Çankaya Üniversitesi Fen-Edebiyat Fakültesi,
Journal of Arts and Sciences Sayı: 5, Mayıs 2006

Chebyshev Polynomial Solution of Nonlinear Fredholm-Volterra Integro-Differential Equations

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Abstract

In this paper, a Chebyshev collocation method [1] is developed to find an approximate solution for nonlinear Fredholm-Volterra integro–differential equation. This method transforms the nonlinear Fredholm-Volterra integro–differential equation into the matrix equation with the help of Chebyshev collocation points. The matrix equation corresponds to a system of nonlinear algebraic equations with the unknown Chebyshev coefficients. Finally, some numerical examples are presented to illustrate the accuracy of the method.

Keywords: *Nonlinear integro-differential equation; Chebyshev series; Colllocation Method.*

Özet

Bu çalışmada, lineer olmayan Fredholm-Volterra integro–diferansiyel denklemlerin yaklaşık çözümlerini bulmak için Chebyshev sıralama yöntemi [1] geliştirilmiştir. Bu yöntem lineer olmayan Fredholm-Volterra integro–diferansiyel denklemini, sıralama noktalarını kullanarak matris denklemine dönüştürür. Bu matris denklemi ise bilinmeyeni Chebyshev katsayıları olan lineer olmayan cebirsel denklem sistemine karşılık gelir. Çalışmanın sonunda yöntemin doğruluğunu göstermek için bazı sayısal örnekler sunulmuştur.

Anahtar Kelimeler: *Lineer Olmayan integro-diferansiyel denklemler; Chebyshev serileri; Sıralama Yöntemi.*

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1. INTRODUCTION

Consider the following nonlinear Fredholm–Volterra integro-differential equation

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda_1 \int_{-1}^{1} \sum_{i=1}^{2} F_i(x,t) y^i(t) dt + \lambda_2 \int_{-1}^{x} \sum_{j=1}^{2} K_j(x,t) y^j(t) dt \quad (1)$$

under the mixed conditions

$$\sum_{j=0}^{m-1} \left[a_{ij} y^{(j)}(-1) + b_{ij} y^{(j)}(1) + c_{ij} y^{(j)}(c) \right] = \mu_i \,, \; i = 0, 1, ..., m\text{-}1, -1 \le c \le 1 \quad (2)$$

where $y(x)$ is an unknown function, the functions $g(x), P_k(x), F_i(x,t)$ and $K_j(x,t)$ are defined on interval $-1 \le x, t \le 1$ and $a_{ij}, b_{ij}, c_{ij}, \lambda_1, \lambda_2, \mu_i$ are constants.

Let us seek the solution of (1) expressed in terms of Chebyshev polynomials as

$$y(x) = \sum_{r=0}^{N}{}' a_r T_r(x), \; -1 \le x \le 1 \quad (3)$$

where a_r, $0 \le r \le N$, are unknown Chebyshev coefficients and N is chosen any positive integer such that $m \le N$. $\sum{}'$ denotes a sum whose first term is halved, $T_r(x)$ denotes the Chebyshev polynomials of the first kind of degree r.

The Chebyshev collocation points defined by

$$x_s = \cos\left(\frac{s\pi}{N}\right), \; s = 0, 1, ..., N \quad (4)$$

are used in the following sections.

2. FUNDAMENTAL RELATIONS

Let us write Eq. (1) in the form

$$D(x) = g(x) + \lambda_1 I(x) + \lambda_2 J(x) \quad (5)$$

where the differential part

$$D(x) = \sum_{k=0}^{m} P_k(x) y^{(k)}(x) \quad (6)$$

Fredholm integral part is

$$I(x) = \int_{-1}^{1} \sum_{i=1}^{2} F_i(x,t) y^i(t) dt \tag{7}$$

and Volterra integral part is

$$J(x) = \int_{-1}^{x} \sum_{j=1}^{2} K_j(x,t) y^j(t) dt \tag{8}$$

2.1. Matrix Representation for Differential Part

Let us assume that the k^{th} derivative of the function (3) with respect to x has the truncated Chebyshev series expansion by

$$y^{(k)}(x) = \sum_{r=0}^{N}{}' a_r^{(k)} T_r(x), \qquad -1 \le x \le 1$$

where $a_r^{(k)}$ ($k = 0, 1, ..., m$) are Chebyshev coefficients.

