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GRUP ALTKROSS MODÜLLER ÜZERİNE BİR ARAŞTIRMA I

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Özet: Grup kross modüllerinin altkross modüllerinde bazı örnekleri ve sonuçları bu makalede verildi.

A SURVEY ON SUBCROSSED MODULES OF GROUPS I

Abstract: In this paper some examples and results of subcrossed of crossed modules of group are given.

Keywords: *crossed module, subcrossed of crossed modules.*

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INTRODUCTION

Crossed modules were introduced by J.H.C. Whitehead in his study on combinatorial homotopy theory [9]. They have found important roles in many areas of mathematics (including homotopy theory, homology and cohomology of groups, algebraic K-theory, cyclic homology, combinatorial group theory, and differential geometry). Possible crossed modules should now be considered one of the fundamental algebraic structures.

Crossed modules of groups are generalization of both normal and subgroups. Any subgroup is a crossed module, so it is of interest to see generalization of group theoretic concepts and structures to crossed modules.

Also areas in which crossed modules have been applied include the theory presentations (see the survey [2]), algebraic K-theory (Loday [5]) and homological algebra ([4]). Now crossed modules can be viewed as 2-dimensional groups (see [1]) and it is therefore of interest to consider counterparts for crossed modules of contents from group theory. K. Norrie in [7] presented the automorphisms of a group N fit into a crossed module $N \to \mathrm{Aut}N$. She explored the corresponding more elaborate structures, for example crossed squares, into which fits the automorphism group of a crossed module.

We will look at the substructures and normal subgroup of crossed modules. To form factor crossed modules we need to work that out with some conditions on normal subgroup. This paper also contains the factorization theorem of morphism between crossed modules.

1. CROSSED MODULES AND EXAMPLES

J.H.C. Whitehead (1949) [9] described crossed modules in various context especially in his investigations the algebraic structures relative homotopy groups. In this section, we recall the definitions and elementary theory of crossed modules of group given by Whitehead [9].

A crossed module (C, G, ∂) [9] consists of groups C and G, an operation of G, on the left of C, written $(g, c) \mapsto {}^{g}c$ and a homomorphism $\partial : C \to G$ of G-groups where G acts on the left of itself by conjugation. The map ∂ must satisfy the rules

CM1: $\partial({}^{g}c) = g\partial(c)g^{-1} \quad g \in G,\ c \in C$

CM2: ${}^{\partial c}c' = cc'c^{-1}.$

The last condition is called the Peiffer identity.

If a group G acts on C and $\partial : C \to G$ satisfies CM1 then it is sometimes convenient to refer to (C, G, ∂) as a precrossed module. For example making $G = NG_0$ acts on $C = NG_1$ via conjugation using s_0 so ${}^{m}c = s_0(m)cs_0(m)^{-1}$, we get that

$$\partial_1 : NG_1 \longrightarrow NG_0$$

is a precrossed module. In such a context the element

$${}^{\partial c}c' \cdot (cc'c^{-1})^{-1}$$

be called the Peiffer commutator of m and m', or more briefly a Peiffer element. Of course the vanishing of these Peiffer elements is equivalent to (C, G, ∂) being a crossed module. The subgroup generated by such elements is known as the Peiffer subgroup of C for the given precrossed module structure on (C, G, ∂).

Given any precrossed module $\partial : C \to G$, one can form an internal directed graph in the category of groups simply by forming the semidirect product $C \tilde{a}\, G$ and taking the source, s, and target, t, to send an element (c,g) to g or $\partial c.g$ respectively. The Peiffer subgroup of M measures the obstruction to the directed graph having an internal category structure. It can easily be seen to be $[\mathrm{Ker}s, \mathrm{Ker}t]$. A morphism $(\alpha,\beta):(C,G,\partial) \to (C',G',\partial')$ of crossed modules consists of homomorphisms $\alpha: C \to C'$, $\beta: G \to G'$ of groups such that $\beta\partial = \partial'\alpha$ and $\alpha({}^{g}c) = {}^{\beta(g)}\alpha(c),\ c \in C,\ g \in G.$ If (C,G,∂) is a crossed module, then C is called a crossed G-module.

