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Complex Lagrangians and Hamiltonians

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Abstract

The aim of this paper is to present a Maple code of complex analogues of Lagrangian and Hamiltonian equations being an important tool in classical mechanics. Firstly, using mathematical and physical structures and operations, complex Lagrangian and Hamiltonian equations are reviewed. And then, it is occurred mechanical equations about the above mentioned systems formed an example with complex coordinates. Also, it is produced a Maple code for complex Lagrangian and Hamiltonian equations. Finally, geometrical and physical results for the complex dynamical equations obtained on underlying space are given.

***Keywords**: Lagrangian and Hamiltonian equations, Lagrangian and Hamiltonian systems, Maple program.*

Özet

Bu makalenin amacı, klasik mekanikte önemli bir araç olan Lagrange ve Hamilton denklemlerinin kompleks benzerlerinin bir Maple kodunu hazırlamaktır. İlk olarak, matematiksel ve fiziksel yapıları ve işlemleri kullanarak, kompleks Lagrange ve Hamilton denklemler ele alındı. Daha sonra, kompleks koordinatlı bir örnek oluşturularak yukarıda bahsedilen sistemlere dair mekanik denklemler ortaya çıkarıldı. Aynı zamanda, kompleks Lagrange and Hamilton denklemleri için bir Maple kodu üretildi. Sonuçta, baz uzayı üzerinde elde edilen kompleks dinamik sistemler için geometrik ve fiziksel sonuçlar verildi.

***Anahtar Kelimeler**: Lagrange and Hamilton denklemleri, Lagrange and Hamilton sistemleri, Maple programı.*

1. Introduction

It is possible to find a good framework for studying Lagrangian and Hamiltonian formalisms of classical mechanics[1,2,3,8]. A renewed interest in higher order

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Lagrangian and Hamiltonian theories has been remarked in last years [6-7]. The modern development of analytical mechanics in terms of intrinsical geometrical properties of differentiable manifolds shows that the dynamics of a Lagrangian and Hamiltonian system is characterized by a suitable vector field defined on the tangent and cotangent bundles (phase-spaces of momentum and velocities) of a given configuration manifold. It is probable to give the more physical examples about Lagrangian and Hamiltonian systems. A well-known example is that the Lagrangian of Einstein theory of relativity does in fact contains second order derivatives of metric field. A classical example of second order Lagrangians appears in the elastic theory of straight beams. As well known, Maple program is an important tool in mathematics and other science fields [4]. Here, we shall use Maple program to model the dynamical systems obtained on complex spaces.

2.Lagrangian and Hamiltonian Dynamics

In this section, a complex version of Lagrangian and Hamiltonian dynamics is presented. Firstly, to introduce complex Lagrangian dynamics we have to define a Lagrangian as a function of the two complex variables of position z and speed $\dot{z}$

$$L(z,\dot{z}) = T(\dot{z}) - U(z) \tag{1}$$

where it is well known that the kinetic energy $T(\dot{z}) = \frac{1}{2}m\dot{z}^2$ is a function of the speed variable and the potential energy is a function of position variable z. The equations given as

$$L1: \quad \mathbf{i}\frac{d}{dt}(\frac{\partial L}{\partial z}) - (\frac{\partial L}{\partial z}) = 0, \quad L2: \quad \mathbf{i}\frac{d}{dt}(\frac{\partial L}{\partial \dot{z}}) + (\frac{\partial L}{\partial \dot{z}}) = 0, \tag{2}$$

are the complex analogues of Euler-Lagrange equations [5].

Secondly, to introduce complex Hamiltonian dynamics, we have to state a Hamiltonian as a function of the two variables, momentum $\dot{z}$ and position z as follows:

$$\text{H}(\bar{z}, z) = \bar{z}\,\dot{z} - \text{L}(z, \dot{z}) \tag{3}$$

The equations determined with

$$H1:\quad \frac{\partial z}{\partial t} = -\mathbf{i}\frac{\partial H}{\partial \bar{z}}, \quad H2:\quad \frac{\partial \bar{z}}{\partial t} = \mathbf{i}\frac{\partial H}{\partial z} \tag{4}$$

are the complex analogues of Hamiltonian equations [5].

Example: A central force field $f(\rho) = A\rho^{\alpha-1} (\alpha \neq 0, 1)$ acts on a body with mass m in a constant gravitational field. Then let us find out the complex Lagrangian and Hamiltonian equations of the motion by assuming the body always on the vertical plane.

