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On Curvature Identities For Para-Hermitian Manifolds

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Abstract

In this paper, firstly it is given the definitions and properties of paracomplex structures. Then using this differential geometric structures we obtain a partial paracomplex generalization of curvature identities for Hermitian manifolds and quasi -Kaehler manifolds known to be complex manifolds and studied by Gray in [2].

Key words: paracomplex structure, paracomplex, para-Hermitian and para-quasi Kaehler manifold, curvature.

Özet

Bu makalede, öncelikle para-kompleks yapıların tanımları ve özellikleri verildi. Daha sonra, bu diferensiyel geometrik yapılar kullanılarak, [2] de Gray tarafından çalışılan ve komplex manifoldlar olarak bilinen Hermit ve yarı-Kahler manifoldları için eğrilik özdeşliklerinin kısmi bir para-kompleks genellemesi elde edildi.

Anahtar Kelimeler: para-kompleks yapı; para-kompleks, para-Hermit ve para-yarı Kahler manifold; eğrilik.

1. Introduction and Notations:

In order to obtain a better understanding of the ideas and results in the survey, we shall now recall some general definitions concerning (almost) paracomplex and (almost) para-Hermitian. From now on, all the manifolds and geometric objects are C^∞ and the sum is taken over repeated indices. Also, we denote by **A** the set of paracomplex

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numbers, by $\Im(M)$ the set of paracomplex functions on M, by $\chi(M)$the set of paracomplex vector fields on M and by Λ_1 the set of paracomplex 1-forms on M.

Definition 1.1: An almost product structure J on a manifold M is a (1,1) tensor field on M such that $J^2=I$. The pair (M,J) is called an almost product manifold.

Definition 1.2: An almost paracomplex manifold is an almost product manifold (M,J) such that the two eigenbundles T^+M and T^-M associated to the eigenvalues +1 and -1 of J, respectively, have the same rank. (Note that the dimension of an almost paracomplex manifold is necessarily even) Equivalently, a splitting of the tangent bundle TM of a manifold M, into the Whitney sum of two subbundles on $T^{\pm}M$ of the same fiber dimension is called an almost paracomplex structure on M.

Definition 1.3: An almost paracomplex structure on a 2m-dimensional manifold M may alternatively be defined as a G- structure on M with structural group $GL(n,R)$x $GL(n,R)$. Let J_0 be matrix representation of J structure. The group G can be described as the invarience group of the matrix J_0, that is, $\alpha\in G$ if and only if $\alpha J_0\alpha^{-1}= J_0$. A paracomplex manifold is an almost paracomplex manifold (M,J) such that the G-structure defined by the tensor field J is integrable [1].

Definition 1.4 : Let be a pseudo- Riemannian metric tensor g on paracomplex manifold M. Then g is called a para-Hermitian metric g on paracomplex manifold M if

$$g(Ju,v)+g(u,Jv)=0 \text{ or } g(Ju,Jv)+g(u,v)=0 \text{ for all } u,v\in T_p(M). \tag{1.1}$$

An almost para-Hermitian manifold $(M\ ,g,J)$ is a differentiable manifold M endowed with an almost product structure J and a pseudo- Riemannian metric g, compatible in the sense that

$$g(JX,Y)+g(X,JY)=0 \text{ or } g(JX,JY)+g(X,Y)=0 \text{ for all } X,Y\in \chi(M)\ . \tag{1.2}$$

An almost para-Hermitian structure on a differentiable manifold M is G- structure on M whose structural group is the representation of the paraunitary group U(n,**A**) given at the end of subsection (2.4) in [1].

Definition 1.5: A para-Hermitian manifold is a manifold with an integrable almost para-Hermitian structure (g,J).

Given an almost para-Hermitian manifold (M,g,J), we shall call para fundamental 2-form (or para-Kaehlerian form) to the 2-covariant skew tensor field Φ defined by

$$\Phi(X,Y)=g(X,JY) \text{ or } \Phi(X,Y)=-g(JX,Y). \tag{1.3}$$

Definition 1.6: An almost para-Hermitian manifold (M,g,J) such that $d\Phi=0$ shall be called an almost para-Kaehlerian manifold.

