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RESEARCH ARTICLE

ON SOME CLASSES OF WEAKLY SUBDIFFERENTIABLE FUNCTIONS

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Abstract

This work presents a theorem that any Lipschitz function is weakly subdifferentiable with $\boldsymbol{x}^*$ component of the weak subgradient is different from $\mathbf{0}_{\mathbb{R}^n}$. This theorem is based on Kasimbeyli's nonlinear cone separation theorem. Also, we show that any positively homogeneous and continuous function is both upper and lower Lipschitz. Additionally, we show that positively homogeneous and lower semicontinuous functions are weakly subdifferentiable that the pair $(\boldsymbol{x}^*, \boldsymbol{c})$ which is a weak subgradient of a function in this case is different from $(\mathbf{0}_{\mathbb{R}^n}, \mathbf{0})$.

Keywords

Nonconvex Optimization,
The Weak Subdifferential,
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1. INTRODUCTION

It is quite famous in convex analysis that at each boundary point a convex set has a supporting hyperplane. This idea leads to one of the central concepts of convex analysis, which is called subgradient. The study of subgradients in convex optimization has been a cornerstone in developing methods for optimality conditions and duality theorems [4,12,13]. However, if the set is not convex there does not exist any supporting hyperplanes at boundary points. Many researchers have tried to generalize for nonconvex problems on optimality conditions. Among these contributions, Azimov and Gasimov's weak subgradient definition emerges as a significant concept, introducing a novel approach to analyzing nonsmooth functions while retaining computational and theoretical practicality and the idea is very useful for analyzing optimality conditions in nonconvex optimization [1,3,5,6,7,8]. In [1,7] they use support cones instead of supporting hyperplanes. Therefore, this enables us to broaden the subdifferentiable class to the lower Lipschitz function class. This motivates our study on a broader class of weakly subdifferentiable functions.

In [10], it has been introduced a distinct separation property in Banach spaces for two closed cones, along with a nonlinear separation theorem applicable to cones having this relation. It also extends traditional dual cones definitions by introducing augmented dual cones. Also, it is well known that any lower Lipschitz function satisfy the weak subdifferentiability [1,2]. Based on the separation theorem

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we establish a theorem that any Lipschitz function is weakly subdifferentiable with x^* component of (x^*, c) is different than zero vector of $\mathbb{R}^n$.

It is proven in [9] that positively homogeneous and continuous function is subdifferentiable. In this work, we show that positively homogeneous and lower semicontinuous function is weakly subdifferentiable.

2. PRELIMINARIES

Consider a normed space $(\mathbb{Y}, \|\cdot\|)$.

$$\mathbb{U} = \{\boldsymbol{y} \in \mathbb{Y}: \|\boldsymbol{y}\| = \mathbf{1}\}$$

is referred to as the unit sphere of $(\mathbb{Y}, \|\cdot\|)$ and,

$$\mathbb{B} = \{\boldsymbol{y} \in \mathbb{Y}: \|\boldsymbol{y}\| \leq \mathbf{1}\},$$

is referred to as the *unit ball* of the space $(\mathbb{Y}, \|\cdot\|)$. The nonempty set $\mathbb{C}$ contained in $\mathbb{Y}$ is considered a *cone* if, for any element $\boldsymbol{y}$ in $\mathbb{C}$ and any non-negative scalar $\boldsymbol{\lambda} \geq \mathbf{0}$, then $\boldsymbol{\lambda y}$ also belongs to $\mathbb{C}$.

A cone $\mathbb{C}$ is *pointed* if

$$\mathbb{C} \cap (-\mathbb{C}) = \{\mathbf{0}_{\mathbb{Y}}\}.$$

A cone *generating* by a set $\mathbb{S}$ is denoted by *cone*($\mathbb{S}$):

$$cone(\mathbb{S}) = \{\boldsymbol{\lambda s}: \boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{s} \in \mathbb{S}\}.$$

Let $\mathbb{C} \neq \{\mathbf{0}_{\mathbb{Y}}\}$ be a convex cone. A nonempty convex subset $\mathbb{D}$ of $\mathbb{C}$ a is referred to as a base for $\mathbb{C}$ if each $\boldsymbol{y} \in \mathbb{C}$ if each $\boldsymbol{y} \in \mathbb{C}\backslash\{\mathbf{0}_{\mathbb{Y}}\}$ has a distinct representation in the form $\boldsymbol{y} = \boldsymbol{\lambda d}$ for some $\boldsymbol{\lambda} > \mathbf{0}$ and some $\boldsymbol{d} \in \mathbb{D}$. Throughtout this work, The norm base of the cone $\mathbb{C}$ is represented by $\mathbb{C}_{\mathbb{U}} = \mathbb{C} \cap \mathbb{U}$=$\{\boldsymbol{y} \in \mathbb{C}: \|\boldsymbol{y}\| = \mathbf{1}\}$.

