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On the Eigenvalues of Integral Operators

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Abstract

In this paper, we obtain asymptotic estimates of the eigenvalues of certain positive integral operators.

Key words: *Positive Integral Operators, Eigenvalues, Hardy Spaces.*

Özet

Bu çalışmada bazı positive integral operatörlerin özdeğerlerinin asimtotik yaklaşımlarını elde edeceğiz.

Anahtar Kelimeler: *Pozitif İntegral Operatörleri, Özdeğerler, Hardy Uzayları.*

1. INTRODUCTION

From now on, let J be a fixed closed subinterval of the real line $\mathbf{R}$. Suppose that D is a simply-connected domain containing the real closed interval J and φ is any function, which maps D conformally onto Δ, where Δ is the open unit disk of complex plane $\mathbb{C}$. Let us define a function K_D on $D \; x \; D$ by

$$K_D(\zeta, z) = \frac{\varphi'(\zeta)^{\frac{1}{2}}\overline{\varphi'(z)^{\frac{1}{2}}}}{1-\varphi(\zeta)\overline{\varphi(z)}} \qquad \text{for all } \zeta, z \in D,$$

for either of the branches of $\varphi'^{\frac{1}{2}}$. The function K_D is independent of the choice of mapping function φ, see [1, p.410]. By restricting the function K_D to the square JxJ we obtain a compact symmetric operator T_D on L^2 defined by

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$$T_D f(s) = \int_J K_D(s,t) f(t) dt \qquad (f \in L^2(J), s \in J).$$

This operator is always positive in the sense of operator theory (i.e. $\langle Tf, f \rangle \geq 0$ for all $f \in L^2(J)$, see [1].

We shall use $\lambda_n(K_D)$ to denote the eigenvalues of T_D.

In this work the following theorem shall be proved in detail.

THEOREM 1.1 If D_1, D_2, D_3 are three half-planes and their boundary lines are not parallel pairwise and if $D = D_1 \cap D_2 \cap D_3$ contains the real closed interval *J*, then

$$\lambda_n(K_D) \cong \lambda_n(K_{D_1} + K_{D_2} + K_{D_3})$$

where $a_n \cong b_n$ means $a_n = O(b_n)$ and $b_n = O(a_n)$.

To prove Theorem 1.1 we will show that

i) $\lambda_n(K_{D_1 \cap D_2 \cap D_3}) = O(\lambda_n(K_{D_1} + K_{D_2} + K_{D_3}))$

ii) $\lambda_n(K_{D_1} + K_{D_2} + K_{D_3}) = O(\lambda_n(K_{D_1 \cap D_2 \cap D_3}))$.

This is a special case of a theorem in [1, Theorem 1] and we give a different proof.

2. PRELIMINARIES

The space $H^\infty(\Delta)$ is just the set of all bounded analytic function on Δ with the uniform norm. For $1 \leq p < \infty$, $H^p(\Delta)$ is the set of all functions *f* analytic on Δ such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta < \infty. \tag{1}$$

The *p*-th root of the left hand side of (1) here defines a complete norm on $H^p(\Delta)$. For more information on this spaces see [2 and 3]. In the case of *p*=2, H^2 be the familiar Hardy space of all functions analytic on Δ with square-summable Maclaurin coefficients.

Let *D* be a simply connected domain in $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ and let φ be a Riemann mapping function for *D*, that is, a conformal map of *D* onto Δ. An analytic function *g* on *D* is said to be of class $E^2(D)$ if there exists a function $f \in H^2(\Delta)$ such that

$g(z)=f\left(\varphi(z)\right)\varphi'(z)^{\frac{1}{2}}$ $(z\in D)$ where $\varphi'^{\frac{1}{2}}$ is a branch of the square root of φ'. We define $\|g\|_{E^2(D)}=\|f\|_{H^2(\Delta)}$. Thus, by construction, $E^2(D)$ is a Hilbert space with

$$\langle g_1,g_2\rangle_{E^2(D)}=\langle f_1,f_2\rangle_{H^2(\Delta)}$$

where $g_i(z)=f_i\left(\varphi(z)\right)\varphi'(z)^{\frac{1}{2}}$, $(i = 1, 2)$ and the map $U_\varphi : H^2(\Delta)\to E^2(D)$ given by