A para-Hermitian manifold (M,g,J) is said to be a para- Kaehlerian manifold if $d\Phi=0$, i.e., Φ is closed.

Definition 1.7: Let be a paracomplex manifold M.. Given by X,Y,X´,Y´ vector fields, by f paraholomorfic function and by [,] Lie bracket on M. Then, N_J is called Nijenhuis tensor of paracomplex structure J defined by equation

$$N_J(X,Y)=[X,Y]-J[JX,Y]-J[X,JY]+[JX,JY]$$

and provided the properties

i) $N_J(X,Y)=-N_J(Y,X)$
ii) $N_J(fX,Y)=N_J(X,fY)=fN_J(X,Y)$
iii) $N_J(X+X',Y)=N_J(X,Y)+N_J(X',Y), N_J(X,Y+Y')=N_J(X,Y)+N_J(X,Y')$.

2. Curvatures for Para-Hermitian Manifolds

Theorem 2.1: We denote by ∇_X covariant derivation, by Φ almost para-Kaehler form and by N_J Nijenhuis tensor on an almost para-Hermitian manifold M. Then, it is provided the equation

$$2g((\nabla_X J)Y,Z)+3d\Phi(X,Y,Z)+3d\Phi(X,JY,JZ)+g(N_J(Y,Z),JX)=0. \quad (2.1)$$

Proof: We have $2g((\nabla_X J)Y,Z)=2g(\nabla_X(JY),Z)+2g(\nabla_X Y,JZ)$.

Then we obtain the equalities

$$\begin{aligned}2g(\nabla_X(JY),Z)&=Xg(JY,Z)+JYg(X,Z)-Zg(X,JY)\\&\quad+g([X,JY],Z)+g([Z,X],JY)+g(X,[Z,JY])\end{aligned} \quad (2.2)$$

$$\begin{aligned}2g(\nabla_X Y,JZ)&=Xg(Y,JZ)+Yg(X,JZ)-JZg(X,Y)\\&\quad+g([X,Y],JZ)+g([JZ,X],Y)+g(X,[JZ,Y]).\end{aligned} \quad (2.3)$$

In the other hand it is

$$\begin{aligned}3d\Phi(X,Y,Z)&=X\Phi(Y,Z)+Y\Phi(Z,X)+Z\Phi(X,Y)\\&\quad-\Phi([X,Y],Z)-\Phi([Y,Z],X)-\Phi([Z,X],Y)\end{aligned} \quad (2.4)$$

$$\begin{aligned}3d\Phi(X,JY,JZ)&=X\Phi(JY,JZ)+JY\Phi(JZ,X)+JZ\Phi(X,JY)\\&\quad-\Phi([X,JY],JZ)-\Phi([JZ,JY],X)-\Phi([JZ,X],JY)\end{aligned} \quad (2.5)$$

$$g(N_J(Y,Z),JX) = \Phi([Y,Z],X) - \Phi(J[JY,Z],X) - \Phi(J[Y,JZ],X) + \Phi([JY,JZ],X). \quad (2.6)$$

From (2.2), (2.3), (2.4), (2.5), and (2.6) equations, the proof is finished.

