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MİNKOWSKİ UZAYINDA LORENTZ KÜRESEL TIMELIKE VE NULL EĞRİLER ÜZERİNE

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Özet: Bu çalışmada E_1^4 Minkowski uzayındaki Lorentz küresel timelike ve null eğrileri karakterize edeceğiz. Ayrıca E_1^4 Minkowski uzayındaki S_1^3 Lorentziyen küresi üzerinde null eğrilerin olmadığını ispatlayacağız.

Anahtar Kelimeler: *Minkowski uzayı, Lorentz küresi, eğrilik ve burulma.*

ON LORENTZIAN SPHERICAL TIMELIKE AND NULL CURVES IN MINKOWSKI SPACE-TIME

Abstract: In this paper, we characterize the Lorentzian spherical timelike and null curves in Minkowski space-time E_1^4**.** Moreover, we prove that there are no null curves lying on the Lorentzian space S_1^3 in Minkowski space-time E_1^4.

Key words: *Minkowski space-time, Lorentzian sphere, curvature and torsion.*

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1.Introduction

Necessary and sufficient conditions for a curve to be a spherical curve in Euclidean space E^3 were given in [7] and [8]. In [2] and [4] the authors characterized the Lorenzian spherical spacelike curves in the Minkowski 3-space E_1^3. A similar characterization of a spacelike, a timelike and a null curves lying on the pseudohyperbolic space H_0^3 in the Minkowski spac-time E_1^4 was obtained in [1]. The corresponding Frenet equations for an arbitrary curve in the Minkowski space-time E_1^4 were given in [6]. By using this equations, in this paper we give some conditions for a timelike and a null curves in Minkowski space-time E_1^4.

2.Preliminaries

Minkowski space-time E_1^4 is the Euclidean 4-space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4.

An arbitrary vector $v = (v_1, v_2, v_3, v_4)$ in E_1^4 may be one of three Lorentzian causal characters; it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike). Also recall that the pseudo-norm of an arbitrary vector $v \in E_1^4$ is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore α is a unit vector if $g(v,v) = \pm 1$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$. The Loretzian sphere of center $m = (m_1, m_2, m_3, m_4)$ and radius $r \in IR^+$ in the space E_1^4 is defined by

$$S_1^3(m,r) = \left\{ a = (a_1, a_2, a_3, a_4) \in E_1^4 : g(a-m, a-m) = r^2 \right\}.$$

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are the tangent, the principal normal, the first binormal and the second binormal fields, respectively. Timelike curve $\alpha(s)$ is said to be parameterized by a pseudo-arclength parameter s, i.e., $g(\alpha'(s), \alpha'(s)) = -1$. In particular, null curve $\alpha(s)$ in E_1^4 is parameterized by a pseudo-arclength parameter s, if $g(\alpha''(s), \alpha''(s)) = 1$, where pseudo-arclenght function s is defined in [9] by

$$s = \int_0^t \left(g(\alpha''(t), \alpha''(t)) \right)^{1/4}.$$

Let $\alpha(s)$ be a curve in the space-time E_1^4, parameterized by arclength function of s. Then for the curve α the following Frenet equations are given in [6]:

Case 1. α is a timelike curve

Then T is timelike vector, so the Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying equations

$$g(T,T) = -1,$$
$$g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1.$$

Case 2. α is a null curve

Then T is null vector, so the Frenet formulae has the form

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_2 & 0 & -k_1 & 0 \\ 0 & -k_2 & 0 & k_3 \\ -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where the first curvature k_1 can take only two values: 0 when α is a straight null line or 1 in all other cases. In this case, the vectors T, N, B_1, B_2 satisfy the equations

$$g(T,T) = g(N,N) = g(B_1,B_1) = 0, g(B_2,B_2) = 1$$
$$g(T,N) = g(T,B_2) = g(N,B_1) = 1,$$
$$g(N,B_2) = g(B_1,B_2) = 0,\ g(T,B_1) = 1.$$

Recall the functions $k_1 = k_1(s),\ k_2 = k_2(s)$ and $k_3 = k_3(s)$ are called respectively, the first, the second and the third curvature of curve $\alpha(s)$.