Definition 2.1: Consider $(\mathbb{Y}, \|\cdot\|)$ as a real normed space where partial ordering is determined by a cone $\mathbb{C}$ which is closed, convex, pointed . The definition of the dual cone $\mathbb{C}^*$ is as follows:

$$\mathbb{C}^* = \{\boldsymbol{z}^* \in \mathbb{Y}^*: \langle \boldsymbol{z}^*, \boldsymbol{z} \rangle \geq \mathbf{0}, \forall \boldsymbol{z} \in \mathbb{C}\}.$$

and quasi interior of $\mathbb{C}^*$ denoted by $\mathbb{C}^{\#}$ is given as follows:

$$\mathbb{C}^{\#} = \{\boldsymbol{z}^* \in \mathbb{Y}^*: \langle \boldsymbol{z}^*, \boldsymbol{z} \rangle > \mathbf{0}, \forall \boldsymbol{z} \in \mathbb{C}\backslash\{\mathbf{0}\}\}.$$

The extended version of these definitions are presented in [10] and called augmented dual cones are given as follows: Let

$$\mathbb{C}^{\boldsymbol{a}*} = \{(\boldsymbol{z}^*, \boldsymbol{a}) \in \mathbb{C}^{\#} \times \mathbb{R}_+: \langle \boldsymbol{z}^*, \boldsymbol{z} \rangle - \boldsymbol{\alpha}\|\boldsymbol{z}\| \geq \mathbf{0}, \forall \boldsymbol{z} \in \mathbb{C}\},$$

$$\mathbb{C}^{\boldsymbol{a}\circ} = \{(\boldsymbol{z}^*, \boldsymbol{a}) \in \mathbb{C}^{\#} \times \mathbb{R}_+: \langle \boldsymbol{z}^*, \boldsymbol{z} \rangle - \boldsymbol{\alpha}\|\boldsymbol{z}\| > \mathbf{0}, \forall \boldsymbol{z} \in \boldsymbol{int}(\mathbb{C})\},$$

and

$$\mathbb{C}^{\boldsymbol{a}\#} = \{(\boldsymbol{z}^*, \boldsymbol{a}) \in \mathbb{C}^{\#} \times \mathbb{R}_+: \langle \boldsymbol{z}^*, \boldsymbol{z} \rangle - \boldsymbol{\alpha}\|\boldsymbol{z}\| > \mathbf{0},\ \forall \boldsymbol{z} \in \mathbb{C}\backslash\{\mathbf{0}_{\mathbb{Y}}\}\}.$$

In the definition of $\mathbb{C}^{\boldsymbol{a}\circ}$ assumes that the interior of the ordering cone $\boldsymbol{int}(\mathbb{C})$ is not empty.

Definition 2.2: A pair (x^*, c) which is in $\mathbb{R}^n \times \mathbb{R}_+$is referred to as a weak subgradient of h at x_0 on the set S provided that

$$\langle x^*, x - x_0 \rangle - c\|x - x_0\| \leq h(x) - h(x_0) \text{ for all } x \in S \tag{1}$$

The weak subdifferential set contains all weak subgradients of h at x_0 and it is represented as $\partial_S^w h(x)$:

$$\partial_S^w h(x_0)=\{(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+ : (1) \text{ is satisfied}\}$$

Remark 2.3: If $\partial_S^w h(x_0) \neq \emptyset$, then h is called the weakly subdifferentiable at x_0. If we let $S = \mathbb{R}^n$ then we ignore the subscript S in $\partial_S^w h(x_0)$, and denote it by $\partial^w h(x_0) = \partial_{\mathbb{R}^n}^w h(x_0)$. It is obvious that if function h is subdifferentiable at x_0 then h is also weakly subdifferentiable at x_0. One can check if $x^* \in \partial h(x_0)$ then by definition $(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$ for every $c \geq 0$. The weak subgradient of h is geometrically interpreted as:
$(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$ is a weak subgradient of h at $x_0 \in X$ if one can found a function