Lemma 2.1: Let be an almost para-Hermitian manifold *M*. Given by $\chi(M)$ Lie algebra and by N_J Nigenhuis tensor of almost paracomplex structure *J* on M. We call a paracomplex manifold if and only if

$$N_J(X,Y)=0 \text{ for all } X,Y \in \chi(M). \quad (2.7)$$

Proof: Let be a paracomplex manifold *M*. In this case, it is $[\nabla_{JX},J]Y = J[\nabla_X,J]Y$. Hence we obtain that *M* is a paracomplex manifold⇔

$$\begin{aligned}
N_J(X,Y) &= [X,Y] - J[JX,Y] - J[X,JY] + [JX,JY] \\
&= \nabla_X Y - \nabla_Y X - J\nabla_{JX} Y + J\nabla_Y JX \\
&\quad - J\nabla_X JY + J\nabla_{JY} X + \nabla_{JX} JY - \nabla_{JY} JX \\
&= J(J\nabla_X)Y - J(J\nabla_Y)X - (J\nabla_{JX})Y + J(\nabla_Y J)X \\
&\quad - J(\nabla_X J)Y + (J\nabla_{JY})X + (\nabla_{JX} J)Y - (\nabla_{JY} J)X \\
&= -J(\nabla_X J - J\nabla_X)Y + J(\nabla_Y J - J\nabla_Y)X \\
&\quad + (\nabla_{JX} J - J\nabla_{JX})Y - (\nabla_{JY} J - J\nabla_{JY})X \\
&= -J[\nabla_X,J]Y + J[\nabla_Y,J]X + [\nabla_{JX},J]Y - [\nabla_{JY},J]X \\
&= -J[\nabla_X,J]Y + J[\nabla_Y,J]X + J[\nabla_X,J]Y - J[\nabla_Y,J]X \\
&= 0. \qquad \square
\end{aligned}$$

Lemma 2.2: Let be a paracomplex manifold *M*. Let $\varepsilon = \pm 1$, and assume that M has $[\nabla_{JX},J] = \varepsilon J[\nabla_X,J]$ for all $X \in \chi(M)$. Then

$$[\nabla_{N_J(X,Y)},J] - [R_{XY} + R_{JXJY},J] + \varepsilon J[R_{JXY} + R_{XJY},J] = 0 \quad (2.8)$$

for all $X,Y \in \chi(M)$.

Proof: The curvature operator R_{XY} is defined by $R_{XY} = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y]$ for all X,Y $\in \chi(M)$. Then we have

$$\begin{aligned}
\left[\nabla_{N_J(X,Y)}, J\right] &= \left[\nabla_{[X,Y]-\varepsilon J[JX,Y]-\varepsilon J[X,JY]+[JX,JY]}, J\right] \\
&= \left[\nabla_{[X,Y]}, J\right] - \varepsilon J\left[\nabla_{[JX,Y]}, J\right] - \varepsilon J\left[\nabla_{[X,JY]}, J\right] + \left[\nabla_{[JX,JY]}, J\right] \\
&= \left[R_{XY} + [\nabla_X, \nabla_Y], J\right] - \varepsilon J\left[R_{JXY} + [\nabla_{JX}, \nabla_Y], J\right] \\
&\quad - \varepsilon J\left[R_{XJY} + [\nabla_X, \nabla_{JY}], J\right] + \left[R_{JXJY} + [\nabla_{JX}, \nabla_{JY}], J\right] \\
&= \left[R_{XY}, J\right] + \left[[\nabla_X, \nabla_Y], J\right] - \varepsilon J\left[R_{JXY}, J\right] - \varepsilon J\left[[\nabla_{JX}, \nabla_Y], J\right] \\
&\quad - \varepsilon J\left[R_{XJY}, J\right] - \varepsilon J\left[[\nabla_X, \nabla_{JY}], J\right] + \left[R_{JXJY}, J\right] + \left[[\nabla_{JX}, \nabla_{JY}], J\right] \\
&= \left[R_{XY} + R_{JXJY}, J\right] - \varepsilon J\left[R_{JXY} + R_{XJY}, J\right] \\
&\quad + \left[[\nabla_X, \nabla_Y] + [\nabla_{JX}, \nabla_{JY}], J\right] - \varepsilon J\left[[\nabla_{JX}, \nabla_Y] + [\nabla_X, \nabla_{JY}], J\right].
\end{aligned} \tag{2.9}$$

