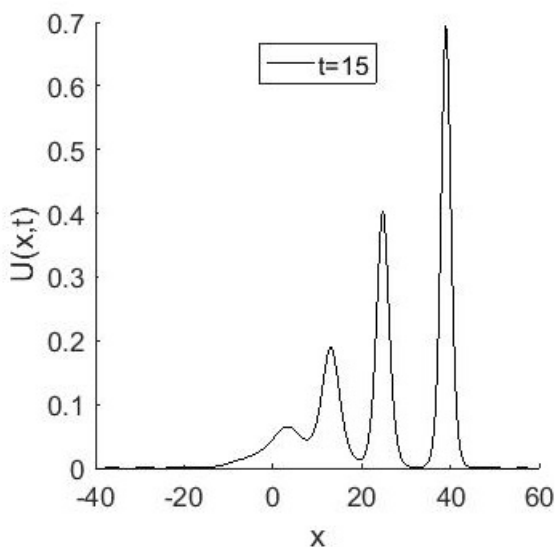


Fig. 6. Wave generation for $t=15$

3.3. Interaction of two solitary waves

The interaction of two positive bell shape solitaries are also studied in the paper [22] using the cosh hyperbolic type initial condition. The following exponential initial condition

$$u(x,0) = -\frac{1}{2} + 2 \frac{(e^{x-5} + 2e^{2x+5}) \left(1 - \frac{1}{9}e^{3x}\right) + \frac{1}{3}e^{3x}(e^{x-5} + 2e^{2x+5})}{(e^{x-5} + e^{2x+5})^2 + \left(1 - \frac{1}{9}e^{3x}\right)^2}$$

is also derived from the analytical solution given in [22]. This initial condition gives two well separated positive bell shaped solitaries of heights 1.49963 and 0.49999 positioned at $x=-2.5$ and $x=7.2$ respectively, at the beginning, Fig. 7. Both

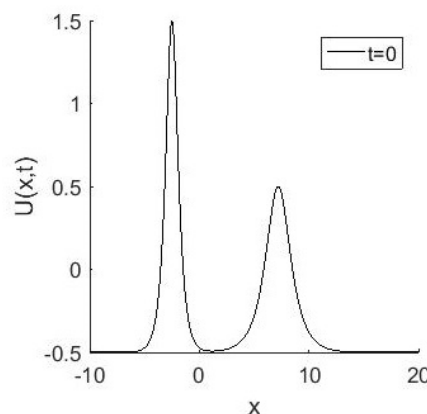


Fig. 7. Interaction of two positive bell shape solitaries for $t=0$

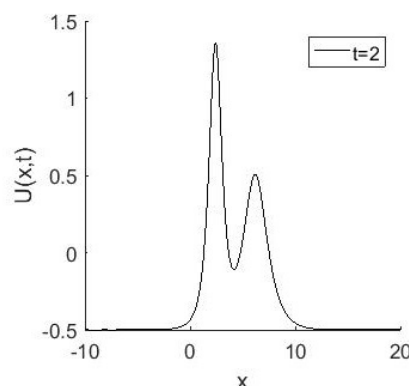


Fig. 8. Interaction of two positive bell shape solitaries for $t=2$

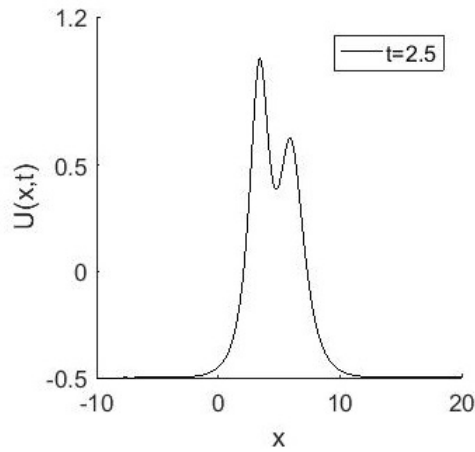


Fig. 9. Interaction of two positive bell shape solitaries for $t=2.5$

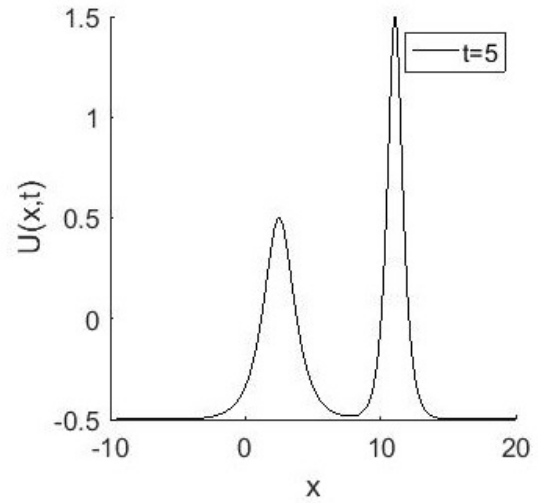


Fig. 11. Interaction of two positive bell shape solitaries for $t=5$

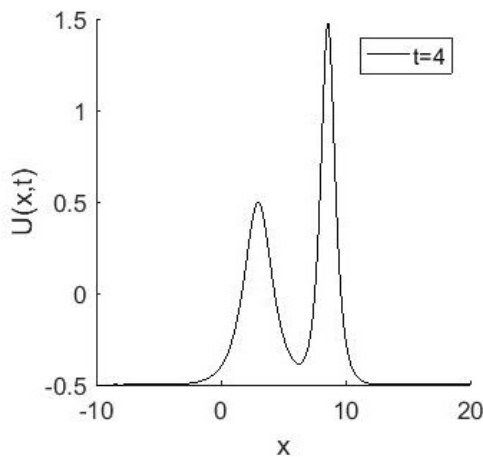


Fig. 10. Interaction of two positive bell shape solitaries for $t=4$

CONCLUSION

The collocation method based on trigonometric quintic B-spline functions is derived for the numerical solutions of some analytical and non-analytical problems for the Gardner equation. The errors between the numerical and the analytical solutions in case the existence of the analytical solutions for the first problem are measured. The perturbation of a single positive bell shaped solitary wave is derived to study wave generation for the Gardner equation successfully in the second problem. As a last test problem, interaction of two solitary waves are studied. The present method simulated the interaction successfully. As a conclusion, trigonometric quintic B-spline collocation method gives numerical solutions of the Gardner equation with high accuracy.

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