$$f(x) = \langle x^*, x - x_0 \rangle - c\|x - x_0\| + h(x_0) \tag{2}$$

which is continuous, concave and, satisfies $h(x) \leq f(x)$, $\forall x \in X$ and $h(x_0) = f(x_0)$. The hypograph of the function f is defined as hypo $(f) = \{(x, a) \in X \times \mathbb{R} \mid f(x) \geq a\}$ and it is a closed cone in $X \times \mathbb{R}$ with its vertex at $(x_0, f(x_0))$. To verify:

$$\begin{aligned} hypo\ (f) - \big(x_0, h(x_0)\big) &= \big\{\big(x - x_0, a - h(x_0)\big) \in X \times \mathbb{R} \mid \langle x^*, x - x_0 \rangle - c\|x - x_0\| \geq a - h(x_0)\big\} \\ &= \{(u, b) \in X \times \mathbb{R} \mid \langle x^*, u \rangle - c\|u\| \geq b\}. \end{aligned}$$

Thus, from (1) and (2) hypo (f) is a supporting cone of the set

$$epi\ (\mathrm{f}) = \{(x, a) \in X \times \mathbb{R} \mid f(x) \leq a\}$$

at the point $(x_0, h(x_0))$ in the way that epi(f) $\subset (X \times \mathbb{R}) \setminus hypo(f)$ and $cl(epi(f)) \cap graph(f) \neq \emptyset$ where $graph\ (f) = \{(x, a) \in X \times \mathbb{R} \mid f(x) = a\}$.

In [1], they derived the weak subdifferential for the specific subclasses of lower Lipschitz functions. Lower Lipschitz function definition is given as follows:

Definition 2.4: A function g from X into $\mathbb{R}$ is referred as "lower locally Lipschitz" at $x_0 \in X$ if there exists a positive constant L and a neighborhood $\mathcal{N}(x_0)$ around x_0 such that

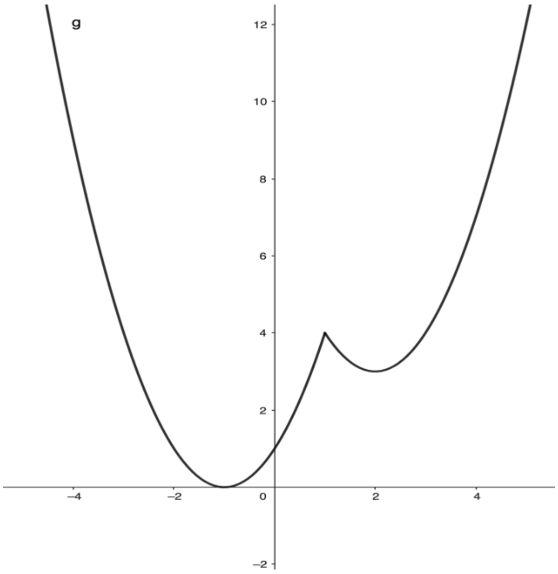
$$-L\|x - x_0\| \leq g(x) - g(x_0), \ \forall x \in \mathcal{N}(x_0). \tag{3}$$

The function g is said to be lower Lipschitz at x_0 where L is called the Lipschitz constant if for all $x \in X$ the inequality (3) holds true.
An example of the weak subdifferential is presented.

Example 2.4: Let $g: \mathbb{R} \to \mathbb{R}$ be given as

$$g(x) = \begin{cases} (x+1)^2 & if\ x \leq 1, \\ (x-2)^2 + 3 & if\ x > 1. \end{cases}$$

The graph of function g is given below.



We want to calculate the weak subdifferentiable of g at $x_0 = 1$. Clearly, function g is not subdifferentiable at $x_0 = 1$.

First, the case $x \leq 1$ is considered. The definition 2.2 implies that:

$$\langle t, x - x_0 \rangle - c\|x - x_0\| \leq g(x) - g(x_0) = (x+1)^2 - g(1)$$
$$t(x-1) - c(1-x) \leq (x^2 + 2x + 1) - 4$$
$$(x-1)(t+c) \leq x^2 + 2x - 3$$

Then $\partial^w g(1)$ for the case $x \leq 1$ obtained as:

$$\partial^w g(1) = \{(t, c) \in \mathbb{R} \times \mathbb{R}_+ : v + c > 4\}$$

Then we consider the case $x > 1$. The weak subdifferential definition indicates that:

$$\langle t, x - x_0 \rangle - c\|x - x_0\| \leq g(x) - g(x_0)$$
$$\langle t, x - 1 \rangle - c|x - 1| \leq (x-2)^2 + 3 - g(1)$$
$$(t-c)(x-1) \leq (x^2 - 4x + 4) + 3 - 4$$
$$(x-1)(t-c) \leq x^2 - 4x + 3$$