If we hold the equation given by (2.9), we obtain

$$\begin{aligned}
\left[\nabla_{N_J(X,Y)}, J\right] - \left[R_{XY} + R_{JXJY}, J\right] + \varepsilon J\left[R_{JXY} + R_{XJY}, J\right] &= \left[[\nabla_X, \nabla_Y] + [\nabla_{JX}, \nabla_{JY}], J\right] \\
&\quad - \varepsilon J\left[[\nabla_{JX}, \nabla_Y] + [\nabla_X, \nabla_{JY}], J\right].
\end{aligned} \tag{2.10}$$

Now, using Jacobi identity on each of terms on the right hand side of (2.10), and put $\left[\nabla_{JX}, J\right] = \varepsilon J\left[\nabla_X, J\right]$, we get

$$\begin{aligned}
&\left[[\nabla_X, \nabla_Y] + [\nabla_{JX}, \nabla_{JY}], J\right] - \varepsilon J\left[[\nabla_{JX}, \nabla_Y] + [\nabla_X, \nabla_{JY}], J\right] \\
&= \left[[\nabla_X, \nabla_Y], J\right] + \left[[\nabla_{JX}, \nabla_{JY}], J\right] - \varepsilon J\left[[\nabla_{JX}, \nabla_Y], J\right] - \varepsilon J\left[[\nabla_X, \nabla_{JY}], J\right] \\
&= -\left[[\nabla_Y, J], \nabla_X\right] - \left[[J, \nabla_X], \nabla_Y\right] - \left[[\nabla_{JY}, J], \nabla_{JX}\right] - \left[[J, \nabla_{JX}], \nabla_{JY}\right] \\
&\quad + \varepsilon J\left[[\nabla_Y, J], \nabla_{JX}\right] + \varepsilon J\left[[J, \nabla_{JX}], \nabla_Y\right] + \varepsilon J\left[[\nabla_{JY}, J], \nabla_X\right] + \varepsilon J\left[[J, \nabla_X], \nabla_{JY}\right] \\
&= -\left[[\nabla_Y, J], \nabla_X\right] + \left[[\nabla_X, J], \nabla_Y\right] - \varepsilon J\left[[\nabla_Y, J], \nabla_{JX}\right] + \varepsilon J\left[[\nabla_X, J], \nabla_{JY}\right] \\
&\quad + \varepsilon J\left[[\nabla_Y, J], \nabla_{JX}\right] - \left[[\nabla_X, J], \nabla_Y\right] + \left[[\nabla_Y, J], \nabla_X\right] - \varepsilon J\left[[\nabla_X, J], \nabla_{JY}\right].
\end{aligned}$$

Hence it follows

$$\left[[\nabla_X, \nabla_Y] + [\nabla_{JX}, \nabla_{JY}], J\right] - \varepsilon J\left[[\nabla_{JX}, \nabla_Y] + [\nabla_X, \nabla_{JY}], J\right] = 0. \tag{2.11}$$

Finally, this finishes the proof of **Lemma 2.2**.

Theorem 2.2: Let be an para-Hermitian manifold *M*. Then it is satisfied the equation

$$\left[R_{XY}, J\right] - J\left[R_{JXY}, J\right] - J\left[R_{XJY}, J\right] + \left[R_{JXJY}, J\right] = 0, \text{ for } X, Y \in \chi(M). \tag{2.12}$$