A crossed module generalizes the concepts of both an ordinary module and that of a normal subgroup. For if Q is a group and A is a Q-(left) module, then (A,Q,I) is a crossed module with I the trivial map $I(a) = 1 \in Q,\ a \in A.$ If G is a group and N is a normal subgroup, then (N,G,i) is a crossed module, with the inclusion i and G acting on N by conjugation.

1.1 Examples

We note once certain consequence of the definition of a crossed module:

(i) the image ∂C is a normal subgroup of G,
(ii) the kernel $\mathrm{Ker}(\partial)$ lies in the center Z of C,
(iii) the operation of G on C induces a natural $(G/\partial C)$-module structure on Z and $\mathrm{Ker}(\partial)$ is a submodule Z,
(iv) the action of G on C induces a natural $(G/\partial C)$-module structure on the commutator factor group $C^{Ab} = C/[C,C],$
(v) the quotient $(G/\partial C)$ is denoted by $\pi_1(\partial).$

Also it is easily checked that the action of G on C induces an action of $\pi_1(\partial)$ on $\mathrm{Ker}(\partial)$ and that $\mathrm{Ker}(\partial)$ is abelian; we denote the $\pi_1(\partial)$-module of $\mathrm{Ker}(\partial)$ by $\pi_1(\partial)$.

It is clear that the crossed modules constitute a category $XMod$: if G is a fixed group, the crossed modules constitute a full subcategory $G - XMod$.

Furthermore the inner automorphism map $\tau: N \to \mathrm{Aut}N$ already mentioned, other standard examples of crossed modules are:
- a G-module M with the zero homomorphism $M \to G$;
- the inclusion of a normal subgroup $N \to G$;
-and any epimorphism $E \to G$ with central kernel.

There are two canonical ways in which a group may be regarded as a crossed module: via the identity map $G \to G$ or via the inclusion of trivial map subgroup.

Now we can give definition of subcrossed module from [7].

Definition: It said that (L',K',∂') is subcrossed module of the crossed module (L,K,∂) if
(i) L' is a subgroup of L and K' is a subgroup of $K,$
(ii) ∂' is restriction of ∂ to L' and
(iii) the action of K' on L' is induced by the action of K on $L.$

1. Let N be any normal subgroup of a group K. Consider an inclusion map (homomorphism on N) $\mathrm{inc}: N \to K$ together with the action of ${}^{k}n = knk^{-1}$. Then (N,K,inc) is a crossed module. Conversely given any crossed K-module $\partial: L \to K$, one can easily verify that $\partial L = N$ is a normal subgroup in K.

2. Let M be any K-module. It can be considered as a K-group with identity map, and then the trivial homomorphism

$1 : M \to K$ is a crossed K-group by ${}^{1m}m' = mm'm^{-1} = 1$ for all $m, m' \in M$.

Conversely, given any crossed module $\partial : L \to K$, then $\mathrm{Ker}\partial$ is a K-module. For this, see Proposition 2.1. We state without proof of the following results (see T. Porter [8]).

3. A simplicial group $\mathbf{G}$ consists of a family of groups $\{G_n\}$ together with face and degeneracy maps

$d_i = d_i^n : G_n \to G_{n-1},\ 0 \le n \le n\ (n \ne 0)$ and

$s_i = s_i^n : G_n \to G_{n-1},\ 0 \le n \le n\ (n \ne 0)$,

satisfying the usual simplicial identities given in [6]. For example it can be completely described as a functor $\mathbf{G} : \Delta^{op} \to Grp_k$ where Δ is the category of finite ordinals $[n] = \{0 < 1 < ... < n\}$ and increasing maps. Assume that a simplicial $\mathbf{G}$ and simplicial subgroup N are given. The inclusion

$$\mathrm{inc} : N \hookrightarrow \mathbf{G}$$

induces a map

$$\pi_0(\mathrm{inc}) : \pi_0(N) \to \pi_0(\mathbf{G}).$$

The action by conjugation of $\mathbf{G}$ on N induces an action of $\pi_0(\mathbf{G})$ on $\pi_0(N)$ Then $(\pi_0(N), \pi_0(\mathbf{G}), \pi_0(\mathrm{inc}))$ is a crossed module. Any crossed module can be obtained as π_0 of a simplicial normal subgroup inclusion, $N \to \mathbf{G}$ as above but we will not include a proof here (see [8]).

