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Research Article

Examination of Mathematical Work Performed in Mathematical Modelling Task in the Light of MWS: The Case of Pre-Service Elementary Mathematics Teachers*

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Abstract – Mathematical modeling plays a crucial role for bridging the gap between mathematics and real-life contexts. The theory of Mathematical Working Spaces (MWS) provides a suitable framework for analyzing the mathematical modeling process by incorporating both epistemological and cognitive dimensions of mathematical work. This study investigates the mathematical work carried out by pre-service elementary mathematics teachers as they engage with modeling-based mathematical tasks, interpreted the lens of MWS. Employing a qualitative case study design, the research involved 15 pre-service teachers selected through criterion sampling. Data were collected via video recordings and written documents capturing the participants' solution processes, which were guided by exploratory questions. The data were analyzed descriptively, focusing on the components of the MWS framework. The findings reveal that pre-service teachers frequently employed components of the epistemological plane during the problem comprehension and simplification stages of the modeling cycle. The mathematization stage predominantly took place within the semiotic-instrumental plane, while the 'working mathematically' stage involved actions spanning across multiple planes. Future research may explore the instructional practices of pre-service teachers as they implement mathematical modeling in classroom environment, guided by the principles of MWS theory.

Keywords: Mathematical working spaces, mathematical modeling, mathematical work, area measurement.

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Introduction

Mathematical literacy is recognized as a crucial skill for developing individuals who understand the role of mathematics in real life, respond constructively to encountered problems, and think deeply about how to apply mathematical knowledge (Altun, 2020; Ministry of National Education [MoNE], 2023). According to the Programme for International Student Assessment (PISA), which evaluates mathematical literacy skills, the performance of students in Türkiye has shown an upward trend. Nevertheless, almost half of the students still perform at the lower proficiency levels (MoNE, 2023). Although fostering mathematical literacy is among the specific objectives of the Mathematics Curriculum (MoNE, 2018), students continue to struggle in achieving higher proficiency levels. At this juncture, the mathematical modeling process, which establishes a link between real life and mathematics, becomes particularly significant.

The mathematical modeling process is regarded as a cyclical progression involving the mathematization of a real-world situation, its translation into the mathematical world, the derivation of a mathematical result, and the interpretation of this result within both mathematical and real-life contexts (Lesh & Doerr, 2003). Various researchers have described this process in different ways (Abrams, 2001; Ang, 2010; Mason, 1988). Borromeo Ferri (2006) and Blum and Leiß (2007) characterize the mathematical modeling cycle through stages such as understanding the problem, simplifying, mathematizing, working mathematically, interpreting, verifying, and presenting, emphasizing the cognitive aspect of problem-solving (Figure 1). Mathematical models and various representations developed during the modeling process facilitate the understanding of abstract mathematical concepts and allow certain real-life situations to be expressed through mathematical language. Moreover, technological tools employed in this process enable the translation of real-world problems into mathematical representations, support the derivation of solutions and aid in interpreting the results within a real-world context (Hall & Lingefjard, 2016).

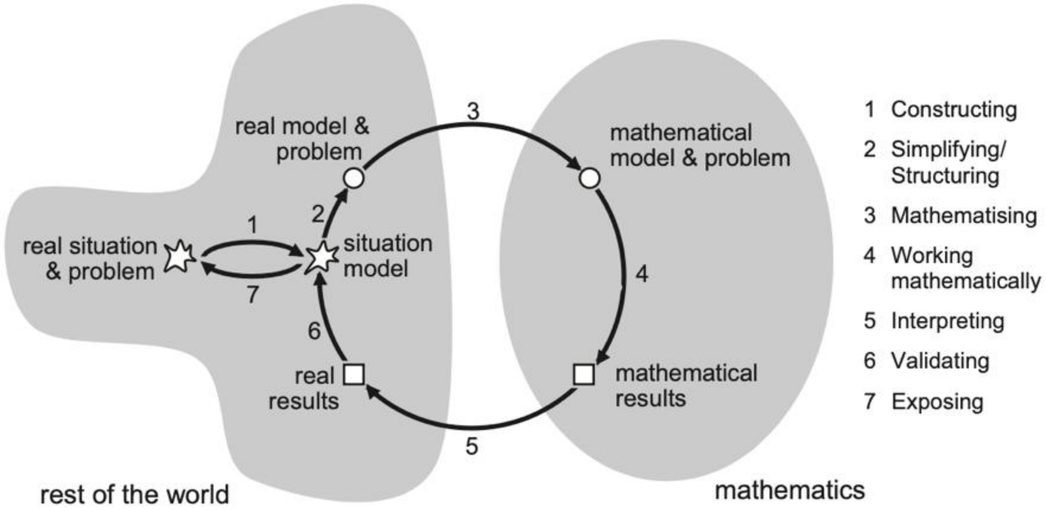


Figure 1 Mathematical Modeling Cycle (Blum & Leiß, 2007)

Hall and Lingefjard (2016) emphasize that reasoning and argumentation skills are essential for translating problem-related data into mathematical expressions that can be processed using technological tools, as well as for interpreting the mathematical results obtained through these tools. They further argue that the development of such skills is more critical than merely using mathematics as a computational instrument. Accordingly, they advocate for the integration of mathematical modeling into classroom practices to enable students to transform real-life situations into mathematical problems, use mathematical language accurately and effectively, interpret and report mathematical finding, and proficiently utilize technological tools and software. Studies examining the competencies of teachers and pre-service teachers in integrating mathematical modeling into classroom settings have consistently highlighted difficulties encountered during the modeling process (Çiltaş, 2011; Hidayat & Iksan, 2018; Korkmaz, 2010; Paolucci & Wessels, 2017).

The Mathematical Working Spaces (MWS) theory (Kuzniak, 2014), which establishes a link between the epistemological and cognitive dimensions of mathematical work, has become an important framework for interpreting, defining, and designing mathematical tasks (Kuzniak et al., 2016). MWS theory explains the actions performed by an individual during a mathematical work through processes known as constructions, which are grounded in the components on the epistemological and cognitive planes, as well as the interactions between these components (Kuzniak et al., 2016).

In the MWS framework, which is typically visualized as a triangular prism, the lower base represents the epistemological plane, while the upper base corresponds to the cognitive plane. On the epistemological plane, the components consist of representamen, artifacts, and theoretical references, which encapsulate the epistemological foundations necessary for mathematical activity. On the cognitive plane, the components include visualization, construction, and proof, which represent the actions performed during mathematical work. The processes that occur through the interaction of components across these two planes are referred to as genesis processes (Kuzniak & Richard, 2014; Kuzniak et al., 2016).

The representation component pertains to the use of mathematical concepts, symbols, and figures associated with those concepts, whereas the visualization component involves re-expressing mathematical representations in different forms, making sense of them, and articulating ideas either verbally or in writing (Kuzniak, 2022). The interaction between representation and visualization components leads to semiotic genesis (Henríquez Rivas et al., 2021; Kuzniak & Nechache, 2021; Verdugo Hernández et al., 2023).

The artifact component concerns the use of technological tools, such as geometric drawing tools or software. Conversely, the construction component pertains to the actions initiated through artifacts, particularly those realized using dynamic geometry software (Kuzniak, 2022). Instrumental genesis arises from the interaction between artifacts and construction components, involving the functional use of artifacts to construct mathematical objects (Henríquez Rivas et al., 2021; Kuzniak & Nechache, 2021; Verdugo Hernández et al., 2023).