Then the solution expressed by (3) and its derivatives can be written in the matrix forms respectively

$$y(x) = \boldsymbol{T}(x)\mathbf{A} \tag{9}$$

and

$$y^{(k)}(x) = \boldsymbol{T}(x)\mathbf{A}^{(k)} \tag{10}$$

It is well known from [6] that the relation between the Chebyshev coefficient matrix $\mathbf{A}$ of $y(x)$ and the Chebyshev coefficient matrix $\mathbf{A}^{(k)}$ of $y^{(k)}(x)$ is given

$$\mathbf{A}^{(k)} = 2^k \mathbf{M}^k \mathbf{A}$$

Then the expression (10) becomes

$$y^{(k)}(x) = 2^k \boldsymbol{T}(x)\mathbf{M}^k \mathbf{A} \tag{11}$$

where

$$\boldsymbol{T}(x) = \begin{bmatrix} T_0(x) & T_1(x) & \dots & T_N(x) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \frac{a_0}{2} & a_1 & \dots & a_N \end{bmatrix}^T$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & N/2 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ for odd } N$$

and

$$\mathbf{M} = \begin{bmatrix} 0 & 1/2 & 0 & 3/2 & 0 & 5/2 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & \cdots & N \\ 0 & 0 & 0 & 3 & 0 & 5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ for even } N$$

Substituting the Chebyshev collocation points into Eq.(6) and using (11), the matrix representation of $D(x_i)$ can be given by

$$D(x_i) = \sum_{k=0}^{m} 2^k P_k(x_i) \boldsymbol{T}(x_i) \mathbf{M}^k \mathbf{A} \tag{12}$$

2. 2. Matrix Representation for Fredholm Integral Part

Let us substitute the Chebyshev collocation points into Eq. (7) to obtain the matrix relation of $I(x_i)$ and assume that, for each x_s, $F_0(x_s,t)$ and $F_1(x_s,t)$ is expanded to the Chebyshev series in the form

$$F_i(x_s,t) = \sum_{r=0}^{N}{}'' f_{ir}(x_s) T_r(t) \ , \quad i = 0\ ,1$$

where a summation symbol with double primes denotes a sum with first and last terms halved and Chebyshev coefficients $f_{ir}(x_s)$ are determined by means of the relation

$$f_{ir}(x_s) = \frac{2}{N} \sum_{j=0}^{N}{}'' F_i(x_s,t_j) T_r(t_j) \ , \ t_j = \cos\left(\frac{j\pi}{N}\right), \ j = 0, 1, ..., N$$

Then the matrix representations of $F_i(x_s,t)$ become

$$F_i(x_s,t) = \boldsymbol{F_i}(x_s)\boldsymbol{T}(t)^T \tag{13}$$

where

$$\boldsymbol{F_i}(x_s) = \begin{bmatrix} \frac{f_{i,0}(x_s)}{2} & f_{i,1}(x_s) & \cdots & f_{i,N-1}(x_s) & \frac{f_{i,N}(x_s)}{2} \end{bmatrix}$$

Besides, $y^2(t)$ function can be written in the matrix form [5],

$$y^2(t) = \overline{\boldsymbol{T}}(t)\mathbf{B} \tag{14}$$

in which

$$\overline{\boldsymbol{T}}(t) = \begin{bmatrix} T_0(t) & T_1(t) & \ldots & T_{2N}(t) \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{b_0}{2} & b_1 & \ldots & b_{2N} \end{bmatrix}^T$$

and the elements b_i of the column matrix B consist of a_i and $a_i = a_{-i}$ as follows:

$$b_i = \begin{cases} \dfrac{\left(a_{\frac{i}{2}}\right)^2}{2} + \displaystyle\sum_{r=1}^{N-\frac{i}{2}} \left(a_{\frac{i}{2}-r}\right)\left(a_{\frac{i}{2}+r}\right), & \text{for even } i \\ \displaystyle\sum_{r=1}^{N-\frac{i-1}{2}} \left(a_{\frac{i+1}{2}-r}\right)\left(a_{\frac{i-1}{2}+r}\right), & \text{for odd } i \end{cases}$$