$$U_\varphi f(z)=f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (f\in H^2(\Delta), z\in D)$$

is an isometric bijection. For more information on this spaces see [1]. If ∂D is a rectifiable Jordan curve then the same formula

$$V_\varphi f(z)=f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (f\in L^2(\partial\Delta), z\in \partial D)$$

defines an isometric bijection V_φ of $L^2(\partial\Delta)$ onto $L^2(\partial D)$, the L^2 space of normalized arc length measure on ∂D where ∂D and $\partial\Delta$ denote the boundary of *D* and Δ respectively. The inverse

$$V_\psi = V_\varphi^{-1} : L^2(\partial D)\to L^2(\partial\Delta)$$

of V_ψ is given by

$$V_\psi g(w)=g(\psi(w))\psi'(w)^{1/2} \quad (g\in L^2(\partial D), w\in\partial\Delta, \psi=\varphi^{-1}).$$

To prove Theorem 1.1 we need the following lemma. This is Corollary 1.3 to Lemma 1.2 in [4].

LEMMA 2.1 Suppose that *D* is a disc or a codisc or a half-plane and $\gamma'\subseteq \overline{D}$ be a circular arc (or a straight line) then for every $g\in E^2(D)$,

$$\frac{1}{2\pi}\int_{\gamma'} |g(z)|^2\,|dz| \le \|g\|^2_{E^2(D)} = \frac{1}{2\pi}\int_{\partial D} |g(z)|^2\,|dz|.$$

Suppose now that *D* contains our fixed interval *J*. By restricting φ to *J* we obtain a linear operator $S_D : E^2(D)\to L^2(J)$ defined by $S_D f(s)=f(s)$ $(f\in E^2(D), s\in J)$. Then S_D is compact operator and $T_D = S_D S^*_D$ is the compact, positive integral operator on *J* with kernel K_D:

$$K_D(s,t)=\frac{\varphi'(s)^{\frac{1}{2}}\overline{\varphi'(t)^{\frac{1}{2}}}}{1-\varphi(s)\overline{\varphi(t)}}$$

for all s,t $\in$ J. This is proved in [1]

DEFINITION 2.1 Let *H* and H' be Hilbert spaces and suppose that *T* is a compact, positive operator on *H*. If $S : H' \to H$ is a compact operator such that $T = SS^*$, then *S* is called a quasi square-root of *T*. We call H' the domain space of *S*.

REMARK 2.2 Suppose that D_1, D_2, D_3 are simply-connected domains containing *J* and let $T_{D_1}, T_{D_2}, T_{D_3}$ be continuous positive operators on a Hilbert space $L^2(J)$ and suppose that for each *i*, S_{D_i} is a quasi square-root of T_{D_i} with domain space $E^2(D_i)$.
If $T_+ = T(K_{D_1} + K_{D_2} + K_{D_3}) = T(\sum_{i=1}^{3} K_{D_i})$
so that $T_+ f(s) = \int_J (K_{D_1}(s,t) + K_{D_2}(s,t) + K_{D_3}(s,t)) f(t)dt \quad (f \in L^2(J), s \in J)$, then $T_+ : L^2(J) \to L^2(J)$ is compact, positive integral operator and T_+ has the quasi square-root

$$S_+ : E^2(D_1) + E^2(D_2) + E^2(D_3) \to L^2(J), \qquad S_+(f_1 + f_2 + f_3) = S_{D_1} f_1 + S_{D_2} f_2 + S_{D_3} f_3$$

so that

$$S_+(f_1 + f_2 + f_3)(s) = S_{D_1} f_1(s) + S_{D_2} f_2(s) + S_{D_3} f_3(s) = f_1(s) + f_2(s) + f_3(s) \qquad (f \in L^2(J), s \in J).$$

LEMMA 2.3 Let T_1, T_2 be compact operators on a Hilbert space *H* and suppose that S_1, S_2 are quasi square-root of T_1, T_2 with domain H_1, H_2 respectively.

i) If there exists a continuous operator $V : H_2 \to H_1$ such that $S_2 = VS_1$ then $(T_2 f, f) \le k(T_1 f, f)$ for some *k*>0 and so $\lambda_n(T_2) = O(\lambda_n(T_1)) \quad (n \ge 0)$.

ii) If there exists continuous operators $V : H_2 \to H_1$ and $W : H \to H$ such that $S_2 = WS_1V$, then $\lambda_n(T_2) = O(\lambda_n(T_1))$.

