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Numerical Solution of a Class of Nonlinear Emden-Fowler Equations by Using Differential Transform Method

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Abstract

In this paper, approximate and/or analytical solutions of the Emden-Fowler type equations in the second-order ordinary differential equations are obtained by applying differential transformation method (DTM). In this work, two examples are given and DTM yields solutions in convergent series forms with easily computable terms by using a reliable algorithm and in most cases iterations leads to the high accuracy of the solutions. Comparisons with the analytical solutions and the solutions found by the DTM show the efficiency in solving equations with singularity. In fact, the numerical results reveal that DTM is very efficient, reliable and accurate.

Keywords: Nonlinear differential equations**,** Singular initial value problems, Differential transform method, Emden-Fowler equations, Numerical solution.

1. Introduction

Many problems in mathematical physics, theoretical physics and chemical physics are modelled by the so-called initial value and boundary value problems in the second-order nonlinear ordinary differential equations. These equations are difficult to be solved analytically and sometimes it is impossible then application must be made to relevant numerical methods such as shooting method, finite difference etc. In recent years, differential transform method has been used to solve this type of equations [6], [7], [9], [10], [11], [12], [13].

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In this work, the differential transform method is used to investigate the numerical and analytical approximate solutions of the nonlinear singular initial value problems of Emden-Fowler Type. Initial value problems in the second order is considered which occur in applied mathematics, astrophysics [1-3] and the numerical solution of the Emden-Fowler equation, and the other linear and nonlinear singular initial value problems, plays very important role because of the singularity behaviour at the origin. In this paper is presented a study of the singular initial value problem of the form

$$y'' + \frac{a}{x} y' + b x^{m-1} y^n = 0, \quad 0 < x \le 1, a \ge 0, \tag{1}$$

under the following initial conditions

$$y(0) = A, \; y'(0) = 0, \tag{2}$$

where A, a, b, m are constants, $n \neq 0$, $n \neq 1$. When $m = 1$, $a = 2$ and $b = 1$ Equation (1) reduces to the Lane –Emden equation [4-13].

In this work, the technique that we used is the differential transformation method, which is based on Taylor series expansion and is a semi numerical-analytical method for solving differential equations. The concept of this method was first introduced by Zhou [5] in 1986, and it was applied to solve linear and nonlinear initial value problems in electric circuit analysis. This method gives an analytical solution in the form of a polynomial and it is different from the higher order Taylor series method. In fact this method takes computationally short time of the necessary derivatives of the data functions rather than Taylor series method. Therefore, differential transform method reduces the size of computational domain and can be applied to many problems easily.

In the present paper, differential transform method is applied to solve the Emden-Fowler equations as a class of nonlinear singular initial value problems and the paper is organised as follows: In Section 2, the differential transform method is introduced. In Section 3, the method is implemented to two examples to illustrate the efficiency and simplicity of the method, and conclusion is given in Section 4.

2. Differential Transform Method

In the following, we introduce the main features of the differential transformation method [2] according to the differential transform of the nth derivative of a function [$y(x)$] in one variable is defined as follows:

$$Y(n) = \frac{1}{n!} \left[\frac{d^n y(x)}{dx^n} \right]_{x=x_0} . \tag{3}$$

In Equation (3), $y\,(x)$ is the original function and $Y\,(n)$ is the transformed function and the differential inverse transform of $Y\,(n)$ is defined as follows:

$$y(x)=\sum_{n=0}^{\infty}Y(n)(x-x_0)^n. \tag{4}$$

In real applications, function $y\,(x)$ is expressed by a finite series and Equation (4) can be written as

$$y(x)=\sum_{n=0}^{k}Y(n)(x-x_0)^n. \tag{5}$$

From Equation (3), we obtain

$$y(x)=\sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!}\left[\frac{d^n y(x)}{dx^n}\right]_{x=x_0}. \tag{6}$$

Actually Equations (6) implies that the concept of differential transform is derived from Taylor series expansion. Although DTM is not able to evaluate the derivatives symbolically, relative derivatives can be calculated by an iterative way which is described by the transformed equations of the original function. In this study Equation (6) also implies $y(x)=\sum_{n=k+1}^{\infty}Y(n)(x-x_0)^n$ is negligibly small. In fact n is decided by the convergence of natural frequency. The following theorems that can be deduced from Equation (3) and (4) are given below [5-6-7-8]:

Theorem 1. If $y(x)=y_1(x)\pm y_2(x)$, then $Y(n)=Y_1(n)\pm Y_2(n)$.

Theorem 2. If $y(x)=cy_1(x)$, then $Y(n)=cY_1(n)$, where c is a constant.

Theorem 3. If $y(x)=\dfrac{d^k y_1(x)}{dx^k}$, then $Y(n)=\dfrac{(n+k)!}{n!}Y_1(n+k)$.

Theorem 4. If $y(x)=y_1(x)\,y_2(x)$, then $Y(n)=\sum_{n_1=0}^{n}Y_1(n_1)Y_2(n-n_1)$.

Theorem 5. If $y(x)=x^k$, then $Y(n)=\delta(n-k)$,

where $\delta(n-k)=\begin{cases}1, & n=k\\ 0, & n\neq k.\end{cases}$

Theorem 6. If $y(x) = y_1(x)\, y_2(x) \ldots y_{k-1}(x)\, y_k(x)$, then

$$Y(n) = \sum_{n_{k-1}=0}^{n} \sum_{n_{k-2}=0}^{n_{k-1}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} Y_1(n_1) Y_2(n_2 - n_1) \ldots Y_{k-1}(n_{k-1} - n_{k-2}) Y_k(n - n_{k-1}).$$

3. Numerical Examples

In this section differential transform method will be applied to some special cases of the class of nonlinear initial value problem by using the proposed algorithm discussed above.