The theoretical references component involves the utilization of mathematical properties, while the proof component focuses on employing these properties during the reasoning process to justify mathematical actions (Kuzniak, 2022). Discursive genesis emerges through the interaction between theoretical references and proof components and encompasses the process of constructing mathematical reasoning and inferences using definitions, theorems, and properties (Henríquez Rivas et al., 2021; Kuzniak & Nechache, 2021; Verdugo Hernández et al., 2023). A diagram illustrating the components and genesis processes of the MWS is presented in Figure 2.

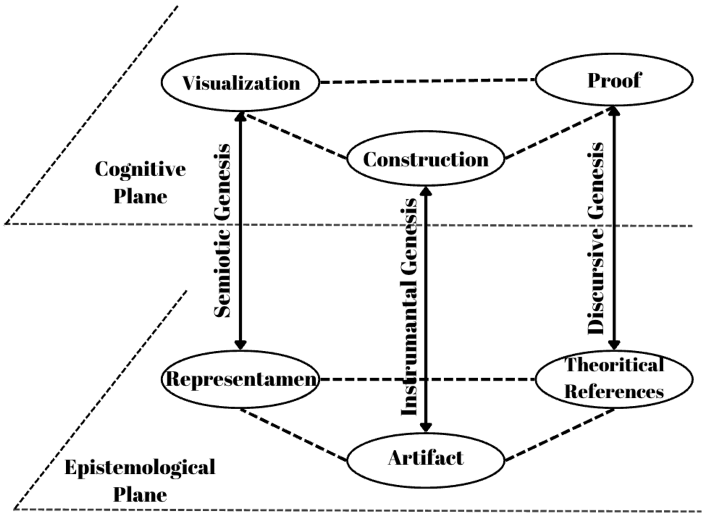


Figure 2 The Components and Formation Processes Of The MWS (adapted from Kuzniak et al. (2016))

The interaction among the three genesis processes gives rise to three distinct vertical planes: semiotic-instrumental vertical plane [Sem-Ins], instrumental-discursive vertical plane [Ins-Dis], and semiotic-discursive vertical plane [Sem-Dis] (Kuzniak, 2022). When discursive reasoning skills are employed in the visualization of mathematical objects, allowing for the interpretation of mathematical data and the visualization of results, mathematical work situated within the [Sem-Dis] plane. If artifacts are utilized during mathematical work primarily for exploration, construction, and investigation without serving as verification tools, the work is situated within the [Sem-Ins] plane. Conversely, when the artifacts are used to develop reasoning or discourse about how and why a constructed structure is created, or to demonstrate geometric constructions related to proofs in theoretical references, mathematical work takes place within the [Ins-Dis] plane (Gómez-Chacón & Kuzniak, 2015; Henríquez Rivas et al., 2023; Henríquez-Rivas & Kuzniak, 2021; Kuzniak & Richard, 2014). A diagram illustrating the vertical planes emerging from the interactions among the genesis processes of MWS is presented in Figure 3.

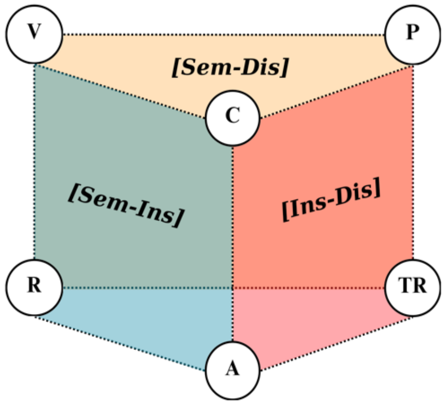


Figure 3 The Vertical Planes of the MWS (adapted from Kuzniak et al. (2016))

MWS can be considered a theoretical framework that enables an in-depth explanation and analysis of the actions undertaken to solve problem situations involving mathematical modeling activities (Delgadillo et al., 2017). Several studies in the literature have explored the integration of mathematical modeling and MWS theory. Derouet et al. (2017) argue that a deeper analysis of the modeling process can be achieved by using the MWS framework to explain how different types of mathematical work are carried out during the mathematization, working mathematically, and interpretation phases, rather than merely considering the modeling cycle as a whole.

Following this perspective, Nechache (2018) and Derouet (2019) examined the genesis processes described in MWS theory during the solution of mathematical tasks related to probability. Similarly, Moutet (2019) and Reyes Avendaño (2020) analyzed mathematical works situated at the intersection of physics and mathematics using the MWS framework. Delgadillo et al. (2017) specifically focused on the mathematization process during the solution of a modeling task, while Psycharis et al. (2021) conducted an a priori analysis of the same task. Lagrange (2021) studied mathematical work aimed at modeling the main cable of a suspension bridge through functional relationships. Recently, Verdugo Hernández et al. (2023) analyzed the modeling activities of engineering students by examining the connections between Blomhøj's modeling cycle and the MWS framework.

Although numerous studies have investigated the relationship between mathematical modeling and MWS theory, it is known that there are only a limited number of studies examining the modeling processes of pre-service elementary mathematics teachers, who are expected to bring mathematical modeling into the classroom environment, through the lens of MWS framework. Moreover, although the MWS theory has been applied various mathematical domains, its systematic use in analyzing the detailed processes involved in mathematical modeling activities remains limited (Lagrange et al., 2022).

This study seeks to fill this gap by applying the MWS framework to investigate how pre-service elementary mathematics teachers interact with epistemological and cognitive components during modeling tasks. By exploring the dynamic interplay among representation, artifact, and theoretical references throughout the modeling process, the study aims to provide new insights into the cognitive structures underlying modeling activities from both theoretical and educational perspective.

Recognizing that the underlying causes of difficulties encountered during mathematical modeling activities can be better understood by analyzing the actions performed by pre-service elementary mathematics teachers, the aim of this study was to examine these actions through the lens of the MWS theory. Accordingly, the research question guiding this study is as follows: What types of interactions occur among the components of MWS framework during the mathematical modeling processes of pre-service elementary mathematics teachers?

Method

Research Design

In this study, which aims to reveal the actions performed by pre-service elementary mathematics teachers during the solution process of a mathematical modeling task by considering the components and genesis processes of the MWS theory. To achieve this aim, a case study design – one of the qualitative research methods – was employed. A case study is a research design that allows for the detailed examination of one or more situations using data collected from various sources (Yıldırım & Şimşek, 2021). The case study approach was chosen to capture the intricate details of the actions performed during mathematical work, to provide comprehensive explorations regarding these actions, and to evaluate the mathematical work based on the interpretations made.

Participants

The study group was determined through criterion sampling, a purposeful sampling method that enables in-depth analysis by selecting information-rich cases. Criterion sampling is applied when individuals, events or objects that meet a predetermined criterion are selected as units of observation (Büyüköztürk et al., 2008). In this study, the criterion for selecting participants was that pre-service teachers had successfully completed a course involving mathematical modeling. This course included instruction on the stages of the mathematical modeling cycle, the characteristics of effective modeling problems, and solution strategies for real-world mathematical tasks. As a result, participants had prior theoretical knowledge and practical experience regarding mathematical modeling processes.