Then $\partial^w g(1)$ for the case $x > 1$ obtained as:

$$\partial^w g(1) = \{(t, c) \in \mathbb{R} \times \mathbb{R}_+ : t - c \leq -2\}.$$

Then finally we obtained that:

$$\partial^w g(1) = \{(t, c) \in \mathbb{R} \times \mathbb{R}_+ : -c + 4 \leq t \leq c - 2\}.$$

3. A CLASS OF WEAKLY SUBDIFFERENTIABLE FUNCTIONS

The below corollary provides a condition for a function to be weakly subddifferentiable.

Corollary 3.1: [1, Corollary 3.1] Assume that $\boldsymbol{h}$ is bounded from below and proper function from $\boldsymbol{X}$ into $\mathbb{R} \cup \{+\infty\}$ and lower locally Lipschitz at $\bar{\boldsymbol{x}}$. Then $\boldsymbol{h}$ is said to be weakly subdifferentiable at $\bar{\boldsymbol{x}}$.

The following theorem express a criteria for a weakly subddifferentiability of a function.

Theorem 3.2: [2, Theorem 1] Assume that function $\boldsymbol{h}$: $\mathbb{R}^{\boldsymbol{n}}$ is finite at $\boldsymbol{x_0}$ then the following conditions are equivalent:

i) $\boldsymbol{h}$ is lower Lipschitz at $\overline{\boldsymbol{x}}$.
ii) $\boldsymbol{h}$ is weakly subdifferentialbe at $\overline{\boldsymbol{x}}$.
iii) $\boldsymbol{h}$ is lower locally Lipschitz at $\overline{\boldsymbol{x}}$ and there exists numbers $\boldsymbol{p} \geq \boldsymbol{0}$ and $\boldsymbol{q}$ such that

$$\boldsymbol{h}(\boldsymbol{y}) \geq -\boldsymbol{p}\|\boldsymbol{y}\| + \boldsymbol{q}, \forall \boldsymbol{y} \in \mathbb{R}^{\boldsymbol{n}}.$$

Lemma 3.3: [9, Lemma 2.7] Let $\boldsymbol{f}$ be bounded from below on some neighborhood of zero and positively homogeneous function from $\boldsymbol{X}$ into $\mathbb{R}$. Then $\boldsymbol{f}$ is a weakly subdifferentiable at $\boldsymbol{0_X}$.

We define the subsequent norm on $\mathbb{R}^{n+1}$.

Let $v \in \mathbb{R}^n$ and $a \in \mathbb{R}$ then

$$\|(v, c)\| = \|v\| + |c|$$

defines a norm on $\mathbb{R}^{n+1}$. One can verify the norm properties easily.
The following definition concerns with the separation of the cones in normed spaces.

Definition 3.4: [10, Definition 4.1] Let $(\mathbb{Y}, \|\cdot\|)$ be a normed space and assume that $\mathbb{C}$ and $\mathbb{K}$ be closed cones taken from $(\mathbb{Y}, \|\cdot\|)$ with norm bases $\mathbb{C}_{\mathbb{U}}$ and $\mathbb{K}_{\mathbb{U}}$, respectively. Suppose that ${\mathbb{K}_{\mathbb{U}}}^{\partial} = \mathbb{K}_{\mathbb{U}} \cap bd(\mathbb{K})$, and let $\widetilde{\mathbb{K}}^{\partial}$ and $\tilde{\mathbb{C}}$ represent the closures of the sets $co({\mathbb{K}_{\mathbb{U}}}^{\partial} \cup \{0_{\mathbb{Y}}\})$ and $co(\mathbb{C}_{\mathbb{U}})$, respectively. The separation relation holds with respect to norm $\|\cdot\|$ for the cones $\mathbb{C}$ and $\mathbb{K}$ if

$$\tilde{\mathbb{C}} \cap \widetilde{\mathbb{K}}^{\partial} = \emptyset. \tag{4}$$

The following lemma is proved in [10] and we rewrite the theorem for the $\mathbb{R}^{n+1}$ case.

