C.B.Ü. Fen Bilimleri Dergisi ISSN 1305-1385 C.B.U. Journal of Science
2.2 (2006) 123 – 128 2.2 (2006) 123 –128

YARI-RİEMANN MANİFOLDUNUN LİGHTLİKE İSOTROPİK ALTMANİFOLDU

Erol YAŞAR[1]

Muğla Üniversitesi, Ula Ali Koçman M.Y.O., 48600 Ula-Muğla, Türkiye.

Özet: Bu makalede, yarı-Riemann manifoldunun isotropic altmanifoldu çalışıldı. Lightlike isotropic altmanifoldun denklem yapıları verildi. Sonra ise *M* isotropic altmanifoldu total geodezik yapan şartlar elde edildi.

LIGHTLIKE ISOTROPIC SUBMANIFOLD OF SEMI-RIEMANNIAN MANIFOLD

Abstract: In this paper, we study isotropic submanifold of semi-Riemannian manifold. We give the structure equations of lightlike isotropic submanifold. Then we obtain that the condition on *M* lightlike isotropic submanifold is totally geodesic.

Key Words and Phrases : *Lightlike isotropic submanifolds, Totally geodesic, Gauss and Codazzi equations.*

MOS Clasification : 53B15 53B30 53C05

---

[1] **Sorumlu Yazar**
yerol@mu.edu.tr

1. Introduction

The theory of submanifolds of a Riemannian or semi-Riemannian manifold is one of the most important topics in differential geometry. It is well known that the primary difference between theory of lightlike submanifold and semi-Riemannian submanifold arises due to the fact that in the first case, a part of the normal vector bundle $TM^{\perp}$ lies in the tangent bundle TM of the submanifold M of $\tilde{M}$, whereas in the second case $TM \cap TM^{\perp}$ ={0}. In 1992, Duggal and Bejancu [3] introduced a half lightlike submanifold M, of codimension 2 and found geometric conditions for the induced connection on M to be metric connection. In 2004, Erol Kılıç and Bayram Şahin [5] studied coisotropic submanifold of a semi-Riemannian manifold. In this study they investigated the integrability condition of the screen distribution and gave a necessary and sufficient condition on Ricci tensor of a coisotropic submanifold to be symmetric.

In the present paper, we have proved that the connection induced from semi-Riemannian manifold of codimension ($m+n$) on lightlike isotropic submanifold is metric. Besides we obtain the structure equations of lightlike isotropic submanifold and proved the theorem on M in semi-Riemannian manifold of constant curvature, of codimension (m+n).

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a real ($2m+n$)-dimensional semi-Riemannian manifold of constant index q such as $m \geq 1, 1 \leq q \leq 2m,$ and M be an m-dimensional submanifold of $\tilde{M}$. In case $\tilde{g}$ is degenerate on the tangent bundle TM of M. We say that M is a lightlike submanifold of $\tilde{M}$ [2]. Throughout this paper we denote the algebra of smooth function on M by *F(M)* and the *F(M)* module of smooth section of a vector bundle E over M by $\Gamma(E)$. The following range of induced is used :

$i, j, k \in \{1,...,r\}$ and $\alpha, \beta, \gamma \in \{r+1,...,n\}$.

For a degenerate tensor field g on M, there exists a locally vector field $\xi \in \Gamma(TM)$, $\xi \neq 0$ such as $g(X, \xi) = 0$ for any $X \in \Gamma(TM)$. Then each tangent space T_xM we have

$$T_xM^{\perp} = \left\{ u \in T_x\tilde{M} \;\middle|\; \tilde{g}(u, v) = 0,\ \forall v \in T_xM \right\}$$

which is degenerate ($m+1$)-dimensional subspace of $T_x\tilde{M}$. The radical (null) subspace of T_xM, denoted by $RadT_xM$, is defined by

$$T_xM^{\perp} = \left\{ \xi_x \in T_x\tilde{M} \;\middle|\; \tilde{g}(\xi_x, X_x) = 0,\ \forall X_x \in T_xM \right\}.$$