Proof: In **Lemma 2.2**, taking ε=1, we have equations

$$
\begin{aligned}
[R_{XY},J] &= \left[\nabla_{[X,Y]} - [\nabla_X,\nabla_Y],J\right] = \left[\nabla_{[X,Y]},J\right] - [[\nabla_X,\nabla_Y],J] \\
&= \left[\nabla_{[X,Y]},J\right] + [[\nabla_Y,J],\nabla_X] + [[J,\nabla_X],\nabla_Y] \\
&= \left[\nabla_{[X,Y]},J\right] + [[\nabla_Y,J],\nabla_X] - [[\nabla_X,J],\nabla_Y]
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
J[R_{JXY},J] &= J\left[\nabla_{[JX,Y]} - [\nabla_{JX},\nabla_Y],J\right] = J\left[\nabla_{[JX,Y]},J\right] - J[[\nabla_{JX},\nabla_Y],J] \\
&= J\left[\nabla_{[JX,Y]},J\right] + J[[\nabla_Y,J],\nabla_{JX}] + J[[J,\nabla_{JX}],\nabla_Y] \\
&= J\left[\nabla_{[JX,Y]},J\right] + J[[\nabla_Y,J],\nabla_{JX}] - [[\nabla_X,J],\nabla_Y]
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
J[R_{XJY},J] &= J\left[\nabla_{[X,JY]} - [\nabla_X,\nabla_{JY}],J\right] = J\left[\nabla_{[X,JY]},J\right] - J[[\nabla_X,\nabla_{JY}],J] \\
&= J\left[\nabla_{[X,JY]},J\right] + J[[\nabla_{JY},J],\nabla_X] + J[[J,\nabla_X],\nabla_{JY}] \\
&= J\left[\nabla_{[X,JY]},J\right] + [[\nabla_Y,J],\nabla_X] - J[[\nabla_X,J],\nabla_{JY}]
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
[R_{JXJY},J] &= \left[\nabla_{[JX,JY]} - [\nabla_{JX},\nabla_{JY}],J\right] = \left[\nabla_{[JX,JY]},J\right] + [[\nabla_{JX},\nabla_{JY}],J] \\
&= \left[\nabla_{[JX,JY]},J\right] + [[\nabla_{JY},J],\nabla_{JX}] + [[J,\nabla_{JX}],\nabla_{JY}] \\
&= \left[\nabla_{[JX,JY]},J\right] + J[[\nabla_Y,J],\nabla_{JX}] - J[[\nabla_X,J],\nabla_{JY}].
\end{aligned}
\tag{2.16}
$$

If we take into consideration equations (2.13), (2.14), (2.15) and (2.16), we find

$$
\begin{aligned}
[R_{XY},J] - J[R_{JXY},J] - J[R_{XJY},J] + [R_{JXJY},J] = &\left[\nabla_{[X,Y]},J\right] + [[\nabla_Y,J],\nabla_X] - [[\nabla_X,J],\nabla_Y] \\
&- J\left[\nabla_{[JX,Y]},J\right] - J[[\nabla_Y,J],\nabla_{JX}] + [[\nabla_X,J],\nabla_Y] \\
&- J\left[\nabla_{[X,JY]},J\right] - [[\nabla_Y,J],\nabla_X] + J[[\nabla_X,J],\nabla_{JY}] \\
&+ \left[\nabla_{[JX,JY]},J\right] + J[[\nabla_Y,J],\nabla_{JX}] - J[[\nabla_X,J],\nabla_{JY}].
\end{aligned}
\tag{2.17}
$$

The right hand side of (2.17) is equal, we write

$$
[R_{XY},J] - J[R_{JXY},J] - J[R_{XJY},J] + [R_{JXJY},J] = \left[\nabla_{N_J(X,Y)},J\right]. \tag{2.18}
$$

Thus, from **Lemma 1.2**, i.e., M is paracomplex manifold if and only if $(X,Y)=0$, the proof is finished. □

Let us put in

$$
R_{WXYZ} = \langle R_{WX}Y, Z\rangle \text{ for } X,Y,Z,W \in \chi(M). \tag{2.19}
$$

Then the sectional curvature of M is defined by equation

$$K_{WX} = R_{WXWX}\left\{\|W\|^2\|X\|^2 - < W, X >^2\right\}^{-1}. \qquad (2.20)$$

Now, we may give the following corollary from **Theorem 2.2**.