Selecting participants with this background ensured that they could meaningfully engage with the cognitive and epistemological components targeted by the MWS. Accordingly, the study group consisted of 15 pre-service elementary mathematics teachers enrolled at a university in western Türkiye. One limitation of the study is the small sample size, which restricts the transferability of the findings. However, the study aimed for depth and contextual understanding rather than statistical generalizability, in line with the principles of qualitative research design. Criterion sampling was used to select participants who had both exposure to and successful completion of a mathematical modeling course, ensuring that all participants processed foundational knowledge and skills related to modeling process. This selection aimed to focus the study on participants capable of engaging with the epistemological and cognitive demands of the modeling tasks analyzed within the MWS theory.

To facilitate collaborative work, the participants were organized into groups of 3-4 pre-service teachers each. The grouping was done voluntarily, while also considering their final course achievement scores in the mathematical modeling course. Pre-service teachers with different levels of achievement were distributed across groups to form heterogeneous groups. The data collection was conducted separately with each group: each group completed the task at same times, independent of the other groups. Each group's modeling process was recorded using a separate video camera, allowing for detailed documentation of their group-specific mathematical work.

Data Collection Process

The data were collected during spring term of 2022-2023 academic year at a university in western Türkiye. The data collection process lasted 60 minutes of a lesson, during which each group of pre-service teachers completed the modeling activities independently. Each group's work was recorded using a separate video camera. These recordings, along with the written solution documents, formed the primary data sources for the study. Prior to data collection, ethical approval was obtained from the ethics committee. Additionally, informed consent was obtained from all participants, who were assured that their participation was voluntary and that their identities would remain confidential.

To design the mathematical modeling activity, the Ministry of National Education Mathematics Curriculum (MoNE, 2018, 2024) was examined, and the concepts suitable for mathematical modeling were identified. The concept of area measurement was selected because it is addressed across almost all levels of mathematics education and allows for rich mathematical exploration using different techniques and geometric shapes.

In constructing the problem situation, the relationship between real-life contexts and mathematics (Blum & Leiß, 2007), as well as the capacity of real-world problems to promote thinking and planning skills (Herget & Torres-Skoumal, 2007), were considered. Additionally, the basic components and principles essential to effective modeling activities were incorporated into the problem design (Bukova Güzel et al., 2016; Chamberlin & Moon, 2005; Tekin-Dede & Bukova-Güzel, 2018).

While preparing the model-eliciting activity, a news article about the project to cover the floor of the Colosseum in Rome, Italy, was selected as the context for the problem situation (Pinkstone, 2020). An introductory text was created to capture the reader's attention and establish the context of the problem based on the news article. A real-life situation that would be meaningful and engaging for individuals was deliberately chosen. Readiness questions, including simple comprehension questions related to the introductory article, were also developed.

In designing the mathematical work intended to solve the problem, it was considered that a mathematical model could be formulated by using mathematical representation, artifact, and theoretical references, while also taking into account the geometric structure of the Colosseum. The problem situation was designed to guide students in creating a mathematical model through collaborative group work, evaluating the validity of their solutions through group discussions, articulating the solution process in detail, and generalizing their solutions to similar contexts. The problem situation is presented as follows: determining the area to be covered by the retractable wooden floor planned to be built above the underground tunnels of the Colosseum, as part of a project initiated by the Italian Ministry of Culture. The introductory article and problem statement are provided in Table 1.

Table 1 Introductory Article and Problem Statement

"Located in Rome, the capital of Italy, the Colosseum is one of the iconic symbols of the Roman Empire. The 2,000-year-old Colosseum is a megastructure that can accommodate more than 50,000 spectators for events such as gladiator fights, public performances, and animal hunting, which were the entertainment of the Roman people. After centuries of ruin due to earthquakes and stone theft, archaeological work in the 19th century removed the base of the structure. Thus, it revealed a network of underground tunnels where gladiators and animals were kept.

In a project prepared by the Italian Ministry of Culture, a retractable wooden floor was planned over the Colosseum's underground tunnels so that visitors could see this gigantic arena through the eyes of the gladiators and organize cultural events inside the Colosseum.

In order to contribute to this project, determine the area to be covered by the floor to be built above the underground tunnels."

In this study, a single modeling problem was selected to enable an in-depth exploration of the mathematical work processes of pre-service teachers, in alignment with the principles of case study research. Focusing on one rich and complex task allowed for a comprehensive analysis within the framework of the MWS. This situation was the limitation of this study. While this approach provided detailed insights into the mathematical work processes, it may limit the generalizability of the findings across different types of modeling tasks.

After each group completed their mathematical work on the modeling task presented in Table 1, their different solution strategies were shared and discussed in a whole-class environment. This discussion took place after the group work and data collection had been completed. Although the mathematical work was carried out in separate sessions with each group, a subsequent classroom presentation was organized to encourage collective reflection on the problem- solving processes.

In addition to the readiness questions accompanying the problem, guided discovery questions were provided to stimulate mathematical work, aligned with the stages of the mathematical modeling cycle and the components of the epistemological plane in the MWS. Three domain experts were consulted during the development of the problem situation and the formulation of the guided discovery questions to ensure the validity and relevance of the designed materials.

A pilot study was conducted with a group separate from the main study group to test implementation of the mathematical task. Based on expert feedback, necessary revisions were made to enhance the clarity of expressions and visuals in the problem statement and guided discovery questions. Following these revisions, the final version of the problem statement and guided discovery questions was applied to the main study group. After an initial period of individual work to comprehend the problem, group work commenced. During the group sessions, the actions taken by the pre-service teachers to solve the problem were closely monitored, necessary notes were taken, and interventions were made when necessary to address situations that hindered the progression of the mathematical work. The purpose of these interventions was to ensure that the mathematical work was continued without disruption (Bukova Güzel et al., 2016).

Each group was provided with a computer to support their modeling activities, which could involve the use of tools such as software and internet resources. A seating arrangement was organized to facilitate effective group collaboration. Pre-service teachers were instructed to verbalize their thinking processes and explain their ideas in detail, even when uncertain.

During the group modeling activities, audio and video recordings were made using a separate camera for each group to ensure the completeness of the data and to capture all interactions. Each group worked independently, and no joint interviews or discussions with all 15 participants were conducted during the data collection phase. In addition to the video recordings, written documents were obtained by asking the participants to record their solutions on worksheets.

Data Analysis

This study aimed to investigate the cognitive processes that occur among the components of the MWS during the mathematical work performed to solve a mathematical task involving mathematical modeling activities. Hankeln and Hersant (2020) assert that the Blum and Leiß modeling cycle is a suitable framework for cognitive analysis. Accordingly, the analysis method proposed by Kuzniak and Nechache (2021), based on cognitive task analysis (Darses, 2001), was utilized.

The data analysis was conducted in three stages. In the first stage, the actions taken to solve the problem were classified. For this purpose, video recordings of the mathematical work were transcribed and put into written form. The transcribed texts were then segmented into sections corresponding to the stages of the mathematical modeling cycle, as the problem-solving actions naturally aligned with these stages. The actions performed during the understanding and simplification stages, as well as those during the interpretation and verification stages, were considered together due to the similarity in their cognitive demands.

In the second stage, the analysis focused on how the actions identified during mathematical work interacted with the components of the MWS framework and how they mapped onto the MWS diagram. Each segment of mathematical work was examined in terms of the epistemological components, genesis processes, and the characteristics of the vertical planes.