The dimension of $RadT_xM$ depends on $x \in M$. The submanifold M of $\tilde{M}$ is said to be r-lightlike submanifold if the mapping

$$RadTM : x \in M \to RadT_xM$$

define a smooth distribution on M of rank r>0, where $RadTM$ is called the radical (null) distribution on M [2]. In this paper, we study lightlike isotropic submanifold where

$$RadTM = \{0\} \ \ and \ \ 1 \leq r \leq m .$$

Therefore, $S(TM) = \{0\}$. Thus we can write (2.1)

$$T\tilde{M}_{|M} = TM \oplus tr(TM) = (TM \oplus ltr(TM)) \perp S(TM^{\perp}).$$

According to the decomposition (2.1), we choose the field of frames $\{\xi_1, ..., \xi_m\}$ and $\{N_1, ..., N_m, W_{m+1}, ..., W_n\}$ on M and $\tilde{M}$, respectively.

Example 2.1 Suppose that (M, g) be a surface of R_2^5 given by equations

$$x^3 = \cos x^1, x^4 = \sin x^1, x^5 = x^2 .$$

We choose a set of vectors $\{\xi_1, \xi_2, u_1, u_2\}$ given by

$$\xi_1 = \partial_2 + \partial_5 ,\ \xi_2 = \partial_1 - \sin x^1 \partial_3 + \cos x^1 \partial_4,$$

$$u_1 = -\sin x^1 \partial_1 + \partial_3,\ \ u_2 = \cos x^1 \partial_1 + \partial_4$$

so that $RadTM = TM = sp\{\xi_1, \xi_2\}$, $TM^{\perp} = sp\{\xi_1, u_1, u_2\}$. Therefore M is an isotropic lightlike submanifold. Construct two null vectors

$$N_1 = \tfrac{1}{2}\{-\partial_1 + \partial_5\} \text{ and}$$

$$N_2 = \tfrac{1}{2}\{-\partial_1 - \sin x^1\partial_3 + \cos x^1\partial_4\}$$

such as $g(N_i, \xi_i) = \delta_{ij}$ for $i, j \in \{1, 2\}$ and $ltr(TM) = sp = \{N_1, N_2\}$.

Let $W = \cos x^1\partial_3 + \sin x^1\partial_4$ be a spacelike vector such that $S(TM^{\perp}) = sp\{W\}$. Thus $\{\xi_1, \xi_2, N_1, N_2, W\}$ is a basis of R_2^5 along *M*.

From (2.1), there exist $\xi_i, W_\alpha \in T_xM^{\perp}$ such that $\tilde{g}(\xi_i, u) = 0,\ \tilde{g}(W_\alpha, W_\beta) \neq 0,\ \forall u \in T_xM^{\perp}$.

Above relation implies that $\xi_i \in TM$ and hence $\xi_i \in RadT_xM$. Thus, locally there exists a lightlike vector fields on *M*, it is also denoted by ξ_i such as

$$\tilde{g}(\xi_i, X) = \tilde{g}(\xi_i, Y) = 0, \forall X \in \Gamma(TM), \forall Y \in \Gamma(TM^{\perp})$$

Consequently, the *m*-dimensional radical distribution *RadTM* of lightlike isotropic submanifld *M* of $\tilde{M}$ is locally spanned by ξ_i. We choose such a non-degenerate distribution on the screen transversal vector bundle $S(TM^{\perp})$ of *M*. Thus we have the following orthogonal direct decomposition

(2.2)

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp}).$$

Thus we choose W_α as a unit vector field and put $\tilde{g}(W_\alpha, W_\beta) = \varepsilon$ where $\varepsilon = \mp 1$.

