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On Basis Property for a Boundary-Value Problem with a Spectral Parameter in the Boundary Condition

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Abstract

In the present work, the properties as completeness, minimality and basis property are investigated for the eigenfunctions of Sturm-Liouville problem with a spectral parameter in the boundary condition.

Key Word: *Sturm-Liouville problem, completeness, minimality and basis, spectral parameter, eigenfunctions.*

A boundary value problem with a spectral parameter in the boundary condition is appeared commonly in mathematical models of mechanic. Consider the vibration problem of homogeneous string. Suppose that there is string on the vertical OX-axis, provided that the one end is fixed to zero point OX-axis and the other end equipped with a mass M corresponds to $x=1$ of OX - axis. So $u(0,t)=0$ in the end equipped with the mass. Tension of the string is expressed by Ku_x, where K is the elastic modulo. When the mass is under the impression of force of gravity and the other forces, on the point $x=1$ the second boundary condition is written as $Mu_{tt}=-Ku_x$ for all t.

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The mathematical model of this problem is represented by the equation

$$\frac{\partial^2}{\partial x^2}u(x,t)=\frac{\partial^2}{\partial t^2}u(x,t),\ 0<x<1,$$

the boundary conditions

$$u(0,t)=0,$$
$$Ku'_x(1,t)+Mu''_{tt}(1,t)=0$$

and the initial condition

$$u(x,0)=\varphi(x),\ \ u'_t(x,o)=\psi(x)\,.$$

Applying the Fourier method to the boundary-value problem, separating the variables by $u(x,t)=y(x)e^{i\mu t}$ we obtain the boundary-value problem

$$y''+\lambda y=0 \tag{1}$$

$$y(0)=0,\quad y'(1)=d\lambda y(1)\,, \tag{2}$$

where d is the positive number determined by the density of the string and $\lambda=\mu^2$. The application of this boundary problem was given in [6-10,19]. Similar problems occur in the heat transfer problems [9, 19].

Our main goal, using the showed works, is to consider the boundary problem

$$-y''+q(x)y=\lambda y\,,\ (0<x<1) \tag{3}$$

$$y(0)=0,\quad y'(0)=d\lambda y(1) \tag{4}$$

which is a general case of the boundary problem (1), (2) and is to investigate the problem of the completeness, minimality and basis property of the eigenfunctions of this boundary value problem, where $q(x)$ is a real valued continuous function in $[0,1]$.

In general, for the equation (1) or (3) when the boundary conditions contain a spectral parameter this problem can’t be interpreted an eigenvalue-eigenfunction problem in the Hilbert space $L_2(0,1)$. From this point of view, in [6,7,14,17,20] the expression of the operator of the boundary problem (3), (4) has been given in the space $L_2(0,1)\times\mathbb{C}$. In [18] the boundary problems for the differential equation of order *n* with a spectral parameter in the boundary condition were considered in the Hilbert spaces generated by distinct factors. In the case when the both of boundary conditions contained a spectral parameter; it was considered the space $L_2(0,1)\times\mathbb{C}^2$ ([2-6,14,17]. In that case, the other spectral properties of Sturm-Liouville operator were investigated in [13]

In [8-11] for distinct cases, it was shown that the eigenfunctions of the spectral problem (1) (2) formed a defect basis in $L_2(0,1)$. In general case, the similar result was obtained in [12]. In the present work the similar results are obtained by different method.

2. An Operator Formulation in the Adequate Hilbert Space

We introduce the special inner product in the Hilbert space $L_2\times\mathbb{C}$ and we give some definitions and lemmas.

We denote by $H=L_2\times\mathbb{C}$, the Hilbert space of all elements $\tilde{y}=\{y(x),a\}$ which is scalar product defined by

$$(\tilde{y},\tilde{y})=\|y\|^2_{L_2(0,1)}+d|a|^2$$

where $y(x)\in L_2(0,1)$, $a\in\mathbb{C}$ and $d>0$. We denote by A the operator is defined in the space H by the equality

$$A\tilde{y}=\left\{-y+q(x)y,\ -\frac{1}{d}y'(1)\right\}$$

and its domain

$$D(A)=\left\{\ \tilde{y}\in H:\tilde{y}=\left\{y(x),y(1)\right\},\ y(0)=0,\ y(x)\in W_2^2[0,1]\ \right\}$$

where $W_2^k[0,1]$ is the Sobolev space.