Proof. See [1, page 407].

3. PROOF OF MAIN RESULT

Suppose that *D* is a simply connected and bounded domain. Let φ be a Riemann mapping function for *D* and suppose that $\psi = \varphi^{-1}$ is the inverse function of φ. An analytic function *f* on *D* is said to be of class $H^\infty(D)$ if it is bounded on *D*.

PROPOSITION 3.1 If $\psi' \in H^1$, then $H^\infty(D) \subseteq E^2(D)$.

Proof. Suppose that $f \in H^{\infty}(D)$. For $z \in D$, define $g(z) = f\left(\psi(z)\right)\psi'(z)^{\frac{1}{2}}$. Then $f \circ \psi \in H^{\infty}(D)$ and $\psi'^{\frac{1}{2}} \in H^2$. It follows that $(f \circ \psi)\psi'^{\frac{1}{2}} \in H^2$. Hence $f \in E^2(D)$.

PROPOSITION 3.2 Suppose that ∂D is a rectifiable Jordan curve, then

i) $\psi' \in H^1$

ii) Each function $f \in E^2(D)$ has a non-tangential limit $\tilde{f} \in L^2(\partial D)$. The map $f \to \tilde{f}$ is an isometric isomorphism and $\|f\|^2_{E^2(D)} = \dfrac{1}{2\pi} \int\limits_{\partial D} \left|\tilde{f}(z)\right|^2 |dz|$.

iii) If *D* is a convex region, it is a Smirnov domain.

iv) If *D* is a Smirnov domain, then polynomials (thus $H^{\infty}(D)$) are dense in $E^2(D)$.

v) $E^2(D)$ coincides with the $L^2(\partial D)$ closure of the polinomials if and only if *D* is a Smirnov domain.

Proof. See [2, pages 44, 170 and 173]. For the definition of Smirnov domain see [2, page 173].

LEMMA 3.4 If *D* is a disc, or codisc or half-plane, then the formula

$$Pf(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

defines a continuous linear operator $P : L^2(\partial D) \to E^2(D)$ with $\|P\| = 1$.

Proof. See [1, page 423].

From now on, suppose now that D_1, D_2, D_3 are three half-planes, and let $D = D_1 \cap D_2 \cap D_3$ contains the real closed interval *J* (see Figure 1). For *k* =1,2,3, let $\gamma_k = \partial D_k \cap \left(D_1 \cap D_2 \cap D_3\right)$.

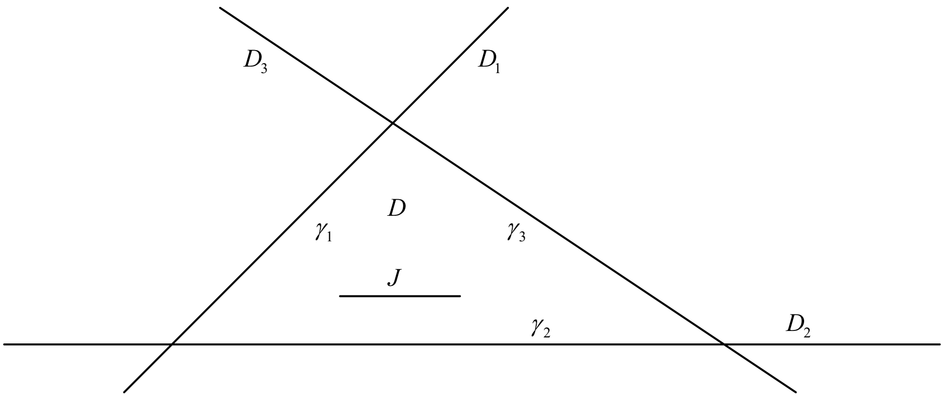

Figure 1.

We shall exhibit continuous operators

$N: E^2(D) \to E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$ and $M: E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3) \to E^2(D)$.