Theorem 2.2 Let $(M, g, S(TM))$ be an isotropic submanifold of ($\tilde{M}$, $\tilde{g}$). Suppose that *U* be a coordinate neighborhood of *M* and $\{\xi_1, ..., \xi_m\}$ be a basis of $\Gamma(TM_{|u})$. Then there exist smooth $\{N_1, ..., N_m\}$ of $T\tilde{M}_{|M}$ such that $\tilde{g}(N_i, \xi_j) = \delta_{ij}$ and

$$\tilde{g}(N_i, N_j) = 0, \tilde{g}(N_i, W_\alpha) = 0,$$

for any $i, j \in \{1, ..., m\}, \alpha \in \{m+1, ..., n\}$ and $W_\alpha \in \Gamma(S(TM^{\perp}))$. Suppose that $\tilde{\nabla}$ be the Levi-Civita connection on $\tilde{M}$. According to (2.1) we have

(2.3)

$$\tilde{\nabla}_X Y = \nabla_X Y + h^s(X, Y),$$

(2.4)

$$\tilde{\nabla}_X N = -A_N X + \nabla^s{}_X N + \nabla^L{}_X N,$$

(2.5)

$$\tilde{\nabla}_X W = -A_W X + \nabla^s{}_X W + \nabla^L{}_X W$$

for any $X, Y \in \Gamma(TM),\ N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$ where $\{\nabla_X Y, A_N X, A_W X\}$ and $\{h^s(X, Y), \nabla^s{}_X N, \nabla^s{}_X W, \nabla^L{}_X N, \nabla^L{}_X W\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. Here $\tilde{\nabla}$ is the metric connection on $\tilde{M}$ but ∇ and ∇^s are linear connections on *M* and tr(*TM*) respectively. Besides, we define the *F(M)*-bilinear mappings

(2.6)

$$\nabla_X{}^L : \Gamma(ltr(TM)) \to \Gamma(ltr(TM)); \nabla_X{}^L(LV) = D_X{}^L(LV),$$

(2.7)

$$\nabla_X{}^s : \Gamma(S(TM^{\perp})) \to \Gamma(S(TM^{\perp})); \nabla_X{}^s(SV) = D_X{}^s(SV),$$

(2.8)

$$D^L : \Gamma(TM) \times \Gamma(S(TM^{\perp})) \to \Gamma(ltr(TM))$$

$$D^L(X, SV) = D^L{}_X(SV),$$

and

(2.9)

$$D^s : \Gamma(TM) \times \Gamma(ltr(TM)) \to \Gamma(S(TM^{\perp}))$$

$$D^s(X, LV) = D^s{}_X(LV)$$

for any $x \in \Gamma(TM),\ V \in \Gamma(tr(TM))$. Since $\{\xi_i, N_j\}$ are locally lightlike sections on $U \subset M$, we define symmetric $F(M)$-bilinear form D^s and 1-forms $p_{ij,} \tau_{i\alpha}, \theta_{\alpha\beta}$ and $V_{\alpha i}$ on U by

$$D^s(X, Y) = \varepsilon_\alpha \tilde{g}(h^s(X, Y), W_\alpha),$$

$$P_{ij}(X) = \tilde{g}(\nabla_X{}^L N_i, \xi_i),$$

$$\tau_{i\alpha} = \varepsilon_\alpha \tilde{g}(D^s(X, N_i), W_\alpha),$$

$$\theta_{\alpha\beta} = \varepsilon_\beta \tilde{g}(\nabla^s{}_X W_\alpha, W_\beta)$$

and

$$V_{\alpha i} = g(D^L(X, W_\alpha), \xi_i)$$

for any $X, Y \in \Gamma(TM)$. It follows that

$$h^s(X,Y) = D^s(X,Y)W_\alpha,$$
$$\nabla_X{}^L N_i = P_{ij}(X)N_j,$$
$$D^s(X, N_i) = \tau_{i\alpha} W_\alpha,$$
$$\nabla^s{}_X W_\alpha = \theta_{\alpha\beta} W_\beta$$
$$D^L(X, W_\alpha) = V_{\alpha i} N_i.$$