4. Suppose that K is the group $\mathrm{Aut}(L)$ of automorphism of some group L. Then the homomorphism $L \to K$ which sends an element $x \in L$ to the inner automorphism $L \to L,\ l \to xlx^{-1}$ is a crossed module.

Each of these examples consists of a group homomorphism with an action of the target group on the source group. Before stating the precise algebraic properties we need by such a homomorphism for it to be a crossed module, let us consider some more substantial examples.

5. Let X be a topological space in which a point x_0 has been chosen. Recall that the fundamental group $\pi_1(X, x_0)$ consists of homotopy classes is a crossed module in the category of crossed modules. Thus it consists of a morphism of crossed modules together with an "action" of the target crossed module on the source crossed module and certain algebraic conditions are satisfied.

6. If

$$C_1 \underset{t,i}{\overset{s}{\longrightarrow}} C_0$$

is an internal category within the category of groups then C_0 acts on $\mathrm{Ker}s$ by conjugation:

$${}^{g}(c) = i(g)ci(g)^{-1}$$

for $g \in C_0, c \in C_1$.

The target map t restricts to $\mathrm{Ker}s$ to give a crossed module $t : \mathrm{Ker}s \to C_0$. Conversely any crossed module $\delta : C \to G$ yields an internal category by taking $C_1 = C$ã G the semidirect product of C and G, and $C_0 = G$; the source map s is given by $s(c, g) = g$ whilst target t is $t(c, g) = \delta c.g$.

7. The original class of example studied by Whitehead [9], come from topology. Let (X, A) be a pair of pointed topological spaces. The second relative homotopy group $\pi_2(X, A)$ of $\pi_1(X, A)$ is obtained as the group of homotopy class of maps from a square I^2, into X such that the boundary of I^2, maps into A, and $I \times \{0\}$, $\{0\} \times I$ and $\{1\} \times I$ are mapped to base point. The homotopies must respect this filtration; full details can be found in any standard text on homotopy theory. The restriction of each homotopy class to $I \times \{1\}$ gives a class of loops within A and this boundary homomorphism

$$\partial : \pi_2(X, A) \to \pi_1(A).$$

There is a natural action of $\pi_1(A)$ on

$\pi_2(X,A)$ so that $(\pi_2(X,A),\pi_1(A),\partial)$ is a crossed module.

8. Let M and N be normal subgroups of G. A non-abelian tensor product $M\otimes N$ has been introduced by R. Brown and J-L. Loday [3]. It is the group generated by the symbols $m\otimes n$ ($m\in M$ and $n\in N$) subjects the relations

$$mm'\otimes n = ({}^m m'\otimes {}^m n)(m\otimes n),$$

$$m\otimes nn' = (m\otimes n)({}^n m\otimes {}^n n'),$$

for $m,m'\in M, n,n'\in N$. In general $M\otimes N$ is a non-abelian group. If however conjugation in G by an element of M (resp. N) leaves all the elements of N (resp. M) fixed, then $M\otimes N$ is precisely the usual abelian tensor product of abeliasined groups $M/M'\otimes N/N'$. For any normal subgroup M and N there is a homomorphism $\partial : M\otimes N\to G$ defined on generators by

$${}^g(m\otimes n) = gmg^{-1}\otimes gng^{-1}.$$

This homomorphism and action is a crossed module.