3. The Lorentzian Spherical Timelike Curves

Theorem 3.1. Let $\alpha(s)$ be a unit speed timelike curve in E_1^4 with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$, $k_3(s) \neq 0$ for each $s \in I \subset IR$. If α lies on S_1^3 then

$$\left.\begin{aligned} &(k_3/k_2)(1/k_1)' \\ &+\left((1/k_3)\left((k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right)\right)' = 0. \end{aligned}\right\}(1)$$

Proof: Assume that α lies on S_1^3 with center m and radius r. Then $g(\alpha - m, \alpha - m) = r^2$ for each $s \in I \subset IR$. Differentiating this equation four times with respect to s and by applying Frenet equations, we get

$$\left.\begin{aligned} &g(T, \alpha - m) = 0 \\ &g(N,\ \alpha - m) = 1/k_1 \\ &g(B_1,\ \alpha - m) = (1/k_2)(1/k_1)' \\ &g(B_2,\ \alpha - m) = (1/k_3)\left[k_2/k_1 + \left((1/k_2)(1/k_1)'\right)'\right] \end{aligned}\right\}(2)$$

Next, decompose the vector $\alpha - m$ with respect to the pseudo orthonormal basis $\{T(s),\ N(s),\ B_1(s), B_2(s)\}$ by

$$\begin{aligned} \alpha(s) - m &= a(s)T(s) + b(s)N(s) \\ &\quad + c(s)B_1(s) + d(s)B_2(s) \end{aligned} \qquad (3)$$

where $a(s), b(s), c(s)$ and $d(s)$ are arbitrary functions. Then by (2) we find

$$\left.\begin{aligned} &g(T, \alpha - m) = -a = 0 \\ &g(N,\ \alpha - m) = b = 1/k_1 \\ &g(B_1,\ \alpha - m) = c = (1/k_2)(1/k_1)' \\ &g(B_2,\ \alpha - m) = d \\ &\qquad = (1/k_3)\left[(k_2/k_1) + \left((1/k_2)(1/k_1)'\right)'\right] \end{aligned}\right\}(4)$$

Therefore, substituting (4) into (3) we find

$$\left.\begin{aligned} \alpha(s) - m &= (1/k_1)N + (1/k_2)(1/k_1)'B_1 \\ &+ (1/k_3)\left[(k_2/k_1) + \left((1/k_2)(1/k_1)'\right)'\right]B_2 \end{aligned}\right\}(5)$$

and so we can write

$$\begin{aligned} &(1/k_1)^2 + \left[(1/k_2)(1/k_1)'\right]^2 \\ &+ \left[(1/k_3)\left((k_2/k_1) + \left((1/k_2)(1/k_1)'\right)'\right)\right]^2 = r^2 \end{aligned} \qquad (6)$$

Now, we may consider the vector $m \in E_1^4$ given by

$$\begin{aligned} m &= \alpha(s) - (1/k_1)N - (1/k_2)(1/k_1)'B_1 \\ &\quad - (1/k_3)\left[(k_2/k_1) + \left((1/k_2)(1/k_1)'\right)'\right]B_2 \end{aligned} \qquad (7)$$

Since α lies on S_1^3 with center m and radius r; m and r must be constant, i.e. m' and r' must be zero. Differentiating (7) with respect to s and by using Frenet formulae, we obtain

$$\begin{aligned} m' = -\Big[&(k_3/k_2)(1/k_1)' \\ &+\left((1/k_3)\left((k_2/k_1) + \left((1/k_2)(1/k_1)'\right)'\right)\right)'\Big]B_2 \end{aligned} \qquad (8)$$

and differentiating (6) with respect to s and by using Frenet formulae, we have

$$\left.\begin{aligned}&(1/k_3)\left[(k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right]\\&\left[(k_3/k_2)(1/k_1)'\right.\\&\left.+\left((1/k_3)\left((k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right)\right)'\right]=2rr'\end{aligned}\right\}(9)$$

and so $m'=0$ and $r'=0$ if and only if

$$(k_3/k_2)(1/k_1)' + \left((1/k_3)\left((k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right)\right)' = 0$$

and so (1) holds.