In the final stage, it was determined whether the mathematical work process was completed based on the cumulative findings from the first two stages, and the genesis and vertical planes that framed the work were identified. Through this comprehensive approach, the interaction between the components of the MWS during the modeling processes was revealed, and a corresponding MWS diagram for each group's mathematical work was constructed.

The study employed a descriptive analysis approach to examine the mathematical work of pre-service teachers within the framework of MWS. The analysis was guided by pre-determined themes derived from the components of the MWS, including representations, artifacts, theoretical references elements of the epistemological plane, as well as semiotic, instrumental, and discursive genesis. Coding units were defined as meaningful episodes in which pre-service teachers carried out specific mathematical actions of verbal/written expressions related to one of more components of the MWS.

To ensure reliability, two researchers independently examined the data and coded episodes based on the established framework. Discrepancies between the coders were discussed and resolved through consensus. When necessary, a third expert was consulted to support the final decision. The validity of analysis was enhanced through the data triangulation (video recordings, worksheets, observational notes), expert consultation, and the inclusion of direct quotes and visuals during the presentation of findings. The researcher was also present as a participant observer during the implementation process (Erlandson et al., 1993; Miles & Huberman, 1994; Yıldırım & Şimşek, 2021).

Findings and Discussions

Mathematical Work of Group 1

The mathematical work of Group 1 began with individual efforts to understand the problem situation. During the individual study phase, participants responded to the readiness questions and directed discovery questions, which were designed to stimulate the development of strategies for solving problem. When group work commenced, it was observed that the pre-service teachers restarted the problem situation through group discussions. They indicated that they fully understood the problem, although minor mistakes were made in identifying the given and required data. Nevertheless, it was noted that Group 1 successfully established a relationship between the given and required data and expressed that they could reach a result close to the actual solution. The pre-service teachers planned to use their mobile phones and the Internet to find alternative photographs of the Colosseum, intending to concretize the geometric shape of the target area by utilizing the dynamic geometry software GeoGebra.

At the stage of understanding and simplifying the problem stage, the use of the Internet to access alternative photographs and transferring these images into GeoGebra indicated the use of artifacts. Identifying the geometric shape to be measured as a circle based on a photograph pointed to the use of mathematical representation. The attempt to concretize the circular shape using the photograph demonstrated the occurrence of semiotic genesis. Simultaneously, relating the real-life measurement of a reference object to its measurement in the photograph through the concept of ratio suggested the use of theoretical references.

Through the examination of various photographs retrieved from the Internet, the group simplified the problem situation by identifying the geometric shape to be measured as a circle and by partially determining the variables necessary for solving the problem. However, an inaccurate assumption was made, as they incorrectly considered the shape to be a perfect circle. Initially, the group reasoned that the necessary data could be obtained by establishing a ratio between the real-life measurement of an object visible in the photograph and its measurement in the image itself. This reasoning was articulated by PT1 as follows:

PT1: ...for example, the length of the truck in the photograph is 2 cm. In real life, the length of this truck would be between 8 m and 13 m. We can ratio the length of the two to the diameter of the circle.

Due to the perspective issues inherent in the original photograph, the pre-service teachers decided to use bird's eye view of the Colosseum, which they accessed via a mobile phone search. This new image provided more realistic data for modeling. While initially considering various strategies, such as associating the height of the Colosseum's arches with human dimensions, the group ultimately decided to base their solution on the truck's size depicted in the photograph. This decision process was reflected in the following dialog:

PT2: Let's go from the truck anyway.

PT1: Let's find the length of the truck horizontally. Was the other shape a circle? Is it a full circle? If it is a full circle, we find the diameter of this place.

PT3: But there will be a difference because of the perspective.

PT1: Let it be that much difference. Let's find an approximation. If we find one side, the other side will be the same amount anyway.

PT2: If we find the length of the diameter of the circle, we already find the measure of its area.

PT1: Yes. Then we find the diameter from the length of the truck.

This dialog shows that by identifying the truck in the photograph as a reference object, the pre-service teachers planned to determine the diameter of the assumed circular shape at the base of the Colosseum using proportional reasoning.

In this mathematization phase, the use of mobile phones and the Internet to search for additional photographs pointed to the use of artifacts, while the proportional reasoning based on the reference object reflected the application of theoretical references. The realization that different data could be obtained through such proportional reasoning was evaluated as evidence of discursive genesis.

Following the mathematization of the problem situation, the pre-service teachers in Group 1 proceeded to the working mathematically phase. They transferred the selected photograph into the GeoGebra dynamic geometry software and identified two point on the base area, which they assumed to be circular. Using GeoGebra's measurement tools, they determined the distance between these two points by selecting the 'length measurement' function (Figure 4). Subsequently, similar actions were performed to measure the length of the truck depicted in the photograph. A proportion constant was established by relating the two measured lengths – the truck length and the distance between the selected points.

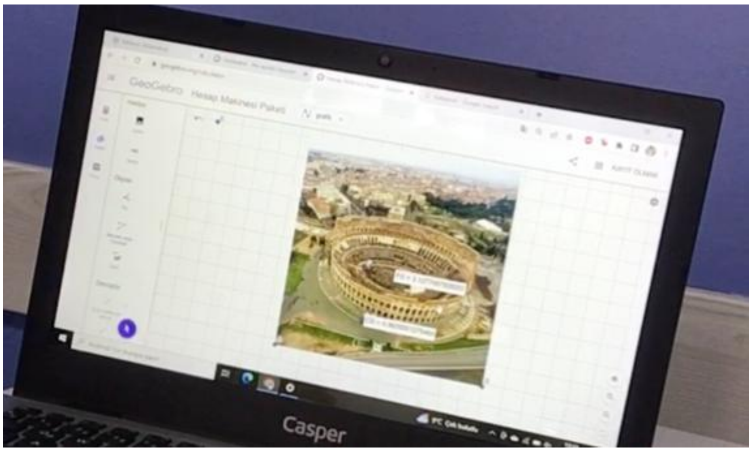


Figure 4 Visual Illustrating Group1's Use of the Length Measurement Tool in GeoGebra

To transfer the data obtained from the photograph into real-life measurements, the pre-service teachers conducted an Internet search to find the average real-life length of a truck. They determined that the real-life length of the truck was approximately 5,5 meters, consistent with their online findings.

Using the proportion constant and the measured distances, they calculated that the diameter of the assumed circular area was approximately 47,3 meters, leading to a radius of about 23,6 meters. After rounding, they accepted the radius as 24 meters. Applying the formula for the area of a circle, they concluded that the area was approximately 1808 square meters.

During the working mathematically phase, the purposeful use of artifacts – such as transferring the photograph into GeoGebra, placing reference points, and applying the measurement tools – demonstrated the realization of instrumental genesis. Additionally, the calculation process, which involved proportional reasoning, rounding strategies, and the application of theoretical references – such as the formula for the area of a circle – reflected the achievement of discursive genesis.

After completing their calculations, the pre-service teachers in Group 1 began to critically evaluate the plausibility of their results. To verify their findings, they conducted an Internet search for official data regarding the dimensions of the Colosseum. They discovered that the total floor area of Colosseum was approximately 24 thousand square meters, with a radius of about 268 meters and a circumference of approximately 500 meters.