9. Let Λ be an associative ring with identity, let $GL(\Lambda)$ be the general linear group, and let $E(\Lambda)$ be the subgroup of $GL(\Lambda)$ generated by the elementary matrices $e_{ij}(\lambda)$ with $i\neq j$ and $\lambda\in\Lambda$ (recall that $e_{ij}(\lambda)$ has 1's on the diagonal, λ in (i,j) position, and 0 elsewhere). The group $E(\Lambda)$ is a normal subgroup of $GL(\Lambda)$ and the non-abelian tensor square $E(\Lambda)\otimes E(\Lambda)$ is known as the Steinberg group and denoted $St(\Lambda)$. The definition of the Steinberg group is equivalent to the usual definition (see Brown-Loday [3]). As a special case of Example 8 we have a crossed module $\partial : St(\Lambda)\to GL(\Lambda)$. It can be shown that $\partial(St(\Lambda)) = E(\Lambda)$. The group $K_1(\Lambda)=\mathrm{Coker}(\partial)$ and $K_2(\Lambda)=\mathrm{Ker}\partial$ are known as first and second algebraic K-theory groups of λ.

2. SOME BASIC GROUP PROPERTIES OF CROSSED MODULES

The following results prove consequences of the definitions of crossed modules and state some properties of those groups.

Proposition 2.*1* *If (L,K,∂) is a crossed K-module, then*
i) $\mathrm{Ker}\partial$ *is a central subgroup of* L,
ii) both $L/[L,L]$ *and* $\mathrm{Ker}\partial$ *have natural* $K/\partial L$ *-module structures.*

Proof: (i) Since for $l\in L$, $a\in\mathrm{Ker}\partial$,

$$\partial({}^l a^{-1}) = l\partial(a^{-1})l^{-1} = ll^{-1} = 1,\ (\partial(a^{-1})=1)$$

as required.
(ii) It is enough to show that ∂L acts trivially on $\mathrm{Ker}\partial$ and $L/[L,L]$. For $a\in\mathrm{Ker}\partial$, $\partial l\in\partial L$, by $\partial({}^l a) = l\partial(a)l^{-1}=1$, ∂L acts trivially on $\mathrm{Ker}\partial$. For $\partial l\in\partial L$, $l'L\in L/[L,L]$ we obtain the following

$$\partial({}^{l'}L = \partial({}^{l'}) = l\partial(l'l^{-1})) = 1,$$

so ∂L acts trivially on $L/[L,L]$. Hence we can unambiguously define maps

$$K/\partial L\times\mathrm{Ker}\partial\to\mathrm{Ker}\partial$$
$$(k\partial l,a)\to k_a$$
$$K/\partial L\times L/[L,L]\to L/[L,L]$$
$$(k\partial l, l[L,L])\mapsto k_l[L,L]$$

and it is routine to check that the turns the abelian groups $\mathrm{Ker}\partial$ and $L/[L,L]$ into $K/\partial L$-modules. $\mathrm{Ker}\partial$ and $L/[L,L]$ have $K/\partial L$-module structure, where $[L,L]$ is a commutator. ■

Recall that definition of the exact sequence of groups.

A sequence of two homomorphisms of groups $A\xrightarrow{f}B\xrightarrow{g}C$ is exact at B if $\mathrm{im}f=\mathrm{Ker}g$. A sequence of abelian groups and homomorphisms

$$\cdots S_{n+1}\xrightarrow{\partial_{n+1}}S_n\xrightarrow{\partial_n}S_{n-1}\longrightarrow\cdots$$

is exact if it is exact at each S_n, that is $\mathrm{Im}\partial_{n+1} = \mathrm{Ker}\partial_n$ for all $n \in \square$. It is clear that every exact sequence is a complex equality ($\mathrm{Im} = \mathrm{Ker}$) implies inclusion ($\mathrm{Im} \subset \mathrm{Ker}$). As a result of previous result we have two exact sequences

$$1 \longrightarrow \mathrm{Ker}\partial \longrightarrow L \longrightarrow \mathrm{Im}\partial \longrightarrow 1$$
$$1 \longrightarrow \mathrm{Im}\partial \longrightarrow K \longrightarrow K/\mathrm{Im}\partial \longrightarrow 1.$$

Proposition 2.2 *The exact sequence*

$$1 \longrightarrow \mathrm{Ker}\partial \longrightarrow L \longrightarrow \mathrm{Im}\partial \longrightarrow 1$$

induced the following exact sequence

$$\mathrm{Ker}\partial \longrightarrow L/[L,L] \xrightarrow{\overline{\partial}} I/[I,I] \longrightarrow 1$$

where $I = \mathrm{Im}$.