Lemma 3.5: Suppose that $\mathbb{C}$ and $\mathbb{K}$ denote two nonempty cones in the space $\mathbb{Y}$. Assume that $\mathbb{C}^{a*} \neq \emptyset$. Then for each $((x^*, a^*), \alpha) \in \mathbb{C}^{a*}$ with $\alpha > 0$, the sublevel sets $S((x^*, a^*), \alpha)$ defined by

$$S((x^*, a^*), \alpha) = \{(x, a) \in \mathbb{Y}: \langle (x^*, a^*), (x, a) \rangle + \alpha\|(x, a)\| \leq 0 \}$$

is a pointed and closed cone that contains $-\mathbb{C}$.

Proof. The proof can be done by following similar steps of the proof of Lemma 3.2 in [9].

We following theorem is presented in [10] and we show that when the cones $\mathbb{C}$ and $\mathbb{K}$ belong to the $\mathbb{R}^{n+1}$ the theorem remains true.

Theorem 3.6: Assume that $\mathbb{C}$ and $\mathbb{K}$ be two closed cones are taken from a reflexive Banach space $(\mathbb{Y}, \|\cdot\|)$. Suppose the cones $-\mathbb{C}$ and $\mathbb{K}$ fulfill the separation relation outlined in definition 3.3,

$$\tilde{\mathbb{C}} \cap \widetilde{\mathbb{K}}^{\partial} = \emptyset.$$

It implies that, $\mathbb{C}^{a\#} \neq \emptyset$ and there exists $((x^*, a^*), \alpha) \in \mathbb{C}^{a\#}$ such that the corresponding sublevel set

$S((x^*, a^*), \alpha)$ of the strongly monotonically increasing sublevel function

$$g(x,a) = \langle (x^*, a^*), (x,a) \rangle + \alpha \|(x,a)\|$$

separates the cones $-\mathbb{C}$ and $bd(\mathbb{K})$ in the following manner

$$\langle (x^*, a^*), (\hat{x}, \hat{a}) \rangle + \alpha \|(\hat{x}, \hat{a})\| < 0 \leq \langle (x^*, a^*), (x,a) \rangle + \alpha \|(x,a)\| \tag{5}$$

for all $(\hat{x}, \hat{a}) \in -\mathbb{C} \backslash \{0_{\mathbb{Y}}\}$ and $(x,a) \in bd(\mathbb{K})$. Then $-\mathbb{C}$ is pointed cone. Conversely, if there exists a pair $((x^*, a^*), \alpha) \in \mathbb{C}^{a\#}$ such that the corresponding sublevel set

$S((x^*, a^*), \alpha)$ of the strongly monotonically increasing sublevel function

$$g(x,a) = \langle (x^*, a^*), (x,a) \rangle + \alpha \|(x,a)\|$$

separates the cones $-\mathbb{C}$ and $bd(\mathbb{K})$ in the following manner of (5) and if either $(\mathbb{Y}, \|\cdot\|)$ is a finite dimensional space or $\mathbb{C}$ is closed and convex cone, then the cones $\mathbb{C}$ and $\mathbb{K}$ fulfill the separation relation in (4).

Proof. We omit the proof since it can be done similarly with the proof in [10, Theorem 4.3].

Now we present a separation relation for an arbitrary closed cone $\mathbb{K}$ which belongs to $\mathbb{R}^{n+1}$.

Lemma 3.7: Let $\mathbb{K}$ be a closed cone in $\mathbb{R}^{n+1}$ and assume that $(\hat{y}, \hat{a}) \notin \mathbb{K}$. Then a vector $(y,a) \in \mathbb{R}^{n+1} \backslash \{0_{\mathbb{R}^{n+1}}\}$ and a positive real number $\alpha \geq 0$ exist such that

$$\langle (y^*, a^*), (\hat{y}, \hat{a}) \rangle + \alpha \|(\hat{y}, \hat{a})\| < 0 \leq \langle (y^*, a^*), (y,a) \rangle + \alpha \|(y,a)\| \qquad \text{for all } (y,a) \in \mathbb{K}$$

Proof. In this proof, the idea is based on [11, Lemma 3.1].