Hence (2.3), (2.4) and (2.5) become

(2.10)

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=m+1}^{n} D^s{}_\alpha(X,Y)W_\alpha,$$

(2.11)

$$\tilde{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{m<n} P_{ij}(X)N_j + \sum_{\alpha=m+1}^{n} \tau_{i\alpha}(X)W_\alpha,$$

(2.12)

$$\tilde{\nabla}_X W_\alpha = -A_{W_\alpha} X + \sum_{i=1}^{m<n} V_{\alpha i}(X)N_i + \sum_{\beta=m+1}^{n} \theta_{\alpha\beta}(X)W_\alpha$$

for any $X, Y \in \Gamma(TM)$. We call D^s the screen second fundamental form of *M* with respect to *tr(TM)*. Both A_{N_i} and A_{W_α} are linear operators on $\Gamma(TM)$. We will see by (2.15) that the first one is *RadTM*-valued, called the shape operations of *M*. Since ξ_i and ξ_j are lightlike vector fields, from (2.10)-(2.12) we obtain

(2.13)

$$D^s(X, \xi_i) = 0,$$

(2.14)

$$D^L(X, \xi_i) = 0,$$

(2.15)

$$\tilde{g}(A_{N_i} X, \xi_i) = \tilde{g}(A_{W_\alpha} X, \xi_i).$$

Further, taking in to account that $\tilde{\nabla}$ is a metric connection and by using (2.10) we obtain

$$0 = (\tilde{\nabla}_X \tilde{g})(Y,Z)$$
$$= X(\tilde{g}(Y,Z)) - \tilde{g}(\tilde{\nabla}_X Y, Z) - \tilde{g}(Y, \tilde{\nabla}_X Z)$$
$$= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$
$$- \sum_{\alpha=m+1}^{n} D^s{}_\alpha(X,Y) g(W_\alpha, Z)$$
$$- \sum_{\alpha=m+1}^{n} D^s{}_\alpha(X,Z) g(W_\alpha, Y)$$
$$= (\nabla_X g)(Y,Z)$$

for any $X, Y, Z \in \Gamma(TM)$. Denote by $\tilde{R}$ and R the curvature tensor of $\tilde{\nabla}$ and ∇ respectively. Then by straightforward calculation and using (2.10), (2.11), (2.12), (2.13), (2.14) and (2.15) we obtain

(2.17)

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \sum_{\alpha=m+1}^{n} \{D^s(X,Y)A_{W_\alpha} Y - D^s(Y,Z)A_{W_\alpha} X + D^s(Y,Z)(\nabla_X W_\alpha) + (\nabla_X D^s)(Y,Z)W_\alpha - D^s(X,Z)(\nabla_Y W_\alpha) + (\nabla_Y D^s)(X,Z)W_\alpha\} + \sum_{i=1}^{m} \sum_{\alpha=m+1}^{n} \{D^s(Y,Z)V_{\alpha i}(X)N_i - D^s(X,Z)V_{\alpha i}(Y)N_i\},$$

(2.18)

$$\tilde{R}(X,Y)W_\beta = \sum_{i=1}^{m} \sum_{\beta=m+1}^{n} \{R^s(X,Y)W_\beta + D^s(Y, A_{W_\beta} X)W_\alpha - D^s(X, A_{W_\beta} Y)W_\alpha + V_{\beta i}(Y)\tau_{i\alpha}(X)W_\beta - V_{\beta i}(X)\tau_{i\alpha}(Y)W_\beta + (\nabla_Y A)(W_\beta, X) - (\nabla_X A)(W_\beta, Y) + V_{\beta i}(X)A_{N_i} Y - V_{\beta i}(Y)A_{N_i} X + (\nabla_X D^L)(Y, W_\beta) - (\nabla_Y D^L)(X, W_\beta)\},$$