We can easily obtain that the boundary problem (3), (4) is equivalent to the spectral problem

$$A\tilde{y}=\lambda\tilde{y}\,. \tag{5}$$

Lemma 1. The eigenvalues of the boundary problem (3), (4) with multiplicity coincide with the eigenvalues of the operator *A*; for every one chain of the eigenfunctions $\tilde{y}_0, \tilde{y}_1, ..., \tilde{y}_n$ corresponding to the eigenvalue λ_0 coincide with the eigenfunctions $y_0, y_1, ..., y_n$ corresponding to the eigenvalue λ_0 of the operator *A*, and vice versa.
The similar statement is true for the associated functions.

Proof: Writing the expression of $\tilde{y}$ and $A\tilde{y}$ in (3) we directly obtain the proof of lemma. Specially, it can't be obtained from the general Lemma 1.4 in [18].

Lemma 2. [18] Let $\{e_k\}_0^\infty$ and $\{e_k^*\}_0^\infty$ be complete orthogonal systems in the Hilbert space *H*. If *P* is a orthogonal projection and *codimP* = *N*, then it can be omitted *N* elements from the system $\{Pe_k\}_0^\infty$ and the rest of the elements of the system $\{Pe_k\}_0^\infty$ form a minimal and complete system.

To show the eigenfunctions of the boundary problem (3), (4) form a basis in $L_2(0,1)$ we have to compare these eigenfunctions with a biorthogonal basis. For this we use the oscillation theorem. The following theorem is proved in [2].

Theorem 1 [2] There is an unboundedly increasing sequence $\{\lambda_n\}_{n=0}^\infty$ of eigenvalues of the boundary value problem (3), (4): $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$. Moreover, the eigenfunction $y_n(x)$ corresponding to λ_n has exactly *n* simple zeros in the interval $[0,1]$.

Lemma 3. The operator *A* is symmetric in the Hilbert space *H*.
Proof. Let $\tilde{y}, \tilde{z} \in D(A)$. Using two times the integration by parts we obtain

$$(A\tilde{y}, \tilde{z}) = \int_0^1 (-\overline{z}'' + q(x)\overline{z})ydx + \overline{z}(0)y'(0) + \overline{z}(1)y(1)$$
$$= (\tilde{y}, A\tilde{z}).$$

Hence the operator *A* is symmetric. The lemma is proved.

2. The Basis Property of Eigenfunctions

Theorem 2. The eigenfunctions of the operator A form a orthonormal basis in the Hilbert space $H = L_2 \times \mathbb{C}$.

Proof. It can be easily obtained that the operator A has at most countable eigenvalues λ_n which have the asymptotic form

$$\lambda_k = (k\pi)^2 + O(\frac{1}{k^2})$$

as $k \to \infty$ [2]. Then, for any number λ which is a not eigenvalue and arbitrary $\tilde{f} \in H$ it can be found a element $\tilde{y} \in D(A)$ satisfying the condition $(A - \lambda I)\tilde{y} = \tilde{f}$. Thus, the operator $A - \lambda I$ is invertible except for the isolated eigenvalues. Without loss of generality we assume that the point $\lambda = 0$ is a not eigenvalue. Then we obtain that the bounded inverse operator A^{-1} is defined in *H*. The operator A is selfadjoint since it is symmetric and invertible. Thus, the selfadjoint operator A^{-1} has countable many eigenvalues which are convergent to zero at infinity. So, the selfadjoint operator A^{-1} is compact. Applying the Hilbert-Schmidt theorem to this operator we obtain that the eigenfunction of the operator A form an orthonormal basis in the Hilbert space *H*. The theorem is proved.

Theorem 3. Let k_0 be an arbitrary fixed nonnegative integer. The system of the eigen-functions $\{y_n\}_0^\infty$ $(n \neq k_0)$ of the boundary problem (3), (4) is a complete and minimal system.

Proof: According to Theorem1 the eigenfunctions $\widetilde{y}_k(x) = \{y_k(x), a\}$ $(a \in \mathbb{C})$ of the boundary problem (3), (4) form a basis in $H = L_2 \times C$. So, the system $\{y_k(x)\}_0^\infty$ is complete and minimal in the space *H*. We denote by *P* the orthoprojection which is defined by the formula $P\widetilde{y}_k(x) = y_k(x)$ in *H*. Thus, of course, *codimP* = 1. Then, by Lemma 2, the system $\{P\widetilde{y}_k(x)\}_0^\infty = \{y_k(x)\}_0^\infty$ whose one element is omitted from forms a complete and minimal system in $H_p = P(H) = L_2(0,1)$. Hence, the eigenfunctions $\{y_k(x)\}_0^\infty$ of the boundary problem (3), (4) are complete and minimal in $L_2(0,1)$. We recall that using the definition of the biorthgonal system, u_n

is looked for as the form $u_n = M(y_n - Ny_{k_0})$ where k_0 is an arbitrary nonnegative integer; M and N are unknown constants. The theorem is proved.