Let $\|(\hat{y}, \hat{a})\| = 1$ and $\alpha = 1 - \frac{\varepsilon^2}{2}$. $\mathbb{K}$ is a closed cone and $(\hat{y}, \hat{a}) \notin \mathbb{K}$ thus there exists $\varepsilon \in (0,1)$ such that

$$N_\varepsilon(\hat{y}, \hat{a}) = \{(y,a) \in \mathbb{R}^{n+1} : \|(y - \hat{y}, a - \hat{a})\| \leq \varepsilon\}$$

Assume that

$$\mathbb{C} = cone(N_\varepsilon(\hat{y}, \hat{a}))$$

and

$$\mathbb{C}_{\mathbb{U}} = \{(y,a) \in \mathbb{U} : \|(y - \hat{y}, a - \hat{a})\| \leq \varepsilon\}.$$

$$\begin{aligned}
(y,a) \in \mathbb{C}_{\mathbb{U}} &\Leftrightarrow \|(y - \hat{y}, a - \hat{a})\|^2 \leq \varepsilon^2 \\
&\Leftrightarrow \|(y - \hat{y})\|^2 + (a - \hat{a})^2 \leq \varepsilon^2 \\
&\Leftrightarrow \|\hat{y}\|^2 + 2\langle y, \hat{y} \rangle + \|y\|^2 + a^2 + 2a\hat{a} + \hat{a}^2 \leq \varepsilon^2 \\
&\Leftrightarrow 2 - 2(\langle y, \hat{y} \rangle + a\hat{a}) \leq \varepsilon^2 \\
&\Leftrightarrow 1 - \frac{\varepsilon^2}{2} \leq \langle y - \hat{y}, a - \hat{a} \rangle \text{ for all } (y,a) \in \mathbb{U} \cap \mathbb{K}.
\end{aligned}$$

The rest of the proof follows similarly.

For a given set $\boldsymbol{S} \subset \mathbb{R}^{\boldsymbol{n+1}}$ and a point at $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{a}}) \in \boldsymbol{S}$, we present the separation theorem.

Theorem 3.8: Assume that $\boldsymbol{S} \subset \mathbb{R}^{\boldsymbol{n+1}}$ be cone shaped at $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{a}}) \in \boldsymbol{S}$. In that case, the cone $\mathbb{C} \subset \mathbb{R}^{\boldsymbol{n+1}}$ exists which is pointed and closed that satisfy

$$(\boldsymbol{S} - \{(\overline{\boldsymbol{y}}, \overline{\boldsymbol{a}})\}) \cap \ \mathbb{C} \backslash \{\boldsymbol{0}_{\mathbb{R}^n}\} = \emptyset$$

and there exists $((\boldsymbol{y}^*, \boldsymbol{a}^*), \boldsymbol{\alpha}) \in (-\mathbb{C})^{\#}$ satisfying

$$\langle (y^*, a^*), (y, a) - (\bar{y}, \bar{a}) \rangle + \alpha \|(y, a) - (\bar{y}, \bar{a})\| \geq 0, \forall (y, a) \in S.$$

Proof: This theorem can be proven easily by following [11, Theorem 3.2].

The subsequent theorem asserts that if a function is positively homogeneous and continuous then it is both lower and upper Lipschitz.

Theorem 3.9: Assume that the function $f: \mathbb{R}^n \to \mathbb{R}$ is a continuous, positively homogeneous. Then f is Lipschitz.

Proof: We know that if f is continuous on $S_1 = \{u \in S: \|u\| = 1\}$ then it attains its minimum and maximum on S_1. Thus there exists real numbers m and M with

$$f(u) \geq m \text{ for all } u \in S_1$$

and

$$f(u) \leq M, \text{ for all } u \in S_1. \tag{6}$$

Take any $x \in S$. Then there exists some $t > 0$ and $x \in S_1$ such that $x = tu$. Therefore,

$$\boldsymbol{f(x) - f(0) = f(x) = f(tu) = tf(u) \geq tm = tm\|u\| = m\|tu\| = m\|x\|} \tag{7}$$

Now, if $\boldsymbol{m > 0}$ then (7) implies that:

$$\boldsymbol{-L\|x\| \leq m\|x\| \leq f(x) - f(0)} \tag{8}$$

where $\boldsymbol{L}$ is an arbitrary positive real number.

If $\boldsymbol{m < 0}$ in (7) then,

$$\boldsymbol{-L\|x\| \leq m\|x\| \leq f(x) - f(0)} \text{ where } \boldsymbol{L > 0} \text{ and } \boldsymbol{-L = m < 0}. \tag{9}$$

Thus (8) and (9) together imply that $\boldsymbol{f}$ is lower Lipschitz. Now since $\boldsymbol{f}$ is bounded above (6) implies that,

$$\boldsymbol{f(x) - f(0) = f(x) = f(tu) = tf(u) \leq tM = tM\|u\| = M\|tu\| = m\|x\|}$$

If $\boldsymbol{M > 0}$, we know that,

$$\boldsymbol{f(x) - f(0) \leq M\|x\| = L\|x\|}. \tag{10}$$

If $\boldsymbol{M < 0}$, then

$$\boldsymbol{f(x) - f(0) \leq M\|x\| \leq L\|x\|} \text{ where } \boldsymbol{L > 0} \text{ and arbitrary.} \tag{11}$$

(9) and (10) imply that $\boldsymbol{f}$ is upper Lipschitz. Thus $\boldsymbol{f}$ is Lipschitz and there exists $\boldsymbol{L > 0}$ such that

$$\boldsymbol{|f(x) - f(0)| \leq L\|x - 0\|}.$$

The proof is completed.