Based on this new information, the pre-service teachers reflected on the accuracy of their earlier calculations. The following commentary by PT3 illustrates this reflection:

PT3: But think about it like this. The round part is really small, because 50 thousand people will fit in here... so we found it approximately correct.

At the interpretation and verification stages, reasoning skills were demonstrated by PT3's reflective commentary, indicating that discursive genesis was achieved. The attempt to validate the obtained result by comparing it with external real-world data, and engaging in reasoning about the relative size of the Colosseum's central arena, exemplified the cognitive processes expected during this phase.

Mathematical Work of Group 2

The pre-service teachers in Group 2 initially worked individually to analyze the problem situation, answering the readiness questions provided. When the group work phase began, the pre-service teachers recognized the need for a photograph taken directly above the Colosseum. They conducted an Internet search using the keyword "bird's eye view" and found a photograph that revealed the target area had an elliptical shape. The following discussion illustrates their collaborative effort to understand and model the problem:

PT4: We use cars here as a reference object. We can also find the diameter of the shape with the length of the line segment of the car. That is, we go from the length of the line segment. Now we will create a circle surrounding the shape, using the center.

PT5: Are we going to take the whole circumference of the colosseum?

PT4: Are we going to take the whole colosseum and subtract the walkable part in the middle?

PT6: Do you think we will think of it as an ellipse or a circle?

PT4: There is an ellipse drawing tool in GeoGebra. We can use that.

PT6: Then we can create the ellipse by marking two points at the endpoints of the shape.

During this discussion, it was observed that the pre-service teachers attempted to simplify the problem situation by utilizing reasoning, planning, and assumption-making skills. Although they encountered difficulties while attempting to create the required shape using GeoGebra's dynamic geometry tools, they managed to place an ellipse on the photograph by exhibiting computer skills such as dragging and zooming. The difficulty stemmed from the fact that the ellipse creation tool in GeoGebra constructs an ellipse based on two focal points, a concept with which the pre-service teachers were unfamiliar.

At the stage of understanding and simplifying the problem, using Internet searches to find a new photograph and transferring the image to GeoGebra indicated the use of artifacts. Identifying the geometric shape as an ellipse represented mathematical representation. Creating the ellipse using GeoGebra tools demonstrated instrumental genesis, and visualizing the mathematical representation over the photograph reflected semiotic genesis. However, although semiotic and instrumental genesis processes were realized, the limited understanding of the theoretical references related to the concept of ellipse made the mathematical work more challenging.

After identifying the geometric characteristics of the problem situation, the pre-service teachers in Group 2 continued to mathematize the problem. They correctly established the relationship between the given information and the problem requirements, as evidenced by the following discussion:

PT4: Now he wants a covering for the rest of the floor. When we look at the shape, it looks like there is a semicircle there.

PT6: Yes, I understand, you are right. We will find the area of the ellipse and subtract the area of that semicircle and find a close value.

PT4: Let's do it like this... Let's create a circle on the ground, the area of this circle will be exactly half of it.

PT6: Yes. That's right.

Through this reasoning process, the pre-service teachers recognized that the walkable area visible in the photograph resembled a semicircle. They decided to model the area to be covered as an ellipse and to subtract the semicircular part from it to approximate the required area.

Using the ellipse creation tool in GeoGebra, they constructed an ellipse that encompassed the Colosseum's base area. Although they faced initial challenges with the focal points, they managed to adjusted the figure by dragging and positioning points manually rather than using the focal points explicitly. To enhance visualization, they colored the created circle and ellipse models using GeoGebra's formatting tools. Thus, they succeeded in creating a mathematical model by mathematizing the problem situation based on their assumptions and visual interpretation (Figure 5).

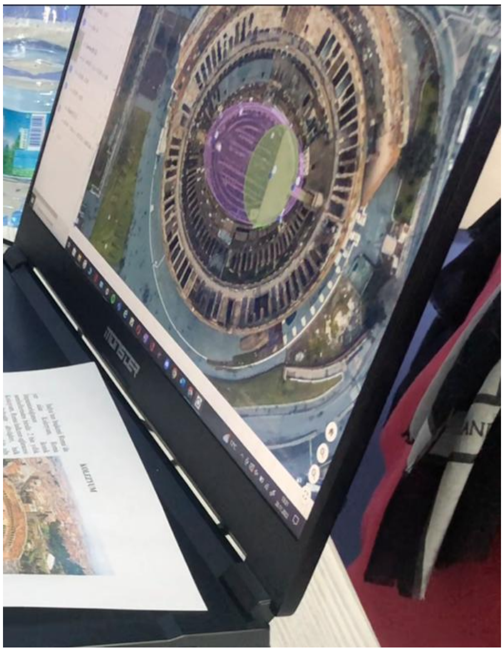


Figure 5 Visual Representing Group 2's Mathematization of the Problem Situation

At the mathematization stage, constructing and visualizing the circle and ellipse using GeoGebra tools demonstrated instrumental genesis, while the graphical visualization of the constructed models reflected semiotic genesis.

To establish a relationship between the dimensions in the photograph and real-world measurements, the pre-service teachers identified an automobile visible in the image and used GeoGebra's dynamic tools to mark its endpoints. They created a line segment using the 'line through two points' tool and measured its length through the algebra window in GeoGebra. Relying on information obtained through Internet research, they assumed that the real-life length of a car was approximately 4.5 meters. After making some corrections on the marked points, the following discussion took place:

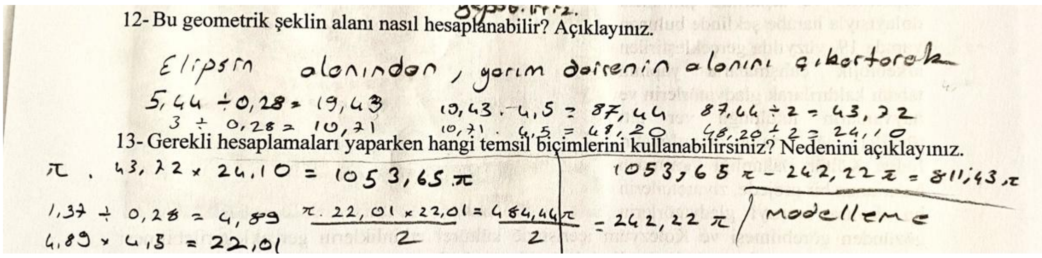
PT5: ...What is the length of the line segment of this? It says 0.28 in the algebra window, what does it mean. I think the length of the line segment is 0.28. Let's find the length of the ellipse as a line segment.

PT4: Now there are two points on the ellipse. Should we take points H and G to determine its length? (Points H and G are the focal points of the ellipse).

PT5: We measure the length from H to G, but it should be a little longer.

PT5: We got 5.44 between H and G. Now let's divide it by 0.28.

Through this dialog, it was evident that the pre-service teachers aimed to determine the necessary scale for transferring measurements from the photograph to real-life dimensions. They realized that two radii were required to calculate the area of an ellipse and constructed the minor radius using the line segment tool in GeoGebra. They then began applying proportional reasoning based on the ratio between real-world and photographic measurements. The worksheet excerpt of the mathematical operations performed is given in Figure 6.


12- Bu geometrik şeklin alanı nasıl hesaplanabilir? Açıklayınız.