(2.19)

$$\tilde{R}(X,Y)N_i = \sum_{i=1}^{m} \sum_{\alpha,\beta=m+1}^{n} \{R^L(X,Y)N_i + \tau_{i\alpha}(Y)V_{\alpha i}(X)N_i - \tau_{i\alpha}(X)V_{\alpha i}(Y)N_i - \tau_{i\alpha}(Y)A_{W_\alpha} X + \tau_{i\alpha}(Y)A_{W_\alpha} X + (\nabla_Y A)(N_i, X) - (\nabla_X A)(N_i, Y) + (\nabla_X D^s)(Y, N_i) - (\nabla_Y D^s)(X, N_i) + D^s(Y, A_{N_i} X) - D^s(X, A_{N_i} Y)$$

for any $X, Y, Z \in \Gamma(TM)$,

$W_\alpha, W_\beta \in \Gamma(S(TM^\perp))$ and $N_i \in \Gamma(ltr(TM))$. Consider the Riemannian curvature of type (0,4) of $\tilde{\nabla}$ and by using (2.17)-(2.19) and the definition of curvature tensor, we derive the following structure equations **:**

(2.20)

$$\tilde{R}(X,Y,Z,N_i) = \sum_{i=1}^{m}\sum_{\beta=m+1}^{n}\{\tilde{g}(R(X,Y)Z,N_i) + \varepsilon_\alpha \tau_{i\alpha}(Y)D^s(X,Z) - \varepsilon_\alpha \tau_{i\alpha}(X)D^s(Y,Z)\},$$

(2.21)

$$\tilde{R}(X,Y,W_\beta,N_i) = \sum_{i=1}^{m}\sum_{\beta=m+1}^{n}\{\tilde{g}((\nabla_Y A)(W_\beta,X) - (\nabla_X A)(W_\beta,Y),N_i) + V_{\beta i}(Y)\tilde{g}(A_{N_i}X,N_i) - V_{\beta i}(X)\tilde{g}(A_{N_i}Y,N_i)\},$$

(2.22)

$$\tilde{R}(X,Y,N_i,N_i^{'}) = \sum_{i=1}^{m}\sum_{\beta=m+1}^{n}\{\tilde{g}((\nabla_X A)(N_i^{'},X) - (\nabla_Y A)(N_i^{'},Y),N_i) + \varepsilon_\alpha \tau^{'}_{i\alpha}(X)\varepsilon_\alpha \tau_{i\alpha}(Y) - \varepsilon_\alpha \tau^{'}_{i\alpha}(Y)\varepsilon_\alpha \tau_{i\alpha}(X)\}.$$

Theorem 2.3 Let M be lightlike isotropic submanifold of an *(2m+n)*-dimensional semi-Riemannian manifold of constant curvature $(\tilde{M}(c),\tilde{g})$, and of codimension *(m+n)*. Then the curvature tensor of M and $\tilde{M}(c)$ related to the following equations **:**

(2.23)

$$R(X,Y)Z = \sum_{i=1}^{m}\sum_{\alpha=m+1}^{n}\{D^s(Y,Z)A_{W_\alpha}X - D^s(X,Z)A_{W_\alpha}Y + D^s(X,Z)V_{\alpha i}(Y)N_i - D^s(Y,Z)V_{\alpha i}(X)N_i + D^s(X,Z)(\nabla_Y W_\alpha) - D^s(Y,Z)(\nabla_X W_\alpha) + (\nabla_Y D^s)(X,Z)W_\alpha - (\nabla_X D^s)(Y,Z)W_\alpha\},$$

(2.24)

$$R^s(X,Y)W_\beta = \sum_{i=1}^{m}\sum_{\beta=m+1}^{n}\{D^s(X,A_{W_\beta}Y)W_\alpha - D^s(Y,A_{W_\beta}X)W_\alpha + V_{\beta i}(X)\tau_{i\alpha}(Y)W_\beta - V_{\beta i}(Y)\tau_{i\alpha}(X)W_\beta + (\nabla_X A)(W_\beta,Y) - (\nabla_Y A)(W_\beta,X) + V_{\beta i}(Y)A_{N_i}X - V_{\beta i}(X)A_{N_i}Y + (\nabla_Y D^L)(X,W_\beta) - (\nabla_X D^L)(Y,W_\beta)\},$$