Theorem 3.2. If a unit speed timelike curve $\alpha(s)$ in E_1^4 with curvatures $k_1(s)\neq 0$, $k_2(s)\neq 0,\ k_3(s)\neq 0$ for each $s\in \mathrm{I}\subset IR$ lies on a Lorentzian sphere in E_1^4 then there exists a differentiable function $f(s)$ such that

$$\left.\begin{aligned}&f(s)=(1/k_3)\left((k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right),\\&f'(s)=-(k_3/k_2)(1/k_1)'.\end{aligned}\right\}(10)$$

Proof: Assume that α lies on S_1^3. Then by Theorem 3.1 (1) holds. Next, let us define the differentiable function $f(s)$ by

$$f(s)=(1/k_3)\left((k_2/k_1)+\left((1/k_2)(1/k_1)'\right)'\right) \quad (11)$$

By using (1) we easily find that the relations (10) are satisfied.

Theorem 3.3. Let $\alpha(s)$ be a unit speed timelike curve in E_1^4 with curvatures $k_1(s)\neq 0$, $k_2(s)\neq 0,\ k_3(s)\neq 0$ for each $s\in \mathrm{I}\subset IR$ and α lies on S_1^3 in E_1^4. Then, there exist constants $A,B\in IR$ such that the following relations hold:

$$\left.\begin{aligned}(1/k_2)(1/k_1)' &= \left[A-\int_0^s (k_2/k_1)\sin\theta ds\right]\sin\left(\int_0^s k_3 ds\right)\\&\quad-\left[B+\int_0^s (k_2/k_1)\cos\theta ds\right]\cos\left(\int_0^s k_3 ds\right)\\ f(s) &= \left(A-\int_0^s (k_2/k_1)\sin\theta ds\right)\cos\theta\\&\quad+\left(B+\int_0^s (k_2/k_1)\cos\theta ds\right)\sin\theta\end{aligned}\right\}(12)$$

Proof: Let us suppose that α lies on S_1^3. By Theorem 3.2, there exists a differentiable function $f(s)$ such that (10) holds. Next let us define the C^2-function $\theta(s)$ by $\theta(s)=\int_0^s k_3 ds$. Moreover, let us define the C^1 functions $g(s)$ and $h(s)$ by

$$\left.\begin{aligned}g(s)&=(1/k_2)(1/k_1)'\sin\theta+f(s)\cos\theta\\&\quad+\int_0^s (k_2/k_1)\sin\theta ds,\\ h(s)&=-(1/k_2)(1/k_1)'\cos\theta+f(s)\sin\theta\\&\quad-\int_0^s (k_2/k_1)\cos\theta ds.\end{aligned}\right\}(13)$$

Differentiating functions $\theta(s), g(s)$ and $h(s)$ with respect to s, we find $\theta'(s)=k_3$, $g'(s)=h'(s)=0$. Hence, $g(s)=A$, $h(s)=B$, $A,B\in IR$. So the relation (13) becomes

$$\left.\begin{aligned}&(1/k_2)(1/k_1)'\sin\theta+f(s)\cos\theta\\&+\int_0^s (k_2/k_1)\sin\theta ds=A,\\&-(1/k_2)(1/k_1)'\cos\theta+f(s)\sin\theta\\&-\int_0^s (k_2/k_1)\cos\theta ds=B.\end{aligned}\right\}(14)$$

Multiplying the first of the equations in (14) with $\sin\theta$ and the second with $-\cos\theta$ and adding, we find that the first equation in (12) holds. Next, by multiplying the first of the equations in (14) with $\cos\theta$ and second with $\sin\theta$ and adding, we get

$$f(s)=\left(A-\int_0^s (k_2/k_1)\sin\theta ds\right)\cos\theta$$
$$+\left(B+\int_0^s (k_2/k_1)\cos\theta ds\right)\sin\theta$$

and that finishes the proof.

4.The Lorentzian Spherical Null Curves

Theorem 4.1. There are no null curves $\alpha(s)$ lying on the Lorentzian sphere S_1^3 in E_1^4.

Proof: Assume that $\alpha(s)$ is a null curve lying on the Lorentzian sphere of center $m \in E_1^4$ and radius $r \in IR^+$. Then we have

$$g(\alpha - m, \alpha - m) = r^2 \qquad (15)$$

for each $s \in I \subset IR$. Differentiating (15) with respect to s we get $g(T, \alpha - m) = 0$. It means that null vector T and spacelike vector $\alpha - m$ are orthogonal vectors in E_1^4 which is a contradiction. So there are no null curves $\alpha(s)$ lying on the Lorentzian sphere S_1^3 in E_1^4.

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