Elipsin alanından, yarım dairenin alanını çıkartarak

5,44 ÷ 0,28 = 19,43   10,43 · 4,5 = 87,44   87,44 ÷ 2 = 43,72

3 ÷ 0,28 = 10,71   10,71 · 4,5 = 48,20   48,20 ÷ 2 = 24,10

13- Gerekli hesaplamaları yaparken hangi temsil biçimlerini kullanabilirsiniz? Nedenini açıklayınız.

π . 43,72 × 24,10 = 1053,65 π   1053,65 π − 242,22 π = 811,43 π

1,37 ÷ 0,28 = 4,89   π . 22,01 × 22,01 = 484,44 π / 2 = 242,22 π / Modelleme

4,89 × 4,5 = 22,01

Figure 6 Excerpt from Group 2’s Worksheet Showing Mathematical Operations

During working mathematically phase, it was observed that the pre-service teachers collected data necessary for solving the problem using GeoGebra’s line segment construction, length measurement, and algebra window tools.

The construction of line segments and use of measurement tools were evaluated as evidence of instrumental genesis. The application of the ellipse area formula represented the use of theoretical references, while the employment of reasoning skills and strategy development throughout the solution process reflected the achievement of discursive genesis.

After completing their mathematical operations, the pre-service teachers obtained an initial result for the area measurement. However, upon reviewing their work, they realized that a computational error had occurred. They re-executed the necessary mathematical operations and corrected their calculations. Following the corrections, the pre-service teachers questioned whether the revised result was reasonable. To validate their findings, they conducted additional Internet research and determined that their final result was close to the actual dimensions of the Colosseum. Based on this validation process, they concluded their mathematical work.

During the interpretation and verification stages, it was observed that the pre-service teachers used artifacts – such as conducting Internet searches – as epistemological tools to verify the plausibility of their solution. However, their validation process remained somewhat limited, as they did not employ alternative strategies or utilize additional components of the MWS framework beyond the initial verification. While they achieved reasoning and partial validation through Internet research, the scope of their mathematical work could have been enriched by engaging with different MWS components or applying multiple verification strategies.

Mathematical Work of Group 3

The pre-service teachers in Group 3 initially worked individually, attempting to answer the readiness and guided discovery questions. When the group work phase began, the pre-service teachers collaboratively discussed and restated the problem situation. To better understand the ‘ground’ referred to in the problem, they examined various photographs of the Colosseum and reached a consensus that the structure resembled the shape of an ellipse. They collectively reviewed the guided discovery questions and agreed that GeoGebra should be used to extract the necessary data for solving the problem, leveraging photographs to obtain various measurements. Accordingly, a photograph was imported into GeoGebra using the photo insertion tool. The pre-service teachers identified the resources needed to solve the problem, including the area formula for an ellipse, a reference object to obtain real-life measurements, the GeoGebra software, and a calculator.

At the stage of understanding and simplifying the problem, it was observed that the pre-service teachers demonstrated comprehension by expressing the problem in their own words. The use of a photograph to identify the region resembling an ellipse indicated the use of the representation component. Importing the photograph into GeoGebra reflected the use of the artifact component, while recognizing the need for the area formula of the ellipse involved the application of the theoretical reference component. Thus, multiple components of the epistemological plane were activated during the problem understanding and simplification phase.

During the mathematization phase, the pre-service teachers in Group 3 selected a truck in the photograph as the reference object. They marked two points on the reference object and measured the distance between them using the algebra window in GeoGebra. Subsequently, they attempted to create an ellipse representing the target area by using the ellipse creation tool. However, due to the ellipse creation tool’s reliance of focal points, the pre-service teachers initially struggled to construct the desired shape. Recognizing that the photograph contained perspective distortion, they decided that a bird’s-eye view image would be more suitable. They imported a satellite image of the Colosseum into GeoGebra and successfully created an ellipse, adjusting it by dragging the focal points to fit the desired region. The group's conversation during this process, along with GeoGebra screenshots, is presented in Figure 7:

PT9: Now I mark two dots on the sides of the shape.

PT10: This shape did not encircle the shape completely.

PT9: Take your time, we will edit it a bit by dragging the dots

PT10: Since this photo is more suitable for drawing the ellipse than the other one, can't we draw the ellipse passing through the edges?

PT11: We cannot draw the ellipse without determining the focal points.

PT8: Yes, we need two focal points to draw an ellipse.

Figure 7 Screenshots of the Geometric Shapes Created by Group 3

Through this process, the pre-service teachers concluded that the problem could be solved by calculating the area of the constructed ellipse. The use of GeoGebra's point and length measurement tools to clarify the reference object reflected the use of the artifact component. The construction of the ellipse its placement through dragging actions were evaluated as evidence of instrumental genesis and semiotic genesis.

After constructing the ellipse, the pre-service teachers attempted to determine its area using the area measurement tool in GeoGebra. PT8 selected the area measurement tool and clicked on the constructed ellipse, calculating its area directly within the software (Figure 8). However, the pre-service teachers soon realized that this measured area corresponded only to the photographic scale and not to real-life dimensions. Recognizing the need for real-world scaling, they created line segments to represent the major and minor radii of the ellipse on the photograph. The lengths of these segments were determined using GeoGebra's algebra window. To convert these measurements into real-life dimensions, they used the ratio obtained from the real-world and photographic measurements of the reference object. By applying proportional reasoning and executing mathematical operations based on these ratios, they eventually calculated the real-life area measurement for the desired region using the formula for the area of an ellipse.

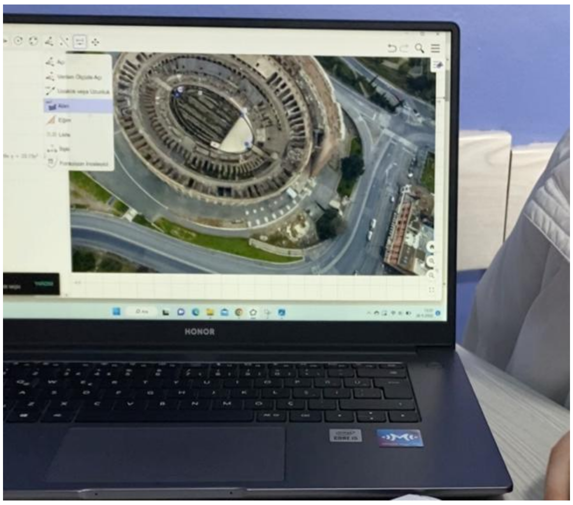


Figure 8 Screenshot Illustrating Group 3's Use of the Area Measurement Tool

During working mathematically phase, the purposeful use of the area measurement tool in GeoGebra was evaluated as an activation of the artifact component. While initially struggling with the direct transfer of measurements to real life, the pre-service teachers shifted strategies: creating line segments and utilizing proportional relationships.

The construction of line segments and real-life scaling reflected instrumental genesis and semiotic genesis, while applying the theoretical relationship between ratio and area concepts, reasoning about measurements, and strategy development corresponded to discursive genesis.

Mathematical Work of Group 4

The pre-service teachers in Group 4 initially worked individually by reading the problem situation. When the group work phase began, they engaged in discussions to answer the guided discovery questions and deepen their understanding of the problem. Through group discussion, the pre-service teachers articulated the problem situation in their own words, initially assuming that the region to be measured resembled a circle. They proposed a plan to identify a reference in the photograph, determine its length in the image, and establish a proportional relationships between this measurement and other necessary dimensions. Thus, they aimed to approximate the desired result using only the provided photograph.