When the relations (9), (13) and (14) are substituted in $I(x_s)$, we have

$$I(x_s) = \boldsymbol{F}_1(x_s)\mathbf{Z}_1\mathbf{A} + \boldsymbol{F}_2(x_s)\mathbf{Z}_2\mathbf{B} \tag{15}$$

where

$$\mathbf{Z}_1 = \int_{-1}^{1} \boldsymbol{T}(\mathbf{t})^{\boldsymbol{T}}\boldsymbol{T}(t)\,dt = \left[z_{ij}\right], \quad i = 0, 1, \ldots, N, \quad j = 0, 1, \ldots, N$$

$$\mathbf{Z}_2 = \int_{-1}^{1} \boldsymbol{T}(\mathbf{t})^{\boldsymbol{T}}\overline{\boldsymbol{T}}(t)\,dt = \left[z_{ij}\right], \quad i = 0, 1, \ldots, N, \quad j = 0, 1, \ldots, 2N$$

and

$$z_{ij} = \int_{-1}^{1} T_i(t)T_j(t)\,dt = \begin{cases} \dfrac{1}{1-(i+j)^2} + \dfrac{1}{1-(i-j)^2}, & \text{for even } i+j \\ 0, & \text{for odd } i+j \end{cases}$$

2. 3. Matrix Presentation for Volterra Integral Part

Firstly, the Chebyshev collocation points are substituted into (8). Similarly the previous section, it is supposed that the kernel function $K_j(x_s,t)$ can be expanded to univarete Chebyshev series with respect to t. Then the matrix form of the kernel function is

$$K_j(x_s,t) = \boldsymbol{K_j}(x_s)\boldsymbol{T}(t)^T, \quad j = 0,1 \quad s = 0,1,\dots,N \tag{16}$$

where

$$\boldsymbol{K}_j(x_s) = \begin{bmatrix} \frac{k_{j,0}(x_s)}{2} & k_{j,1}(x_s) & \cdots & k_{j,N-1}(x_s) & \frac{k_{j,N}(x_s)}{2} \end{bmatrix}$$

Substituting the relations (9), (14) and (16) in $J(x_s)$, the matrix representation of $J(x_s)$ is obtained as

$$J(x_s) = \boldsymbol{K}_1(x_s)\boldsymbol{Z}_1(x_s)\mathbf{A} + \boldsymbol{K}_2(x_s)\boldsymbol{Z}_2(x_s)\mathbf{B} \tag{17}$$

where

$$\boldsymbol{Z}_1(x_s) = \int_{-1}^{x_s} \boldsymbol{T}(\mathbf{t})^{\boldsymbol{T}}\,\boldsymbol{T}(t)dt = \left[z_{ij}(x_s)\right] \; I = 0, 1,\dots, N, \quad j=0,1,\dots, N$$

$$\boldsymbol{Z}_2(x_s) = \int_{-1}^{x_s} \boldsymbol{T}(\mathbf{t})^{\boldsymbol{T}}\,\overline{\boldsymbol{T}}(t)dt = \left[z_{ij}(x_s)\right], \; I = 0, 1,\dots, N, \quad j=0,1,\dots, 2N$$