Proof: To prove that the above sequence is exact we need to show that:

(i) the morphism $\overline{\partial} : L/[L,L] \longrightarrow I/[I,I]$, is onto

(ii) $\mathrm{Ker}\partial$ maps onto the kernel of $\overline{\partial}$ i.e. each element $l[L,L]$ in kernel $\overline{\partial}$ is of the form $l'[L,L]$, for some $l' \in \mathrm{Ker}\partial$. We know that the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Ker}\partial \arrow[r] & L \arrow[r, "\partial"] \arrow[d] & I \arrow[r] \arrow[d] & 1 \\ & & L/[L,L] \arrow[r, "\overline{\partial}"'] & I/[I,I] & \end{tikzcd}$$

is commutative, and ∂ is onto, then $\overline{\partial}$ is onto, and the image of $\mathrm{Ker}\partial \longrightarrow L/[L,L]$ is contained in $\mathrm{Ker}\overline{\partial}$.

(ii) if $l \in [L,L] \in \mathrm{Ker}\overline{\partial}$, then $\overline{\partial}(l[L,L]) = \partial(l)[I,I] = [I,I]$ and $\partial(l) \in [I,I]$. Thus $\partial(l) = \partial(l) = \partial(b)\partial(b')$ for some $b, b' \in L$. This implies that $l(bb')^{-1} \in \mathrm{Ker}\partial$, i.e $(l(bb)^{-1}) = l' \in \mathrm{Ker}\partial$, but then $l[L,L] = l'[L,L]$, so $\mathrm{Ker}\partial$ mapped onto $\mathrm{Ker}\overline{\partial}$. ■

Proposition 2.3 *Let* $\psi : (L,K,\partial) \to (B,K,\beta)$ *be a morphism of crossed* K*-modules. Then* (L,B,ψ) *is a crossed* B*-modules where* B *acts on* L, *via* β.

Proof: We have the following commutative diagram

$$\begin{tikzcd} L \arrow[rr, "\psi"] \arrow[dr, "\partial"'] & & B \arrow[dl, "\beta"] \\ & K & \end{tikzcd}$$

where ψ is a morphism of K-group and B acts on L, via β. i.e. for $l \in L$ and $b \in B$ we have

$${}^{l}b = \beta({}^{l}b).$$

Now we need to check that ψ is a morphism of B-group and satisfies the conditions CM1 and CM2. Let $l \in L$ and $b \in B$, then we have

$$\psi({}^{l}b) = \psi(\beta({}^{l}b)) = {}^{\psi(l)}\beta(b) = {}^{\psi(l)}b.$$

Also for $l, l' \in C$

$${}^{\psi(l)}l' = {}^{\beta\psi(l)}l' = {}^{\partial(l)}l' = {}^{l}l'.$$

Similarly ${}^{l}\psi(l') = {}^{l}l'$. Thus the axioms of a crossed module are satisfied. ■

Thus by Proposition 2.1 $\psi(L)$ is a normal subgroup in B.

Proposition 2.4 *Let* (L,B,∂) *be a crossed* B*-module and* (B,K,β) *be a crossed* K*-module such that* K *acts on* L, *where the action is compatible with* B*-action on* L, *then* $(L,K,\beta\partial)$ *is a crossed* K*-module.*

Proof: The only thing we need to check that is the Peiffer identity. If $l, l' \in L$, then

$$\beta\partial({}^{l}l') = l\beta\partial(l'l^{-1}l') = \beta({}^{l}l') = {}^{l}l'.$$

Thus ∂ is a crossed module. ■

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