At the stage of understanding and simplifying the problem, it was observed that the representation component was activated by conceptualizing the region as a circle. Additionally, the use of theoretical references – such as the formulas relating to the area and diameter of a circle – was integrated into their planning. The artifact component was also utilized at this stage through the intention to employ GeoGebra dynamic geometry software to obtain necessary data. The actions performed by the pre-service teachers at this stage demonstrated engagement primarily within the epistemological plane of the MWS.

After selecting a reference object in the photograph and estimating its real-life length as approximately 3 meters, the pre-service teachers located the original image on the Internet and attempted to import it into GeoGebra. Although they initially experienced difficulties using the dragging method, they successfully transferred the photograph through copy-paste functionality. Using GeoGebra's tools, they constructed a line segment by marking two point on the reference object, and measured its length via the algebra window. They then created another line segment across the ground area of Colosseum, which they initially assumed to represent the diameter of a circle. The center of the circle was marked using the midpoint, and a circle was drawn using the center-and-point circle tool.

Upon further examination of the photograph, the pre-service teachers realized that the ground area resembled an ellipse more closely than a circle, particularly because the photograph lacked a perfect bird's-eye view. At this point, PT13 utilized the ellipse creation tool to draw an ellipse on the photograph. Observing the resulting shape, the other pre-service teachers agreed that the region should be modeled as an ellipse, and decided that the area of this ellipse should be calculated (Figure 9).

Figure 9 Screenshots of Group 4's Mathematization of the Problem Situation

In the mathematization phase, the use of the photograph to construct geometric representations reflected the representation component, while the application of GeoGebra's tools demonstrated engagement with the artifact component. The purposeful use of GeoGebra to construct line segments, midpoints, circles, and ellipses indicated that instrumental genesis was realized. Additionally, the visualization of mathematical representations through the construction of geometric shapes on the photograph was evaluated as the realization of semiotic genesis.

After constructing the ellipse model, the pre-service teachers proceeded to solve the mathematical model they had created. They initially attempted to determine the area of the ellipse directly by selecting the area measurement tool in GeoGebra and clicking on a point within the ellipse. However, when attempting to relate the obtained area measurement to the real-life context, they realized that a direct comparison between the area value and the linear length of the reference object would not be meaningful. Recognizing this issue, they shifted their strategy. Instead of using the area measurement directly, they decided to construct two line segments representing the major and minor radii of the ellipse. These lengths were measured using GeoGebra's algebra window. By associating these lengths with the real-life length of the reference object through proportional reasoning, they could calculate the actual dimensions of the ellipse and subsequently its real-life area. The following discussion illustrates their decision-making process:

PT15: Shouldn't we find the radius lengths of the ellipse?

PT14: We need to use the radii of the ellipse in the area formula of the ellipse.

PT13: Then we should find the area covered by the car and determine the ratio over the area.

PT12: How can we find the area of the car, we only measured its length.

PT13: Yes, that's right.

PT14: Then we will have to use the length of the car. We use the length to find the radius.

PT12: To find the area of the ellipse, we need to find the short radius and the long radius, I looked on the internet.

PT14: Okay, then we need to find both of the lengths.

PT13: Then we will draw line segments by putting points on the other sides.

PT12: Yes, let's calculate.

Through this strategic shift, the pre-service teachers determined the real-life equivalents of the radii measurements and successfully calculated the approximate real-life area of the region by applying the area formula for an ellipse. The mathematical operations performed in line with this process are shown in Figure 10.

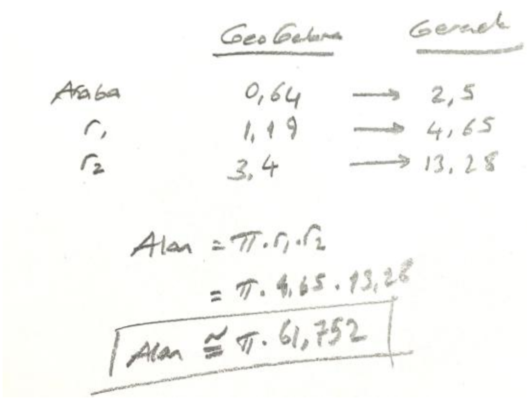


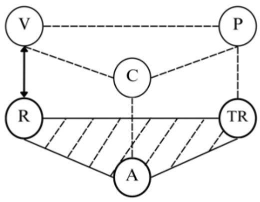
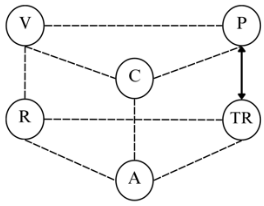
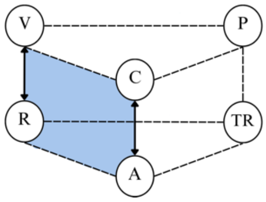
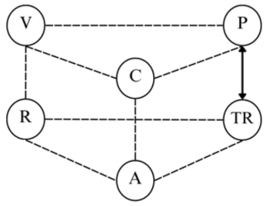
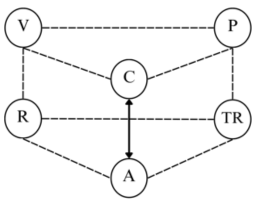
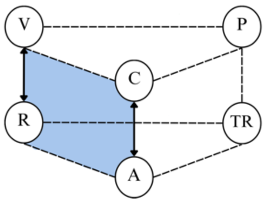
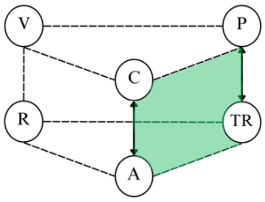
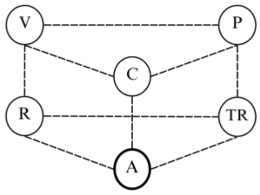
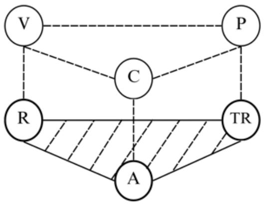
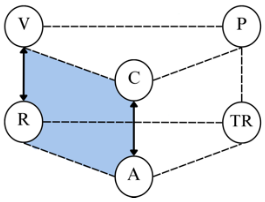
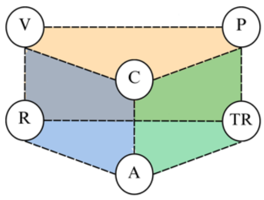
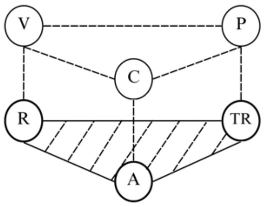
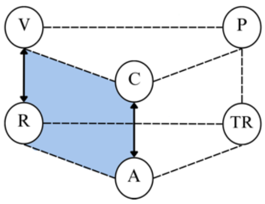
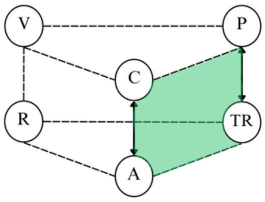
Figure 10 Excerpt from the Worksheet on Mathematical Operations Performed by Group 4

During the working mathematically phase, the use of the area measurement tool, the construction of line segments for the major and minor radii, and the use of GeoGebra's dynamic geometry tools demonstrated a purposeful and structured engagement with artifacts, realizing instrumental genesis. Furthermore, the use of the ratio concept to transfer measurements from the photograph to real-life dimensions, the application of the theoretical references related to the ellipse, and the reasoning processes involved in connecting measurements to real-world quantities indicated the realization of discursive genesis.