Now we consider the case $d < 0$. In the space $H = L_2 \otimes \mathbb{C}$ for any $\tilde{y} \in H$ the scalar product is defined by the formula

$$(\tilde{y}, \tilde{y}) = \|y\|^2_{L_2(0,1)} - \frac{1}{d}|a|^2$$

where $d < 0$. Let $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ where *I* is the unit operator in the space *H*.

Lemma 4. The operator *A* is *J*-selfadjoint in the Hilbert space *H*.

Proof. Let $\tilde{y}, \tilde{z} \in D(A)$. Using two times the integration by parts we obtain

$$(JA\tilde{y}, \tilde{z}) = \int_0^1 (-\overline{z}'' + q(x)\overline{z})y dx + \overline{z}'(1)y(1) + \overline{z}(0)y'(0)$$
$$= (\tilde{y}, JA\tilde{z}).$$

Hence the operator *A* is *J*-symmetric. The lemma is proved.

Using the idea in Theorem 1 and considering that the operator *J* is a bounded operator it can be shown that the operator *B*=*JA* is invertible. Since the operator B^{-1} is symmetric and invertible the operator B^{-1} is selfadjoint. The selfadjoint operator B^{-1} has at most countable many eigenvalues which converge to zero at infinity. Hence, the operator B^{-1} is compact. Then applying Theorem 2.12 in [1] to the operator *B* we obtain that the eigenfunctions of the *J*-selfadjoint operator *A* form a Riesz basis in the space $H = L_2 \otimes \mathbb{C}^2$. Consequently we proved the following theorem.

Theorem 4. The eigenfunctions of the operator *A* form a Riesz basis in the space $H = L_2 \otimes \mathbb{C}^2$.

It can be obtain that Theorem 3 is satisfied in this case too.

Now we prove that the following theorem is satisfied for two cases above.

Theorem 5. Let k_0 be an arbitrary fixed nonnegative integer. The system of the eigen-functions $\{y_n\}_0^\infty$ $(n \neq k_0)$ of the boundary problem (3), (4) forms a basis in $L_2(0,1)$.

Proof : The proof of the theorem depends on the theorem regarding basis property which was given in [8-12]. The eigenvalues λ_n of the boundary problem (1), (2) for sufficiently large *n* have the form

$$\lambda_n = (\pi n)^2 + O(1) \text{ f}$$

and corresponding eigenfunctions y_n have the form

$$y_n(x) = \sqrt{2}\sin n\pi x + O(\frac{1}{n}) .$$

Compare the system $\{y_n\}_0^\infty$ $(n = 1, 2, ...;\ n \neq k_0)$ with the known system $\{\sqrt{2}\sin n\pi x\}$, $n = 1, 2, ...$ which is an orthonormal basis for $L_2(0,1)$. As similarly in [8], we obtain that the system $\{y_n\}_0^\infty$ $(n = 1, 2, ...;\ n \neq k_0)$ is quadratically close to the system $\{\sqrt{2}\sin n\pi x\}$ $(n = 1, 2, ...)$. According to Theorem 2 the system is complete and minimal in $L_2(0,1)$. Using the Bari theorem regarding quadratic convergence, we have that this system forms a basis in $L_2(0,1)$.

Remark: If both of the boundary conditions contain a spectral parameter λ, it is shown that, the system of the eigenfunctions of this boundary problem obtained omitting two elements from this system forms a Riesz basis in $L_2(0,1)$.

Conclusion

Besides the example given in this work, there are boundary conditions containing spectral parameter boundary-value problems having large thermally conductive materials that are surrounded by in thermal conduction phenomena is found [19, p.189]. Similar problems are found that mathematically describes the vibration of a mechanically charged membrane [19, p.152]. Therefore, the problem of investigations is being continued [5, 13, 15, 16]. To investigate characteristics of the eigenfunctions for the boundary-value problem that is dealt with in our investigation such as completeness, minimality and basis property, a different method is used.

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