and

$$z_{ij}(x) = \int_{-1}^{x} T_i(t)T_j(t)dt$$

$$= \frac{1}{4}\begin{cases} 2x^2 - 2 & \text{, for } i+j=1 \\ \frac{T_{i+j+1}(x)}{i+j+1} - \frac{T_{i+j-1}(x)}{i+j-1} - \frac{1}{i+j+1} + \frac{1}{i+j-1} + x^2 - 1 & \text{, for } |i-j|=1 \\ \frac{T_{i+j+1}(x)}{i+j+1} + \frac{T_{i-j-1}(x)}{i-j-1} + \frac{T_{1+i-j}(x)}{1+i-j} + \frac{T_{1-i+j}(x)}{1-i+j} + 2\left[\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right] & \text{, for even } i+j \\ \frac{T_{i+j+1}(x)}{i+j+1} + \frac{T_{i-j-1}(x)}{i-j-1} + \frac{T_{1+i-j}(x)}{1+i-j} + \frac{T_{1-i+j}(x)}{1-i+j} - 2\left[\frac{1}{1-(i+j)^2} + \frac{1}{1-(i-j)^2}\right] & \text{, for odd } i+j \end{cases}$$

2. 4. Matrix Representation for the Conditions

Using the relation (11), the matrix form of the conditions defined in (2) can be written as

$$\sum_{j=0}^{m-1} 2^j \left(a_{ij}\boldsymbol{T}(-1) + b_{ij}\boldsymbol{T}(1) + c_{ij}\boldsymbol{T}(c) \right) \mathbf{M}^j \mathbf{A} = \mu_i, \quad i=0, 1, \ldots, N$$

Let us define U_i as

$$\mathbf{U}_i = \sum_{j=0}^{m-1} 2^j \left(a_{ij}\boldsymbol{T}(-1) + b_{ij}\boldsymbol{T}(1) + c_{ij}\boldsymbol{T}(c) \right) \mathbf{M}^j = \begin{bmatrix} u_{i0} & u_{i1} & \ldots & u_{iN} \end{bmatrix}$$

Thus, the matrix forms of conditions (2) become

$$\mathbf{U}_i \mathbf{A} = \mu_i \tag{18}$$

3. METHOD OF SOLUTION

To construct the fundamental matrix equation corresponding to Eq. (1), the Chebyshev collocation points are substituted in (5) and then using the matrix relations (12), (15) and (17), it is obtained for $s = 0, 1, \ldots, N$

$$\sum_{k=0}^{m} 2^k P_k(x_s) \boldsymbol{T}(x_s) \mathbf{M}^k \mathbf{A} = g(x_s) + \lambda_1 \left(\boldsymbol{F}_1(x_s) \mathbf{Z}_1 \mathbf{A} + \boldsymbol{F}_2(x_s) \mathbf{Z}_2 \mathbf{B} \right)$$
$$+ \lambda_2 \left(\boldsymbol{K}_1(x_s) \boldsymbol{\mathcal{Z}}_1(x_s) \mathbf{A} + \boldsymbol{K}_2(x_s) \boldsymbol{\mathcal{Z}}_2(x_s) \mathbf{B} \right)$$

Thereby, the fundamental matrix is gained of the form

$$\left(\sum_{k=0}^{m} 2^k \mathbf{P}_k \mathbf{T} \mathbf{M}^k - \lambda_1 \mathbf{F}_1 \mathbf{Z}_1 - \lambda_2 \mathbf{K}_1 \boldsymbol{\mathcal{Z}}_1 \right) \mathbf{A} - \left(\lambda_1 \mathbf{F}_2 \mathbf{Z}_2 + \lambda_2 \mathbf{K}_2 \boldsymbol{\mathcal{Z}}_2 \right) \mathbf{B} = \mathbf{G} \tag{19}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}, \mathbf{K}_n = \begin{bmatrix} \boldsymbol{K}_n(x_0) & 0 & \cdots & 0 \\ 0 & \boldsymbol{K}_n(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{K}_n(x_N) \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} \boldsymbol{T}(x_0) \\ \boldsymbol{T}(x_1) \\ \vdots \\ \boldsymbol{T}(x_N) \end{bmatrix}, \ \mathbf{F}_n = \begin{bmatrix} \boldsymbol{F}_n(x_0) \\ \boldsymbol{F}_n(x_1) \\ \vdots \\ \boldsymbol{F}_n(x_N) \end{bmatrix}, \ \boldsymbol{\mathcal{Z}}_n = \begin{bmatrix} \boldsymbol{\mathcal{Z}}_n(\mathrm{x}_0) \\ \boldsymbol{\mathcal{Z}}_n(\mathrm{x}_1) \\ \vdots \\ \boldsymbol{\mathcal{Z}}_n(\mathrm{x}_\mathrm{N}) \end{bmatrix} \text{ for } n = 1, 2$$

This equation corresponds to a system of N+1 nonlinear algebraic equations with unknown Chebyshev coefficients a_0, a_1, …, a_N.