The following theorem shows that positively homogeneous and lower semicontinuous functions are weakly subdifferentiable and it is worth to emphasize that the pair $\boldsymbol{(x^*, c)}$ in this case is different from $\boldsymbol{(0_{\mathbb{R}^n}, 0)}$.

Theorem 3.10: Let $f: \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and positively homogeneous function on the cone S. Then f is weakly subdifferentiable at $x = 0$, that is there exists $(x^*, c) \in (\mathbb{R}^n \times \mathbb{R}_+) \setminus \{(0_{\mathbb{R}^n}, 0)\}$ such that

$$\langle x^*, x\rangle - c\|x\| \leq f(x) - f(0) \text{ for all } x \in S.$$

Proof: Since $\boldsymbol{f}$ is positively homogeneous it implies that $\boldsymbol{f(0) = 0}$. $\boldsymbol{f}$ is bounded below on $\boldsymbol{S_1 = \{x \in S\colon \|x\| = 1\}}$ since $\boldsymbol{f}$ is lower semicontinuous. Consider an arbitrary element $\boldsymbol{x^*} \in \mathbb{R}^{\boldsymbol{n}} \setminus \{\boldsymbol{0}\}$. Then it implies that $\boldsymbol{y = \langle x^*, x\rangle}$ is continuous and thus bounded from below on $\boldsymbol{S_1}$. Then there exists a sufficiently large number $\boldsymbol{c > 0}$ such that

$$\boldsymbol{\langle x^*, u\rangle - c\|u\| \leq f(u) - f(0)} \text{ for all } \boldsymbol{x \in S_1}. \tag{12}$$

Consider an arbitrary element $\boldsymbol{x \in S}$. Since $\boldsymbol{S}$ is a cone, then there exists $\boldsymbol{t > 0}$ and $\boldsymbol{u \in S_1}$ such that $\boldsymbol{x = tu}$. For this $\boldsymbol{u}$, multiply both sides of (12) by $\boldsymbol{t > 0}$. Then it yields to

$$\boldsymbol{\langle x^*, x\rangle - c\|x\| \leq f(x) - f(0)} \ , \forall \boldsymbol{x \in S}.$$

The proof is completed.

Remark 3.11: Theorem 3.8 and Corollary 3.9 show that any lower Lipschitz function is weakly subdifferentiable. However, it is proved for $\boldsymbol{y^*}$ component of the weak subgradient $\boldsymbol{(x^*, \alpha)}$ when $\boldsymbol{x^* = 0_{\mathbb{R}^n}}$. The following theorem demonstrate that every Lipschitz function is weakly subdifferentiable with $\boldsymbol{x^*}$ component of the weak subgradient is different from $\boldsymbol{0_{\mathbb{R}^n}}$.

The following theorem shows that a class of weakly subdifferentiable functions.

Theorem 3.12: Assume that $f: \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz continuous where L is a Lipschitz constant. Then f is weakly subdifferentiable at $x_0 \in int\big(dom(f)\big)$, that is $\partial^w f(x_0) \neq \emptyset$ and there exists $(x^*, \alpha) \in \partial^w f(x_0)$ with $x^* \neq 0_{\mathbb{R}^n}$ and $\alpha > 0$.

Proof: The proof is built upon nonlinear cone separation theorem [10]. Assume that $x_0 \in int\big(dom(f)\big)$. Since $(x_0, f(x_0))$ belongs to the boundary of $epi(f) \subset \mathbb{R}^n \times \mathbb{R}$, we can separate it from $int\big(dom(f)\big)$ by a closed pointed cone. By Theorem 3.8 there exists $((x^*, a^*), \alpha) \in (-\mathbb{C})^{\#}$such that

$$\langle (x^*, a^*), (x, a) - (x_0, a_0)\rangle + \alpha\|(x, a) - (x_0, a_0)\| \geq 0 \text{ for all } (x, a) \in epi(f).$$

$$\langle (x^*, a^*), (x - x_0, a - a_0)\rangle + \alpha\|(x - x_0, a - a_0)\| \geq 0 \text{ for all } (x, a) \in epi(f).$$

