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A Note on the Chain Rule on Time Scales

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Abstract

It is known, in general, that the chain rule on general time scale derivatives does not behave well as in the case of usual derivative. However, we discuss some special cases where the time scale derivative has the usual chain rule. The results are analyzed for both the delta and nabla time scales derivatives.

Key words: *Time scales derivative, q-time scale, H-time scale, Forward jump operator, Backward jump operator,rd-continuous, ld-continuous, Chain rule.*

1. Introduction and Preliminaries

The calculus on time scales was introduced ([1], [2]) to unify the continuous and discrete analysis. As usually expected when we generalize some theory we lose some nice properties. One of the basic concepts that researchers has to care about when deal with differential equations, variational calculus and so on, on time scales is chain rule. For that reason we try here to summarize and formulate some cases where the chain rule obeys the order, so that it might be useful to use in next works in the future. For the theory of delta derivative on time scales we refer to [1] and for the one for nabla derivative we refer to [2].

A time scale is an arbitrary nonempty closed subset of the real line. Thus real numbers ,, and natural numbers, are examples of time scales. Throughout this article and following [1] , the time scale will be denoted by . Here are down some examples of time scales on which we will study the chain rule in this article.

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Examples

1. For $q \in \Re^+$, let $T_q = \begin{cases} \{q^n : n \in N\} \cup \{0\} & 0 < q < 1 \\ \{q^n : n \in N\} & q \geq 1 \end{cases}$. When $0 < q < 1$, the theory of *q*-calculus is obtained [q-sur] with *0* an improper accumulation point, where the backward $q-derivative$ is usually defined. When $q > 1$ the time scale T_q will have ∞ as an improper point, where the forward $q-derivative$ is usually defined.

2. For $q \in \Re^+$ and $h \in \Re^+$, let

$$T_q^h = \begin{cases} \left\{ q^k + \sum_{i=0}^{k-2} q^i h : k \geq 2, k \in N \right\} \cup \left\{ \frac{h}{1-q} \right\} & 0 < q < 1 \\ \left\{ q^k + \sum_{i=0}^{k-2} q^i h : k \geq 2, k \in N \right\} & q > 1 \end{cases}. \quad (1)$$

The forward jump operator $\sigma : T \to T$ is defined by

$$\sigma(t) = \inf\{s \in T : s > t\} \quad (2)$$

while the backward jump operator $\rho : T \to T$ is defined by

$$\rho(t) = \sup\{s \in T : s < t\}, \quad (3)$$

where, $\inf \Phi = \sup T$ (i.e $\sigma(t) = t$ if *T* has maximum *t*) and $\sup \Phi = \inf T$ (i.e $\rho(t) = t$ if *T* has minimum *t*). It is to be noted that if $T = T_q$ then $\sigma(t) = \begin{cases} q^{-1}t & 0 < q < 1 \\ qt & q > 1 \end{cases}$ and $\rho(t) = \begin{cases} qt & 0 < q < 1 \\ q^{-1}t & q > 1 \end{cases}$. Also if $T = T_q^h$, $q > 1$ then $\sigma(t) = qt + h$ and $\rho(t) = q^{-1}(t-h)$. A point $t \in T$ is called right scattered if $t < \sigma(t)$, left-scattered if $\rho(t) < t$ and isolated if $\rho(t) < t < \sigma(t)$. In connection the forward and backward graininess functions $\mu, \nu : T \to [0, \infty)$ are defined, respectively, by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$.

In order to define the forward and backward time scale derivative, we need the sets T^κ and ${}^\kappa T$ which are derived from the time scale T as follows: If *T* has a left-scattered maximum *M*, then $T^\kappa = T - \{M\}$ and otherwise $T^\kappa = T$. If T has a right-scattered minimum *m*, then ${}^\kappa T = T - \{m\}$ and otherwise ${}^\kappa T = T$.

Assume $f : T \to \Re$ and $t \in T^\kappa$. Then the forward time-scale derivative is the number $f^\Delta(t)$ (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood *U* of *t* (i.e $U = (t-\delta, t+\delta)$ for some $\delta > 0$) such that

$$\left| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \qquad \forall s \in U. \quad (4)$$

Moreover, we say that *f* is (delta) differentiable on T^κ provided that $f^\Delta(t)$ for all $t \in T^\kappa$.