(2.25)

$$R^L(X,Y)N_i = \sum_{i=1}^{m}\sum_{\alpha,\beta=m+1}^{n}\{\tau_{i\alpha}(X)V_{\alpha i}(Y)N_i - \tau_{i\alpha}(Y)V_{\alpha i}(X)N_i + \tau_{i\alpha}(Y)A_{W_\alpha}X - \tau_{i\alpha}(X)A_{W_\alpha}Y + (\nabla_X A)(N_i,Y) - (\nabla_Y A)(N_i,X) + D^s(X,A_{N_i}Y) - D^s(Y,A_{N_i}X)\} + (\nabla_Y D^s)(X,N_i) - (\nabla_X D^s)(Y,N_i)$$

Proof. By using the definition of constant curvature $\tilde{M}(c)$ and (2.17), (2.18) and (2.19) we obtain (2.23), (2.24) and (2.25).

Definition 2.4. A lightlike isotropic submanifold *(M,g)* of a semi-Riemannian manifold $(\tilde{M},\tilde{g})$ is said to be totally umbilical in $\tilde{M}$ if there is a smooth vector field H^s such as

(2.26)

$$h^s(X,Y) = H^s\tilde{g}(X,Y), \forall X,Y \in \Gamma(TM).$$

Corollary 2.5. Lightlike isotropic submanifold of $(\tilde{M},\tilde{g})$ is totally geodesic if M is totally umbilical , i.e.,

$$h^s(X,Y) = 0, \forall X,Y \in \Gamma(TM).$$

Then we have

Theorem 2.6. Let (M,g) be a isotropic submanifold of $(\tilde{M},\tilde{g})$ of codimension (*m+n*) if M is totally umbilical in $\tilde{M}$ then

(2.27)

$$\tilde{R}(X,Y)Z = R(X,Y)Z, \forall X,Y,Z \in \Gamma(TM).$$

Proof. By using (2.17) and Corollary 2.3 we get (2.27).

Corollary 2.7. Under the hypothesis of theorem we have

(2.28)

$$\tilde{R}(X,Y)W_\beta = R^s(X,Y)W_\beta + (\nabla_Y A)(W_\beta, X) - (\nabla_X A)(W_\beta, Y),$$

(2.29)

$$\tilde{R}(X,Y)N_i = (\nabla_Y A)(N_i, X) - (\nabla_X A)(N_i, Y).$$

References

[1] Bejancu, A., Null Hypersurfaces in Semi-Euclidean Space, Saitama Math. J. Vol. 14, 25-40, (1996).

[2] Duggal, K., and Bejancu, A., Lightlike Submanifold of Semi-Riemannian Manifold and Applications, Kluver Academic Pub., (1996).

[3] Duggal, K. and Jin, D.H., Half Lightlike Submanifolds of Codimension 2, Math. J. Toyama Uni., 22, 121-161, (1999)

[4] Duggal, K.L. and Bejancu A., Lightlike Submanifold of Semi-Riemannian Manifold Applications, Acta Appl. Math 38, 197-215, (1995).

[5] Kılıç, E., Şahin, B., H.B. Karadağ, Güneş, R., Coisotropic Submanifold of a Semi-Riemannian Manifold, Turk J. Math., 28, 335-352, (2004).

[6] Katsuno, K., Null Hypersurfaces in Lorentzian Manifolds I, Math. Proc. Camb. Phil. Soc. 88, 175-182, (1980).

[7] O' Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, London, (1983).

Geliş Tarihi: 13/10/2005 *Kabul Tarihi: 30/01/2006*