Finally, to obtain the solution of Eq.(1) under the mixed conditions (2), m equations in nonlinear algebraic system (19) are replaced with m equations in linear algebraic equations system (18). Therefore, Chebyshev coefficients are determined by solving the new nonlinear algebraic system.

The method also can be developed for the problem defined on the domain [0,1]

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda_1\int_0^1\sum_{i=1}^{2} F_i(x,t)y^i(t)dt + \lambda_2\int_0^x\sum_{j=1}^{2} K_j(x,t)y^j(t)dt \qquad (20)$$

The solution of this equation under the mixed conditions is found in terms of shifted Chebyshev polynomials $T_r^*(x)$ of the form

$$y(x) = \sum_{r=0}^{N}{}' a_r^* T_r^*(x), \qquad 0 \le x \le 1$$

where $T_r^*(x) = T_r(2x-1)$.

It is followed the previous procedure using the collocation points defined by

$$x_s = \frac{1}{2}\left[1+\cos\left(\frac{s\pi}{N}\right)\right], \quad s = 0,1,\ldots,N \qquad (21)$$

and the relation

$$\mathbf{A}^{*(k)} = 4^k \mathbf{M}^k \mathbf{A}^*, \quad k = 0, 1, \ldots, N$$

where

$$\mathbf{A}^* = \begin{bmatrix} \frac{a_0^*}{2} & a_1^* & \cdots & a_N^* \end{bmatrix}^T .$$

Then we obtain the fundamental matrix equation for (20) as

$$\sum_{k=0}^{m} 4^k \mathbf{P}_k \mathbf{T}^* \mathbf{M}^k \mathbf{A}^* - \lambda_1\left[\mathbf{F}_1^*\mathbf{Z}_1^*\mathbf{A}^* + \mathbf{F}_2^*\mathbf{Z}_2^*\mathbf{B}^*\right] - \lambda_2\left[\mathbf{K}_1^*\mathbf{Z}_1^*\mathbf{A}^* + \mathbf{K}_2^*\mathbf{Z}_2^*\mathbf{B}^*\right] = \mathbf{G}$$

Moreover, the matrix forms of the conditions become

$$\sum_{j=0}^{m-1} 4^j\left(a_{ij}\boldsymbol{T}^*(-1) + b_{ij}\boldsymbol{T}^*(1) + c_{ij}\boldsymbol{T}^*(c)\right)\mathbf{M}^j\mathbf{A}^* = \mu_i, \quad i=0, 1,\ldots, N$$

It is easily seen that $\mathbf{T} = \mathbf{T}^*$, $\mathbf{Z}_n = 2\mathbf{Z}_n^*$ and $\mathcal{Z}_n = 2\mathcal{Z}_n^*$ for $n = 0$,1, because of the properties of the Chebyshev polynomials.

5. NUMERICAL EXPERIMENTATIONS

The efficiency of the presented method is shown in following three examples. Results were computed using the program written in Mathcad 2000 Professional.

Example1. Let us consider two examples of nonlinear Fredholm- Volterra integro-differential equation. These problems has been solved by Taylor polynomials for $N = 4$ and $N = 5$ respectively in [3].

$$\textbf{a)}\ y''(x) - xy'(x) + xy(x) = g(x) + \int_{-1}^{1} xty(t)dt + \int_{-1}^{x} (x-2t)y^2(t)dt \tag{22}$$

where $g(x) = \frac{2}{15}x^6 - \frac{1}{3}x^4 + x^3 - 2x^2 - \frac{23}{15}x + \frac{5}{3}$ and $y(0) = -1,\ y'(0) = 0$.