Similarly, the backward time-scale derivative is the number $f^\nabla(t)$, $t \in {}^\kappa T$ (provided it exists) with the property that given any $\varepsilon > 0$, there exists a neighborhood *U* of *t* (i.e $U = (t-\delta, t+\delta)$ for some $\delta > 0$) such that

$$\left| [f(s) - f(\rho(t))] - f^\nabla(t)[s - \rho(t)] \right| \leq \varepsilon |s - \rho(t)|, \qquad \forall s \in U, \quad (5)$$

and moreover, we say that *f* is (nabla) differentiable on ${}^{\kappa}T$ provided that $f^{\nabla}(t)$ for all $t\in {}^{\kappa}T$.
The following two theorems are valid for the forward and backward time-scale derivatives:

Theorem 1 Assume $f:T\to\Re$ is a function and let $t\in T^{\Delta}$. Then we have the following:

(i) If *f* is delta differentiable at *t*, then *f* is continuous at *t*.

(ii) If f is continuous at *t* and *t* is right-scattered, then f is (delta) differentiable at *t* with $f^{\Delta}(t)=\dfrac{f(\sigma(t))-f(t)}{\mu(t)}$. (6)

(iii) If *t* is right-dense (i.e $\sigma(t)=t$) then *f* is (delta) differentiable at *t* if and only if the limit $\lim_{s\to t}\dfrac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}. \tag{7}$$

(iv) If f is (delta) differentiable at t, then

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t). \tag{8}$$

Theorem 2 Assume $f:T\to\Re$ is a function and let $t\in {}^{\kappa}T$. Then we have the following:

(i) If *f* is (nabla) differentiable at *t*, then *f* is continuous at *t*.

(ii) If f is continuous at *t* and *t* is left-scattered, then f is (nabla) differentiable at *t* with $f^{\nabla}(t)=\dfrac{f(t)-f(\rho(t))}{\nu(t)}$.

(iii) If *t* is left-dense (i.e $\rho(t)=t$) then *f* is (nabla) differentiable at *t* if and only if the limit, $\lim_{s\to t}\dfrac{f(s)-f(t)}{s-t}$ exists as a finite number. In this case

$$f^{\nabla}(t)=\lim_{s\to t}\frac{f(s)-f(t)}{s-t}. \tag{9}$$

(iv) If f is (nabla) differentiable at t, then

$$f(\rho(t))=f(t)-\nu(t)f^{\nabla}(t). \tag{10}$$

The following product formulas are useful:

$$[f(t)g(t)]^{\Delta}=f^{\Delta}(t)g(\sigma(t))+f(t)g^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t) \tag{11}$$

$$[f(t)g(t)]^{\nabla}=f^{\nabla}(t)g(\rho(t))+f(t)g^{\nabla}(t)=f^{\nabla}(t)g(t)+f(\rho(t))g^{\nabla}(t) \tag{12}$$

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In case of the time scales T_q and T_q^h given above in examples (1) and (2) all the nodes are isolated except for the limit points and the jump operators σ and ρ are inverses to each other and hence the following relations between the (delta) derivative and the (nabla) derivative hold

(a) $f^{\Delta}(\rho(t))=f^{\nabla}(t)$ (b) $f^{\nabla}(\sigma(t))=f^{\Delta}(t)$.

For a function $f:T\to\Re$, $f^{\Delta^n}(t)$ will mean the nth (delta) derivative of *f* at $t\in T^{\kappa^n}$ and a similar notation can be assigned also for the (nabla) derivative.

Definition 1 A function $f : T \to \Re$ is called rd-continuous provided it is continuous at right-dense points in *T* and its left-sided limits exist (finite) at left-dense points in *T,* and is called ld-continuous provided it is continuous at left-dense points in *T* and its right-sided limits exist (finite) at right-dense points in *T*.
Continuous functions are clearly rd-continuous and ld-continuous. Also it is easy to see that σ is an example of an rd-continuous function which is not continuous and ρ is an example of an ld-continuous function which is not continuous. This is, of course, true if the time scale *T* contains left-dense right-scattered nodes and right-dense left-scattered nodes, respectively.

Theorem 3 [1] (Existence of delta antiderivatives). Every rd-continuous function has delta antiderivative. In particular, if $t_0 \in T$, then F defined by

$$F(t) := \int_{t_0}^{t} f(s)\Delta s, \qquad \text{for } \ t \in T \tag{13}$$

is a delta antideivative of *f*.
Theorem 4 (Existence of nabla antiderivatives). Every ld-continuous function has nabla antiderivative. In particular, if $t_0 \in T$, then F defined by

$$F(t) := \int_{t_0}^{t} f(s)\nabla s, \qquad \text{for } \ t \in T \tag{14}$$

is a nabla antideivative of *f*.