To define *N* suppose first that *G*={*f*: *f* is a polynomial in $E^2(D)$}. Since *D* is convex, $\overline{G} = E^2(D)$. If $f \in G$, then for all $z \in D$, Cauchy's Integral Formula gives

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\gamma_3} \frac{f(w)}{w-z}dw$$

For $f \in G$ and $1 \le k \le 3$, define a function f_k on D_k by

$$f_k(z) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{f(w)}{w-z}dw \qquad (z \in D_k),$$

and define a function $\tilde{f}_k$ on ∂D_k by

$$\tilde{f}_k(z) = \begin{cases} f(\zeta), & \text{if } \zeta \in \gamma_k \\ 0, & \text{if } \zeta \in \partial D_k - \gamma_k \end{cases} \qquad (z \in \partial D_k).$$

LEMMA 3.5 If $f \in G$ and $1 \le k \le 3$ then

i) $\tilde{f}_k \in L^2(\partial D_k)$ and $f_k \in E^2(D_k)$.

ii) The formula $V_1 f = (f_1, f_2, f_3)$ defines a continuous linear operator $V_1 : G \to E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$ and so that V_1 has an extension N by continuity to $E^2(D)$.

Proof. *i*) Let φ_k be a Riemann mapping function for D_k and suppose V_{φ_k} and U_{φ_k} are as in Section 2. The map $P_k : L^2(\partial D_k) \to E^2(D_k)$ given by

$$P_k f(z) = \frac{1}{2\pi i} \int_{\partial D_k} \frac{f(\zeta)}{\zeta - z} d\zeta$$

is a continuous linear operator with $\|P_k\| = 1$ (from Lemma 3.4). Since $f_k = P_k \tilde{f}_k$, it follows that $f_k \in E^2(D_k)$. So $(f_1, f_2, f_3) \in E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$. Since

$$\|f_k\|^2_{E^2(D_k)} = \|P_k \tilde{f}_k\|^2_{E^2(D_k)} \le \|\tilde{f}_k\|^2_{L^2(\partial D_k)} = \frac{1}{2\pi} \int_{\gamma_k} |f(\zeta)|^2 |d\zeta| \le \|f\|^2_{E^2(D)}$$

and

$$\|V_1 f\|^2_{E^2(D_1)\oplus E^2(D_2)\oplus E^2(D_3)} = \|(f_1, f_2, f_3)\|^2_{E^2(D_1)\oplus E^2(D_2)\oplus E^2(D_3)}$$

$$= \|f_1\|^2_{E^2(D_1)} + \|f_2\|^2_{E^2(D_2)} + \|f_2\|^2_{E^2(D_3)} \le 3\|f\|^2_{E^2(D)}$$

it follows that the map $f \to (f_1, f_2, f_3)$ is a continuous linear operator $G \to E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$. Now suppose that N is an extension by continuity to $E^2(D)$. Note that then $\|N\|^2 \le 3$.

If we denote $F = H^\infty(D_1) \oplus H^\infty(D_2) \oplus H^\infty(D_3)$ then $\overline{F} = E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$.

LEMMA 3.6 The map $V_2 : F \to E^2(D)$, is given by

$$V_2(f_1, f_2, f_3)(z) = f_1(z) + f_2(z) + f_3(z), \qquad ((f_1, f_2, f_3) \in F, z \in D),$$

is a continuous operator so that V_2 has an extension M by continuity to $E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$.

Proof. If $(f_1, f_2, f_3) \in F$ then by Propositions 3.1 and 3.2, $f_i \in H^\infty(D) \subseteq E^2(D)$ $(1 \le i \le 3)$ and $V_2(f_1, f_2, f_3) \in H^\infty(D) \subseteq E^2(D)$. So we have