Example 1. We will consider the nonlinear singular initial value problem

$$y'' + \frac{3}{x} y' + 2x^2 y^2 = 0 \tag{7}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{8}$$

The initial condition in equation (8) can be transformed at $x_0 = 0$ as

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{9}$$

Now, applying the above theorems to equation (7), we obtain the following recurrence relation

$$Y(k+1) = -\frac{2}{(k+1)(k+2)} \times \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1) Y(k_2 - k_1)\, \delta(k - k_2 - 3) \tag{10}$$

Using Equations (9), (10) and (5), for n = 20, the following series solution is obtained:

$$y_{20}(x) = 1 - \frac{1}{12} x^4 + \frac{11}{240} x^8 - \frac{1}{60480} x^{12} + \frac{1}{136080} x^{16} - \frac{97}{342144000} x^{20} \tag{11}$$

In this study, we did not give the value of $y_{30}(x)$, $y_{40}(x)$ and $y_{50}(x)$ due to the length of expansion of the series.

Table 1. Comparison of numerical results with the Maple11 solution for different values of n

x	$n=10$	$n=20$	$n=30$	$n=40$	*Maple11* Solution
0.1	0.99999166	0.99999166	0.99999166	0.99999166	0.99999166
0.2	0.99986667	0.99986667	0.99986667	0.99986667	0.99986667
0.3	0.99932527	0.99932527	0.99932527	0.99932527	0.99932527
0.4	0.99786939	0.99786939	0.99786939	0.99786939	0.99786939
0.5	0.99480794	0.99480789	0.99480789	0.99480789	0.99480789
0.6	0.98926998	0.98926959	0.98926959	0.98926959	0.98926958
0.7	0.98023186	0.98022937	0.98022937	0.98022937	0.98022937
0.8	0.96656571	0.96655342	0.96655342	0.96655342	0.96655340
0.9	0.94711861	0.94706857	0.94706857	0.94706857	0.94706857
1.0	0.92083333	0.92065852	0.92065853	0.92065853	0.92065853

Example 2. We will consider the nonlinear singular initial value problem

$$y'' + \frac{8}{x} y' + xy^2 = x^5 + x^4 \tag{12}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{13}$$

The initial condition in equation (13) can be transformed at $x_0 = 0$ as

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{14}$$

Now, applying the above theorems to equation (12), we obtain the following recurrence relation

$$Y(k+1) = -\frac{2}{(k+1)(k+8)} \left\{ \delta(k-6) + \delta(k-5) - \sum_{k_2=0}^{k} \sum_{k_1=0}^{k_2} Y(k_1) Y(k_2 - k_1) \delta(k - k_2 - 2) \right\}. \tag{15}$$

Table 2. Comparison of numerical results with the Maple11 solution for different values of n

x	$n = 10$	$n = 20$	$n = 30$	$n = 35$	*Maple11* Solution
0.1	0.99996668	0.99996668	0.99996668	0.99996668	0.99996668
0.2	0.99973433	0.99973433	0.99973433	0.99973433	0.99973433
0.3	0.99911219	0.99911219	0.99911219	0.99911219	0.99911219
0.4	0.99793933	0.99793933	0.99793933	0.99793933	0.99793933
0.5	0.99612622	0.99612622	0.99612622	0.99612622	0.99612622
0.6	0.99372096	0.99372097	0.99372097	0.99372097	0.99372097
0.7	0.99100452	0.99100463	0.99100463	0.99100463	0.99100463
0.8	0.98861874	0.98861927	0.98861927	0.98861927	0.98861928
0.9	0.98772971	0.98773191	0.98773191	0.98773191	0.98773192
1.0	0.99022826	0.99023588	0.99023587	0.99023587	0.99023588

Evaluating the recurrence relation in Equation (15) and the transformed boundary conditions in Equation (14), $Y(k)$ for $k \geq 2$ are easily obtained and then using the inverse transformation rule in Equation (5), we get the some series solution is evaluated up to $n = 35$ but we only present the following series solution for $n = 20$ due to the length of expression, the value of $y_{30}(x)$ and $y_{35}(x)$ is not given.

$$y_{20}(x) = 1 - \frac{1}{30}x^3 + \frac{8}{585}x^6 + \frac{1}{98}x^7 - \frac{37}{187200}x^9 - \frac{1}{8330}x^{10} + \frac{367}{64022400}x^{12} + \frac{23}{6497400}x^{13}$$
$$- \frac{55049}{85830030000}x^{15} - \frac{2867}{3586564800}x^{16} - \frac{1}{3918432}x^{17} + \frac{1618343}{102996036000000}x^{18} \tag{16}$$
$$+ \frac{164117}{8858815056000}x^{19} + \frac{29}{5289883200}x^{20}$$

Comparisons of numerical results with the numerical solutions of the nonlinear singular initial value problems of Emden-Fowler type equations given in (7-8) and (12-13) evaluated using Maple11 commands for $n = 10$, $n = 20$, $n = 30$ and $n = 40$ are reported in Table 1 and Table 2. It can be seen from Table 1 and 2; the results obtained with differential transform method for $n = 40$ and $n = 35$ respectively are eight digits accurate.

4. Conclusion

In this work, we calculated the numerical and/or analytical solutions of some non-linear singular initial value problems of Emden-Fowler Type by using the differential transform method (DTM). Two equations are solved and approximate solutions are found. Our present method is a very fast convergent, effective and reliable tool for solving the Emden-Fowler equations as singular initial value problem. Indeed it is observed that differential transform method avoids the tedious work needed by traditional techniques and then we got more accurate numerical solutions as shown in Table 1 and 2.

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