Complex Lagrangian and Hamiltonian functions of the system are, respectively,

$$L = \frac{1}{2}m\dot{z}\dot{\bar{z}} - \frac{A}{\alpha}(\sqrt{z\bar{z}})^{\alpha} + \mathbf{i}\,mg\frac{(z-\bar{z})\sqrt{z\bar{z}}}{(z+\bar{z})\sqrt{1-\frac{(z-\bar{z})^2}{(z+\bar{z})^2}}},$$

$$H = \frac{1}{2}m\dot{z}\dot{\bar{z}} - \frac{A}{\alpha}(\sqrt{z\bar{z}})^{\alpha} - \mathbf{i}\,mg\frac{(z-\bar{z})\sqrt{z\bar{z}}}{(z+\bar{z})\sqrt{1-\frac{(z-\bar{z})^2}{(z+\bar{z})^2}}}.$$

Then, using (2) and (4), the so-called complex-Lagrangian and complex-Hamiltonian equations of the motion on the complex-mechanical systems, can be obtained, respectively, as follows:

$$L1:\quad \mathbf{i}\frac{d}{dt}K - K = 0, \quad L2:\quad \mathbf{i}\frac{d}{dt}V + V = 0,$$

such that

$$K = -\frac{A}{2z}(\sqrt{z\bar{z}})^{\alpha} + \mathbf{i}\frac{mg(z-\bar{z})\bar{z}}{2\sqrt{z\bar{z}}(z+\bar{z})W} + \mathbf{i}\frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W}$$

$$-\mathbf{i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2W} - \mathbf{i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})\left(-\frac{(z-\bar{z})}{(z+\bar{z})^2}+\frac{(z-\bar{z})^2}{(z+\bar{z})^3}\right)}{(z+\bar{z})W^3}$$

$$V = \frac{1}{2}m\dot{\bar{z}}$$

and

$$H1: \quad \frac{dz}{dt} = \textbf{-i}\Big(\frac{A}{2\bar{z}}(\sqrt{z\bar{z}})^{\alpha} \textbf{-i}\frac{mg(z-\bar{z})z}{2\sqrt{z\bar{z}}(z+\bar{z})W} \textbf{+i}\frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W}$$

$$\textbf{+i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2 W} \textbf{+i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})\Big(\frac{(z-\bar{z})}{(z+\bar{z})^2}+\frac{(z-\bar{z})^2}{(z+\bar{z})^3}\Big)}{(z+\bar{z})W^3}\Big),$$

$$H2: \quad \frac{d\bar{z}}{dt} = \textbf{i}\Big(\frac{A}{2z}(\sqrt{z\bar{z}})^{\alpha} \textbf{-i}\frac{mg(z-\bar{z})\bar{z}}{2\sqrt{z\bar{z}}(z+\bar{z})W} \textbf{-i}\frac{mg\sqrt{z\bar{z}}}{(z+\bar{z})W}$$

$$\textbf{+i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})}{(z+\bar{z})^2 W} \textbf{+i}\frac{mg\sqrt{z\bar{z}}(z-\bar{z})\Big(-\frac{(z-\bar{z})}{(z+\bar{z})^2}+\frac{(z-\bar{z})^2}{(z+\bar{z})^3}\Big)}{(z+\bar{z})W^3}\Big).$$

Where $W = \sqrt{1-\frac{(z-\bar{z})^2}{(z+\bar{z})^2}}$.

3. Maple code of the above example

m=constant, g=constant, A=constant, alpha=constant, z=z(t), zE=zE(t), zT=zT(t),zTT=zTT(t),

zET=zET(t), zETT=zETT(t); Diff((z),t)=zT, Diff((zE),t)=zET, Diff(zT,t)=zTT,

Diff(zET,t)=zETT;

L:=(1/2)*m*zT*zET-(A/alpha)*((z*zE))^ (alpha/2)-*g*sqrt(z*zE)*(sin(arctan(-I*(z-zE)/(z+zE))));

L1:=I*Diff(diff(L,z),t)=diff(L,z); L2:=I*Diff(diff(L,zT),t)=-diff(L,zT);

H:=(1/2)*m*zT*zET+(A/alpha)*((z*zE))^(alpha/2)+m*g*sqrt(z*zE)*(sin(arctan(-I*(z-zE)/(z+zE))));

H1:=Diff(z,t)=-I*diff(H,zE); H2:=Diff(zT,t)=I*diff(H,z);

4. Conclusion

Taking into consideration the above, Lagrangian and Hamiltonian systems in classical mechanics can be intrinsically clarified on an underlying complex space. If it is solved complex Euler- Lagrange and Hamiltonian differential equations given the above, the solutions are easily seen to be paths of vector fields on complex space. Also, Maple program helps to find numeric solutions of dynamical systems obtained on the mentioned space.

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