2.The Chain Rule and Special Cases

Recall that if *f* and *g* are differentiable real-valued functions defined on $\Re$, then the chain rule states :

$$\frac{d}{dt} f(g(t)) = f'(g(t)).g'(t). \tag{15}$$

Question: For which functions $f : \Re \to \Re$, $g : T \to \Re$ and time scales T the following relation is valid:

$$h^{\Delta}(t) = f^{\Delta}(g(t)).g^{\nabla}(t) \ \text{ or } \ h^{\nabla}(t) = f^{\nabla}(g(t)).g^{\nabla}(t), \tag{16}$$

where $h(t) = f(g(t))$.
Clearly from the usual chain of rule the above relation is true for the time scale $T = \Re$ for any two diffrentiable functions *f* and *g* with $h^{\Delta}(t) = h^{\nabla}(t) = h'(t)$. But if you let your time scale be the naturals *N*, then $f^{\Delta}(t) = \Delta f(t) := f(t+1) - f(t)$ and $f^{\nabla}(t) = \nabla f(t) := f(t) - f(t-1)$, and this case you can find many examples of sequences *f* and *g* for which the above relation is not valid. However, if take any sequence f and let $g(t) = t+1$ for example then the above relation in the question is valid for the delta derivative.
For the sake of completeness we state the general time scale delta chain of rule.
Theorem 5 [1] Assume $g : \Re \to \Re$ is continuous, $g : T \to \Re$ is delta differentiable on T^{κ}, and $f : \Re \to \Re$ is continuously differentiable. Then there exists a number $c \in [t, \sigma(t)]$ with

$$(f \circ g)^{\Delta}(t) = f \ \ (g(c)).g^{\Delta}(t). \tag{17}$$

Definition 2 [4] A time scale T is said to have the *H-property* if its forward jumping operator has the form: $\sigma(t) = qt + h$, for some $q \in \Re^{+}$ and $h \in \Re$. We call such a time sacle an H-time scale.

It is easy to see that $\Re$, N, $,T_q$ and T_q^h are all examples of H-time scales.

Proposition 1 Let T be an H-time scale , $f : T \to \Re$ is two times delta differentiable function and $g(t) = \sigma(t)$. Then

$$(f \circ g)^{\Delta}(t) = f^{\Delta}(g(t)).g^{\Delta}(t), \quad t \in T^{\kappa^2}. \tag{18}$$

Proof. By (8) we have $(f \circ \sigma)^{\Delta} = \left[f(t) + \mu(t) f^{\Delta}(t)\right]^{\Delta}$. Then (11) leads to

$$(f \circ \sigma)^{\Delta} = f^{\Delta}(t) + \mu^{\Delta}(t) f^{\Delta}(\sigma(t)) + \mu(t) f^{\Delta^2}(t) \tag{19}$$

and applying (8) again for the function f^{Δ} we get

$$(f \circ \sigma)^{\Delta} = (1 + \mu^{\Delta}(t)) f^{\Delta}(\sigma(t)). \tag{20}$$

Then the result follows by noting that $\mu^{\Delta}(t) = \sigma^{\Delta}(t) - 1$.
Similarly, by the help of (10) and (12) we can prove

Proposition 2 Let T be an H-time scale , $f : T \to \Re$ is two times nabla differentiable function and $g(t) = \rho(t)$. Then

$$(f \circ g)^{\nabla}(t) = f^{\nabla}(g(t)).g^{\nabla}(t), \quad t \in {}^{\kappa^2}T \ . \tag{21}$$

Remark 1 In fact, Proposition 1 and Proposition 2 are valid for time scales whose forward jump operator and backward jump operator are delta and nabla differentiable, respectively. In particular, the are valid for the time scales T_q and T_q^h.

Lemma 1 Let T be a an H-time scale then $(\sigma^n)^{\Delta}(t) = (\sigma^{\Delta}(t))^n$, for all $n \in N$ and $t \in T^{\kappa}$.
We now generalize Proposition 1 and Proposition 2, for the H-time scale with $g(t) = \sigma^n(t)$ (means the composition of σ n times) and $g(t) = \rho^n(t)$, respectively.