$$\textbf{b)}\ y'(x) + xy(x) = g(x) + \int_{0}^{1} (x+t)y(t)dt + \int_{0}^{x} (x-t)y^2(t)dt \tag{23}$$

where $g(x) = \frac{-1}{30}x^6 + \frac{1}{3}x^4 + x^3 - 2x^2 - \frac{5}{3}x + \frac{3}{4}$ and $y(0) = -2$.

a) Let us take $N = 2$ for solution of Eq. (22) and seek the solution $y(x)$ as a truncated Chebyshev series

$$y(x) = \sum_{r=0}^{2}{}' a_r T_r(x), \qquad -1 \le x \le 1$$

Fundamental matrix equation of this problem defined in Section 3 is

$$\left(4\mathbf{TM}^2 - 2\mathbf{P}_1\mathbf{TM} + \mathbf{P}_0\mathbf{T} - \mathbf{F}_1\mathbf{Z}_1\right)\mathbf{A} - \mathrm{K}_2\mathcal{Z}_2\mathbf{B} = \mathbf{G}$$

and condition equations are

$$\boldsymbol{T}(0)\mathbf{A} = -1 \quad \text{and} \quad 2\boldsymbol{T}(0)\mathbf{MA} = 0$$

This matrix equation corresponds to nonlinear algebraic system as follows:

$$\frac{1}{2}a_0 - \frac{2}{3}a_1 + a_2 - \frac{1}{2}a_0{}^2 - \frac{2}{3}a_1{}^2 - \frac{14}{15}a_2{}^2 + \frac{4}{3}a_0a_1 + \frac{8}{15}a_1a_2 + \frac{2}{3}a_0a_2 = \frac{-16}{15}$$

$$4a_2 - \frac{1}{4}a_0{}^2 - \frac{1}{2}a_1{}^2 - \frac{1}{3}a_2{}^2 + \frac{2}{3}a_0a_1 + \frac{4}{15}a_1a_2 = \frac{5}{3}$$

$$\frac{-1}{2}a_0 + \frac{8}{3}a_1 - a_2 = 0 \tag{24}$$

and condition equations are

$$\frac{1}{2}a_0 - a_2 = -1$$

$$a_1 = 0 \tag{25}$$

In system (24), first and second equations are replaced by condition equations in (25) and new linear algebraic system is obtained. This system is solved easily, so we have

$$y(x) = x^2 - 1$$

which is the exact solution of Eq.(22).

b) Let us consider solution of Eq. (23) for $N = 2$ and seek the solution $y(x)$ as a truncated Chebyshev series

$$y(x) = \sum_{r=0}^{N}{}' a_r^* T_r^*(x), \qquad 0 \le x \le 1 \tag{26}$$

The fundamental matrix equation of this problem defined in Section 3 is

$$\left(4\mathbf{P}_1\mathbf{T}^*\mathbf{M} + \mathbf{P}_0\mathbf{T}^* - \mathbf{F}_1^*\mathbf{Z}_1^*\right)\mathbf{A}^* - \mathrm{K}_2^*\mathbf{Z}_2^*\mathbf{B}^* = \mathbf{G}$$

and for condition equation is

$$\boldsymbol{T}^*(0)\mathbf{A}^* = -2$$

The matrix equation corresponds to nonlinear algebraic system as follows:

$$\frac{-1}{4}a_0^* + \frac{17}{6}a_1^* + \frac{19}{2}a_2^* - \frac{1}{8}a_0^{*2} - \frac{1}{6}a_1^{*2} - \frac{7}{30}a_2^{*2} + \frac{1}{6}a_0^*a_1^* + \frac{1}{15}a_1^*a_2^* + \frac{1}{6}a_0^*a_2^* = \frac{103}{60}$$

$$\frac{-1}{4}a_0^* + \frac{11}{6}a_1^* - \frac{1}{6}a_2^* - \frac{1}{32}a_0^{*2} - \frac{1}{16}a_1^{*2} - \frac{1}{24}a_2^{*2} + \frac{1}{12}a_0^*a_1^* + \frac{1}{30}a_1^*a_2^* = \frac{2359}{1920}$$