Corollary 1.1: Let M be a para-Hermitian manifold. Then there exist the equalities

a) $R_{WXYZ} + R_{JWJXJYJZ} + R_{JWJXYZ} + R_{JWXJYZ} + R_{JWXYJZ} + R_{WJXJYZ} + R_{WJXYJZ} + R_{WXJYJZ} = 0$,

such that $W, X, Y, Z \in \chi(M)$.

b) $K_{WX} + K_{JWJX} \pm K_{WJX} \mp K_{JWX} = R_{WXWX} + R_{JWJXJWJX}$

for $\|W\| = \|X\| = 1$ and $< W, X >= 0$, such that $W, X, Y, Z \in \chi(M)$.

Proof: a) By (2.19), we write equations

$$R_{WXJYJZ} =< R_{WX}JY, JZ >=< JR_{WX}Y, JZ >= - < R_{WX}Y, J^2Z >= - < R_{WX}Y, Z >= -R_{WXYZ}. \qquad (2.21)$$

Similarly, we see that

$$R_{JWJXJYJZ} = -R_{JWJXYZ}, R_{JWXJYZ} = -R_{JWXYJZ}, R_{WJXJYZ} = -R_{WJXYJZ}. \qquad (2.22)$$

Taking into consideration (2.21) and (2.22) equalities, the proof is completed. □

b) From (2.20), we obtain the following equalities:

$$\begin{aligned} K_{WX} &= R_{WXWX}\left\{\|W\|^2\|X\|^2 - < W, X >^2\right\}^{-1} = R_{WXWX} \\ K_{JWJX} &= R_{JWJXJWJX}\left\{\|JW\|^2\|JX\|^2 - < JW, JX >^2\right\}^{-1} = R_{JWJXJWJX} \\ K_{WJX} &= R_{WJXWJX}\left\{\|W\|^2\|JX\|^2 - < W, JX >^2\right\}^{-1} = R_{WJXWJX}\left\{-1 - < W, JX >^2\right\}^{-1} \\ K_{JWX} &= R_{JWXJWX}\left\{\|JW\|^2\|X\|^2 - < JW, X >^2\right\}^{-1} = R_{JWXJWX}\left\{-1 - < W, JX >^2\right\}^{-1} \end{aligned} \qquad (2.23)$$

Using hypothesis and (2.23), we see that

$$K_{WX} + K_{JWJX} \pm K_{WJX} \mp K_{JWX} = R_{WXWX} + R_{JWJXJWJX}.$$

Thus, the proof is completed. □

Analogously Hermitian manifold conformally equlivalent to $\mathbf{C}^n$ are considered, para- Hermitian manifolds conformally equivalent to $\mathbf{A}^n$ may be considered. Then, In particular new examples of manifold with constant paraholomorphic sectional curvature δ are written. Now, we may put as follows:

Theorem 2.3: Given by M para-Hermitian manifold and by δ constant paraholomorphic sectional curvature. Then

$$R_{WXWX} + R_{JWJXJWJX} + R_{WJXWJX} + R_{JWXJWJX} = 2\delta\{<W,X>^2 + <JW,X>^2\}, \text{ for } W,X \in \chi(M).$$

Proof: Put in the relation between curvature tensor and sectional curvature as follows:

$$R(X,U,X,U) = \frac{\delta}{4}\{<X,X><U,U> - <X,U>^2 + 3<JX,U>^2\} + \frac{5}{8}\lambda(X,U,X,U) + \frac{1}{8}\lambda(X,JU,X,JU). \quad (2.24)$$

Then, we have

$$\begin{aligned} \lambda(X,U,X,U) &= R(X,U,X,U) + R(X,U,JX,JU) = 0 \\ \lambda(X,JU,X,JU) &= R(X,JU,X,JU) + R(X,JU,JX,U) = 0 \end{aligned} \quad (2.25)$$

Using (2.24 and (2.25) we obtain

$$R(X,U,X,U) = \frac{\delta}{4}\{<X,X><U,U> - <X,U>^2 + 3<JX,U>^2\}. \quad (2.26)$$