$$\begin{aligned}
\|V_2(f_1,f_2,f_3))\|^2_{E^2(D)} &= \|f_1+f_2+f_3\|^2_{E^2(D)} \\
&= \|f_1\|^2_{E^2(D)} + \|f_2\|^2_{E^2(D)} + \|f_3\|^2_{E^2(D)} + 2\,\mathrm{Real}\langle f_1, f_2\rangle_{E^2(D)} \\
&\quad + 2\,\mathrm{Real}\langle f_1, f_3\rangle_{E^2(D)} + 2\,\mathrm{Real}\langle f_2, f_3\rangle_{E^2(D)} \\
&\le \|f_1\|^2_{E^2(D)} + \|f_2\|^2_{E^2(D)} + \|f_3\|^2_{E^2(D)} + (\|f_1\|^2_{E^2(D)} + \|f_2\|^2_{E^2(D)}) \\
&\quad + (\|f_1\|^2_{E^2(D)} + \|f_3\|^2_{E^2(D)}) + (\|f_2\|^2_{E^2(D)} + \|f_3\|^2_{E^2(D)}) \\
&= 3(\|f_1\|^2_{E^2(D)} + \|f_2\|^2_{E^2(D)} + \|f_3\|^2_{E^2(D)}) \\
&\text{(by Lemma 2.1)} \\
&\le 3(3\|f_1\|^2_{E^2(D_1)} + 3\|f_2\|^2_{E^2(D_2)} + 3\|f_3\|^2_{E^2(D_3)}) \\
&= 9\|(f_1,f_2,f_3)\|^2_{E^2(D_1)\oplus E^2(D_2)\oplus E^2(D_3)}
\end{aligned}$$

Hence $\|V_2\|^2 \le 9$ and V_2 is a continuous linear operator. Let now M be extension by continuity to $E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$. Note that then $\|M\|^2 \le 9$.

PROOF OF THEOREM 1.1

i) Suppose that V_1 is as in Lemma 3.5. Note that here $T_+ = S_+ S_+^*$ and $T_D = S_D S_D^*$. By definition of V_1, we have $S_D f = S_+ V_1 f$ for every $f \in G$. Thus, by continuity of V_1, $S_D f = S_+ N f$ for every $f \in E^2(D)$ and so $S_D = S_+ N$.

So for $g \in L^2(J)$,

$$\begin{aligned}
\langle S_D S_D^* g, g\rangle &= \|S_D^* g\|^2 = \|N^* S_+^* g\|^2 \\
&\le \|N^*\|^2 \|S_+^* g\|^2 = \|N\|^2 \langle S_+ S_+^* g, g\rangle \\
&\le 3\langle S_+ S_+^* g, g\rangle.
\end{aligned}$$

That is, $S_D S_D^* \leq 2 S_+ S_+^*$. Hence by Lemma 2.3

$$\lambda_n(S_D S_D^*) \leq 3\lambda_n(S_+ S_+^*)$$

as required.

ii) Suppose that V_2 is as in Lemma 3.6. By definition of V_2, it follows that $S_+(f_1, f_2, f_3) = S_D V_2(f_1, f_2, f_3)$ for every $(f_1, f_2, f_3) \in H^\infty(D_1) \oplus H^\infty(D_2) \oplus H^\infty(D_3)$. Thus, by continuity of V_2,

$$S_+(f_1, f_2, f_3) = S_D M(f_1, f_2, f_3) \text{ for every } (f_1, f_2, f_3) \in E^2(D_1) \oplus E^2(D_2) \oplus E^2(D_3)$$

and so $S_+ = S_D M$. So for $g \in L^2(J)$, we have

$$\begin{aligned} \langle S_+ S_+^* g, g\rangle &\leq \|M\|^2 \langle S_D S_D^* g, g\rangle \\ &\leq 9\langle S_D S_D^* g, g\rangle. \end{aligned}$$

i.e, $S_+ S_+^* \leq 9 S_D S_D^*$. Consequently, from Lemma 2.3,

$$\lambda_n(S_+ S_+^*) \leq 9\lambda_n(S_D S_D^*).$$

REFERENCES

Little, G., "Equivalences of positive integrals operators with rational kernels", Proc. London. Math. Soc. (3) 62 (1991), 403-426.

Duren, P. L., "Theory of spaces", Academic Press, New York, 1970

Koosis, P., "Introduction to spaces", Cambridge University Press, Cambridge, Second Edition, 1998.

Soykan Y., "An Inequality of Fejer-Riesz Type", Çankaya University, Journal of Arts and Sciences, Issue: 5, May 2006, 51-60.