Conclusions and Suggestions

In this study, the process by which pre-service elementary mathematics teachers solved a mathematical task involving mathematical modeling activities was examined through the components of the MWS theory. The stages of the mathematical modeling cycle were taken into account when analyzing the actions performed during the task. Table 2 presents the MWS diagrams corresponding to the mathematical work produced by the pre-service teachers in each group.

Table 2 MWS Diagrams of the Mathematical Works Performed by the Groups in the Mathematical Modeling Process

	Understanding and simplifying	Mathematization	Working mathematically	Interpreting - verifying
Group 1				
Group 2				
Group 3				-
Group 4				-

When Table 2 is examined, it can be seen that the actions performed during the understanding and simplifying stages took into account the epistemological basis of the mathematical concepts embedded in the problem. At this stage, Group 1 achieved semiotic genesis, while Group 2 achieved instrumental genesis, enriching their work at this stage.

At the mathematization stage, it was observed that pre-service teachers commonly engaged in both instrumental genesis and semiotic genesis. During this stage, they utilized the features of dynamic geometry software to perform geometric constructions that supported the visualization of the problem situation. Through these actions, mathematical models were created by practically and accurately employing the artifact component withing the epistemological plane.

According to the findings of Verdugo Hernández et al. (2023), this type of mathematical work predominantly took place within the [Sem-Ins] plane, where semiotic and instrumental genesis were generally realized, following an exploratory approach for the interpretation and verification of results. Similarly, Flores Salazar and Almonacid Adriano (2020) reported that mathematical work during the solution of modeling tasks was frequently conducted within the [Sem-Ins] plane.

In the present study, it was found that Group 1 carried out their mathematical work predominantly within the [Sem-Ins] plane, Group 4 and Group 2 operated within the [Ins-Dis] plane, and Group 3 engaged mathematical work in all three planes. At this stage, the realization of discursive genesis – in which pre-service teachers utilized theoretical references and reasoning strategies – significantly contributed to the completion of their mathematical work. However, it was noted that semiotic genesis was not achieved by two of the groups. Verdugo Hernández et al. (2023) emphasized that achieving semiotic genesis related to the mathematical concepts involved is crucial for the successful progression of the modeling cycle. In this respect, it can be inferred that realizing semiotic genesis plays a vital role in successfully completing mathematical work correctly.

During the interpretation and evaluation phase – where the mathematical results obtained are interpreted in the context of real-life situations – it was observed that only Group 1 successfully realized discursive genesis. This action can be considered supportive of the modeling process, as it involved discourse that included reasoning processes and the application of mathematical concept properties. At this stage, it was observed that Group 2 utilized only the artifact component by conducting an Internet search to access information related to the real-life situation. Meanwhile, the other two groups did not perform any significant actions during this phase. The main reason for this situation appeared to be that the pre-service teachers were unable to use their time effectively and thus lacked sufficient time to engage in the interpretation and verification stages.

Although the primary aim of this study was to document the activation of MWS components across different modeling stages, preliminary observations suggest that groups demonstrating a more integrated use of representation and theoretical reference components tended to exhibit more coherent mathematization and validation processes. Conversely, groups with fragmented activation of epistemological plane components greater difficulties in progressing through the mathematical modeling cycle.

The mathematical task presented to pre-service teachers in this study was developed within the context of the concept of area measurement. In future studies, mathematical tasks involving different geometric shapes related to the concept of area measurement could be designed, and the implementation process of these tasks can be examined through MWS. In addition, in studies involving different concepts from the discipline of mathematics and associating mathematical concepts with other disciplines, MWS can be used as an analysis tool. In this study, it was observed that pre-service teachers in different groups experienced difficulties during the modeling process. In order to determine the reasons for the difficulties and bottlenecks in the modeling process, studies can be conducted to reveal how the interaction between the epistemological and cognitive components of the MWS is realized. Moreover, further studies could be conducted to examine the instructional practice plans of pre-service teachers who will transfer mathematical modeling to the classroom environment in the future within the framework of the MWS theory. Although this study focused on descriptive analysis based on the MWS, future research could incorporate in-depth content analysis or cognitive analysis to provide a more detailed examination of the modeling practices of pre-service teachers. Additionally, future studies could conduct a process-oriented analysis that systematically examines how interactions among MWS components influence the success and challenges of different groups during the mathematical modeling cycle.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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Research involving Human Participants and/or Animals

This research involves human participants. The ethic committee permissions were obtained with the letter dated 28.02.2023 and numbered 19928322/108.01/233477 of the Ethics Committee for Science and Engineering of Balikesir University.

Matematiksel Modelleme Sürecinin Matematiksel Çalışma Alanı Işığında İncelenmesi: İlköğretim Matematik Öğretmen Adayları Örneği

Özet:

Matematiksel modelleme, gerçek yaşam durumları ile matematik arasındaki köprüyü kurmak için önemli bir süreçtir. Matematiksel Çalışma Alanları teorisi, matematiksel çalışmanın epistemolojik ve bilişsel taraflarını ele alarak matematiksel modelleme sürecinin açıklanmasında uygun bir araç olarak görülmektedir. Bu çalışmada, modelleme etkinliği içeren matematiksel görevlerin çözümü için ilköğretim matematik öğretmen adayları tarafından gerçekleştirilen matematiksel çalışmalar Matematiksel Çalışma Alanları teorisi doğrultusunda incelenmiştir. Nitel araştırma yöntemlerinden durum çalışması deseninin temel alındığı çalışmada, ölçüt örnekleme ile belirlenen 15 ilköğretim matematik adayı ile gerçekleştirilmiştir. Yönlendirilmiş keşif çalışma soruları ile desteklenen matematiksel modelleme etkinliği probleminin çözüm sürecine ilişkin veriler video kayıtları ve çözüm süreçlerini içeren dokümanlar ile elde edilmiştir. Elde edilen veriler Matematiksel Çalışma Alanları teorisinin bileşenleri temel alınarak betimsel olarak yorumlanmıştır. Çalışmanın sonucunda, matematiksel modelleme döngüsünün problemi anlama ve basitleştirme aşamalarında öğretmen adayları tarafından sıklıkla epistemolojik düzlem bileşenlerinin kullanıldığı, matematikselleştirme aşamasının ağırlıklı olarak semiyotik-enstrümantal düzlemde gerçekleştirildiği, matematiksel olarak çalışma aşamasının farklı düzlemlerde gerçekleştirilen eylemler ile yürütüldüğü görülmüştür. İleriki çalışmalarda, matematiksel modellemeyi sınıf ortamına aktaracak öğretmen adaylarının gerçekleştireceği öğretim uygulamalarının Matematiksel Çalışma Alanları teorisi çerçevesinde incelenmesine yönelik çalışmalar gerçekleştirilebilir.

Anahtar kelimeler: Matematiksel çalışma alanı, matematiksel modelleme, matematiksel çalışma, alan ölçme.

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