So, with respect to (2.26) it follows the following equations:

$$\begin{aligned} R_{WXWX} &= \frac{\delta}{4}\{<W,W><X,X> - <W,X>^2 + 3<JX,W>^2\} \\ R_{JWJXJWJX} &= \frac{\delta}{4}\{<JW,JW><JX,JX> - <JW,JX>^2 + 3<X,JW>^2\} \\ R_{WJXWJX} &= \frac{\delta}{4}\{<W,W><JX,JX> - <W,JX>^2 + 3<X,W>^2\} \\ R_{JWXJWX} &= \frac{\delta}{4}\{<JW,JW><X,X> - <JW,X>^2 + 3<JX,JW>^2\} \end{aligned} \quad (2.27)$$

From (2.27), we have

$$R_{WXWX} + R_{JWJXJWJX} + R_{WJXWJX} + R_{JWXJWX} = \frac{\delta}{4}\left\{\begin{array}{l} \|W\|^2\|X\|^2 - <W,X>^2 + 3<JW,X>^2 \\ \|W\|^2\|X\|^2 - <W,X>^2 + 3<JW,X>^2 \\ -\|W\|^2\|X\|^2 - <JW,X>^2 + 3<W,X>^2 \\ -\|W\|^2\|X\|^2 - <JW,X>^2 + 3<W,X>^2 \end{array}\right\}$$

$$= \frac{\delta}{4}\left\{4<W,X>^2 + 4<JW,X>^2\right\}$$

$$= \delta\left\{\left\{<W,X>^2 + <JW,X>^2\right\}\right\}.$$

Finally, the proof finishes. □

Theorem 2.4: Let be a para-quasi Kaehler manifold M. Then it is satisfied the equation

$$[R_{XY} + R_{JXJY}, J] - J[R_{JXY} + R_{XJY}, J] = 2J[\nabla_{[\nabla_X, J]Y}, J] - 2J[\nabla_{[\nabla_Y, J]X}, J], \text{ for } X, Y \in \chi(M).$$

Proof: Using the properties that M is para-quasi Kaehler manifold $\Leftrightarrow [\nabla_{JX}, J] = -J[\nabla_X, J]$. Then we calculate that

$$\begin{aligned} N_J(X,Y) &= [X,Y] - J[JX,Y] - J[X,JY] + [JX,JY] \\ &= \nabla_X Y - \nabla_Y X - J\nabla_{JX} Y + J\nabla_Y JX \\ &\quad - J\nabla_X JY + J\nabla_{JY} X + \nabla_{JX} JY - \nabla_{JY} JX \\ &= J(J\nabla_X)Y - J(J\nabla_Y)X - (J\nabla_{JX})Y + J(\nabla_Y J)X \\ &\quad - J(\nabla_X J)Y + (J\nabla_{JY})X + (\nabla_{JX} J)Y - (\nabla_{JY} J)X \\ &= J(\nabla_X J - J\nabla_X)Y - J(\nabla_Y J - J\nabla_Y)X \\ &\quad - (\nabla_{JX} J - J\nabla_{JX})Y + (\nabla_{JY} J - J\nabla_{JY})X \\ &= J[\nabla_X, J]Y - J[\nabla_Y, J]X - [\nabla_{JX}, J]Y + [\nabla_{JY}, J]X \qquad (2.28) \\ &= J[\nabla_X, J]Y - J[\nabla_Y, J]X + J[\nabla_X, J]Y - J[\nabla_Y, J]X \\ &= 2J[\nabla_X, J]Y - 2J[\nabla_Y, J]X = 2J\{[\nabla_X, J]Y - [\nabla_Y, J]X\} \end{aligned}$$

In **Lemma 2.2**, for ε = –1, we have equality

$$[R_{XY} + R_{JXJY}, J] - J[R_{JXY} + R_{XJY}, J] = [\nabla_{N_J(X,Y)}, J]. \qquad (2.29)$$

From (2.28) and (2.29), the proof is completed. □

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