Proposition 4 Let T be an H-time scale , $f : T \to \Re$ is two times delta differentiable function and $g(t) = \sigma^n(t)$, for $n \in N$. Then

$$(f \circ g)^{\Delta}(t) = f^{\Delta}(g(t)).g^{\Delta}(t), \quad t \in T^{\kappa^2}. \tag{22}$$

Proof. We follow by induction. The case $n = 1$ is true by Proposition 1. Assume the result is true for $n = k$. Then, (8) applied to $(f \circ \sigma^k)(t)$ implies that

$$(f \circ \sigma^{k+1})^{\Delta}(t) = (f \circ \sigma^k \circ \sigma)^{\Delta}(t) = \left[(f \circ \sigma^k)(t) + \mu(t)(f \circ \sigma^k)^{\Delta}(t)\right]^{\Delta}, \tag{23}$$

and hence by (11) we obtain

$$(f \circ \sigma^{k+1})^{\Delta}(t) = \left[(f \circ \sigma^k)^{\Delta}(t)\right] + \left[\mu^{\Delta}(t)(f \circ \sigma^k)^{\Delta}(\sigma(t)) + \mu(t)(f \circ \sigma^k)^{\Delta^2}(t)\right] \tag{24}$$

Again (8) implies that

$$\begin{aligned}(f \circ \sigma^{k+1})^{\Delta}(t) &= (f \circ \sigma^k)^{\Delta}(t) + \mu^{\Delta}(t)(f \circ \sigma^k)^{\Delta}(\sigma(t)) + (f \circ \sigma^k)^{\Delta}(\sigma(t)) - (f \circ \sigma^k)^{\Delta}(t). \\ &= (\mu^{\Delta}(t) + 1)(f \circ \sigma^k)^{\Delta}(\sigma(t)) = \sigma^{\Delta}(t)(f \circ \sigma^k)^{\Delta}(\sigma(t))\end{aligned} \tag{25}$$

From the induction step $n = k$, it follows that

$$(f \circ \sigma^{k+1})^{\Delta}(t) = \sigma^{\Delta}(t).\left(f^{\Delta}(\sigma^k(t)).(\sigma^k)^{\Delta}(t)\right)(\sigma(t)) \tag{26}$$

By Lemma 1 and that $(\sigma^k)^\Delta(t)$ is constant we conclude that

$$(f\circ\sigma^{k+1})^\Delta(t)=(\sigma^{k+1})^\Delta(t).f^\Delta(\sigma^{k+1}(t)), \qquad (27)$$

and the proof is complete.

Similarly and by the help of (10) we can also state a chain rule result for the nabla derivative.

Proposition 5 Let T be an H-time scale , $f:T\to\Re$ is two times delta differentiable function and $g(t)=\rho^n(t)$, for $n\in N$. Then

$$(f\circ g)^\nabla(t)=f^\nabla(g(t)).g^\nabla(t), \quad t\in T^{\kappa^2}.$$

For the purpose of comparisons we state the following chain rule in the theory of q-calculus, which is Lemma 2.1 of [3] .

Proposition 6 Let $g(x)=cx^k$, where c and k are constants and f any function. Then

$$D_q(f\circ g)(x)=\left(D_{q^k}(f)\right)(g(x)).D_q(g)(x) \qquad (28)$$

where $D_q(f)(x)$ means the nabla derivative of f on the time scale T_q, $0<q<1$.

As direct cosequences of Proposition 4 and Proposition 5, we state the following two useful corollaries

Corollary 1 Let T be an H-time scale and $f:T\to\Re$ be an rd-continuous function. If $h(t)=\sigma^n(t)$, $g(t)=\sigma^k(t)$ and $F(t)=\int_{h(t)}^{g(t)}f(s)\Delta s$, $n\in N$, $k\in N$. Then

$$F^\Delta(t)=f(g(t)).g^\Delta(t)-f(h(t)).h^\Delta(t). \qquad (29)$$

Corollary 2 Let T be an H-time scale and $f:T\to\Re$ be an ld-continuous function. If $h(t)=\rho^n(t)$, $g(t)=\rho^k(t)$ and $F(t)=\int_{h(t)}^{g(t)}f(s)\nabla s$, $n\in N$, $k\in N$. Then

$$F^\nabla(t)=f(g(t)).g^\nabla(t)-f(h(t)).h^\nabla(t). \qquad (30)$$

REFERENCES

Bohner M, Peterson A, Dynamic Equations on Time Scales: An introduction with applications. Birkhauser, Boston, (2001).

Bohner M, Peterson A, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, (2002).

Ernst T, The history of q-calculus and a new method (Licentiate Thesis), U.U.D.M. Report 2000: 16; http://www.math.uu.se/thomas/Lics.pdf.

Rui A. C. Ferreira and Delfim F. M. Torres, Higher-Order Calculus of Variations on Time Scales.arXiv: 0706.3141v2 [math.OC] 30 Sep 2007.