$$\frac{-1}{4}a_0^* + \frac{11}{6}a_1^* - \frac{47}{6}a_2^* = \frac{3}{4} \tag{27}$$

and condition equation is

$$\frac{1}{2}a_0^* - a_1^* + a_2^* = -2 \tag{28}$$

When the first equation the system (27) is replaced by Eq. (28), new nonlinear algebraic system is obtained. Taking starting points $a_i^* = 0$ (i = 0, 1, 2) the solution of the system is obtained and we have

$$y(x) = -1.625T_0^*(x) + 0.5T_1^*(x) + 0.125T_2^*(x)$$

or

$$y(x) = x^2 - 2$$

which is the exact solution of Eq.(23).

Example 2. Consider the nonlinear Volterra integro-differential equation

$$y'(x) = -1 + \int_0^x y^2(t)dt \, , \, y(0) = 0 \tag{29}$$

Using the method in Section 3, Eq. (29) is solved for $N = 6$. The solution of this example can be found analytically by reducing to differential equation, but the analytical solution is not represented by the elementary functions. However, it can be represented by hypergeometric functions. The numerical solutions of Eq. (29) were given by Sepehrian-Razzaghi [4] and by Avudainayagam-Vani [2]. A comparison of these solutions with the present solution is given in Table 1.

Table 1. Numerical results of Example (2)

x	Wavelet-Galerkin Method	Walsh Series Method (m=60)	Presented Method N=6	Exact Solution
0.0000	0.0000	0.00000	0.00000	0.00000
0.0625	-0.0625	-0.06250	-0.06250	-0.06250
0.1250	-0.1250	-0.12498	-0.12498	-0.12498
0.1875	-0.1874	-0.18740	-0.18740	-0.18740
0.2500	-0.2497	-0.24967	-0.24967	-0.24967
0.3125	-0.3117	-0.31171	-0.31171	-0.31171
0.3750	-0.3734	-0.37336	-0.37336	-0.37336
0.4375	-0.4345	-0.43446	-0.43446	-0.43446
0.5000	-0.4948	-0.49482	-0.49482	-0.49482
0.5625	-0.5542	-0.55423	-0.55423	-0.55423
0.6250	-0.6124	-0.61243	-0.61243	-0.61243
0.6875	-0.6692	-0.66916	-0.66917	-0.66917
0.7500	-0.7242	-0.72415	-0.72415	-0.72415
0.8125	-0.7771	-0.77709	-0.77709	-0.77709
0.8750	-0.8277	-0.82766	-0.82767	-0.82767
0.9375	-0.8756	-0.87557	-0.87557	-0.87557
1.0000	-0.9205	-0.92047	-0.92047	-0.92048

$$y'''(x) - xy''(x) + \sin xy(x) = e^x\left(1 - x + \sin x\right) - 2 + \int_{-1}^{1} e^{-2t} y^2(t)dt$$

with the conditions $y(0) = y'(0) = y''(0) = 1$.

Let us suppose that $y(x)$ is approximated by Chebyshev series

$$y(x) = \sum_{r=0}^{7}{}' a_r T_r(x), \qquad -1 \le x \le 1$$

Using the procedure in Section 3, we find the approximate solution of this equation. A comparison of the obtained solution with the exact solution at the collocation points is given in Table 2.

Table 2. Numerical results of Example (3)

x	Presented Method	Exact solution e^x
x_0	2.718281	2.718282
x_1	2.519044	2.519044
x_2	2.028115	2.028115
x_3	1.466214	1.466214
x_4	0.682029	0.682029
x_5	0.493068	0.493069
x_6	0.396976	0.396976
x_7	0.367879	0.367879

6. CONCLUSIONS

In this work, Chebyshev collocation method has applied to nonlinear integro-differential equation. The study has showed that solving Fredholm part is easier than Volterra part. An interesting feature of this method is that the analytical solution is obtained for smaller N as shown in the Example 1. Moreover, this method gives better approximate solutions than the other methods as shown in the Example 2. One of the advantages of this method that solution is expressed as a truncated Chebyshev series, then $y(x)$ can be easily evaluated for arbitrary values of x.

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