By using the norm defined on $\mathbb{R}^{n+1}$ we have,

$$\langle x^*, x - x_0\rangle + a^*(a - a_0) + \alpha\|(x - x_0\| + \alpha|a - a_0| \geq 0 \text{ for all } (x, a) \in epi(f).$$

If we substitute $f(x_0) = a_0$ it yields to,

$$\langle x^*, x - x_0\rangle + a^*(a - f(x_0)) + \alpha\|x - x_0\| + \alpha|a - f(x_0)| \geq 0 \text{ for all } (x, a) \in epi(f).$$

Or equivalently,

$$\langle x^*, x - x_0\rangle + a^*(f(x) - f(x_0)) + \alpha\|(x - x_0\| + \alpha|f(x) - f(x_0)| \geq 0 \text{ for all } x \in dom(f).$$

By the assumption that f is a Lipschitz function, it implies that:

$$\alpha|f(x) - f(x_0)| \leq \alpha L\|x - x_0\| .$$

Then we have,

$$\langle x^*, x - x_0\rangle + a^*\big(f(x) - f(x_0)\big) + \alpha\|x - x_0\| + \alpha L\|x - x_0\| \geq$$
$$\langle x^*, x - x_0\rangle + a^*\big(f(x) - f(x_0)\big) + \alpha\|x - x_0\| + \alpha|f(x) - f(x_0)| \text{ for all } x \in dom(f).$$

Hence, we obtain

$$\langle x^*, x - x_0\rangle + a^*\big(f(x) - f(x_0)\big) + (\alpha + \alpha L)\|x - x_0\| \geq 0 \text{ for all } x \in dom(f).$$

Thus,

$$\langle x^*, x - x_0\rangle + (\alpha + \alpha L)\|x - x_0\| \geq -a^*\big(f(x) - f(x_0)\big) \text{ for all } x \in dom(f).$$

Then finally,

$$\left\langle -\frac{x^*}{a^*}, x - x_0\right\rangle - \left(\frac{\alpha + \alpha L}{a^*}\right)\|x - x_0\| \leq f(x) - f(x_0) \text{ for all } x \in dom(f).$$

Thus, $\left(-\frac{x^*}{a^*}, \frac{\alpha+\alpha L}{a^*}\right) \in \partial^w f(x_0)$. The proof is completed.

4. CONCLUSION

In this work, we showed that a class of functions which are weakly subdifferentiable and with an important distinction that x^* component of the weak subgradient of the pair (x^*, α) in this case is different from $0_{\mathbb{R}^n}$. As shown in Theorem 3.12, we proved that any Lipschitz function is weakly subdifferentiable with a distinct weak gradient component. Our results extend the class of weakly subdifferentiable functions by introducing a new criterion based on nonlinear cone seperation.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Samet Bila: Formal analysis, Writing- original draft, Investigation, Conceptualization. **Refail Kasımbeyli:** Conceptualization, Visualization, Supervision.

REFERENCES

[1] Azimov AY, Gasimov RN. On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization, International Journal of Applied Mathematics, 1, 1999, pp. 171–192.

[2] Azimov AY, Gasimov RN. Stability and duality of nonconvex problemsvia augmented lagrangian, Cybernatics and System Analysis, 3(3), 2002, pp. 412–421.

[3] Bila S, Kasimbeyli R. On the some sum rule for the weak subdifferential and some properties of augmented normal cones, Journal of Nonlinear and Convex Analysis, 24(10), 2023, pp. 2239–2257.

[4] Borwein JM, Lewis AS. Convex Analysis and Nonlinear Optimization, CMS Books in Mathematics, Springer Science+Business Media, Inc., New York, 2006.

[5] Dinc Yalcin G, Kasimbeyli R. Weak subgradient method for solving non-smooth nonconvex optimization problems, Optimization, 70(7), 2021, pp. 1513-1553.

[6] Gasimov RN. Duality in nonconvex optimization, Ph.D. Dissertation, Department of Operations Research and Mathematical Modeling, Baku State University, Baku, 1992.

[7] Gasimov RN. Augmented Lagrangian duality and nondifferentiable optimization methods in nonconvex programming, J. Global Optimization, 24, 2002, pp. 187–203.

[8] Kasimbeyli R, Mammadov M. Optimality conditions in nonconvex opti-mization via weak subdifferentials, Nonlinear Analysis: Theory, Methods and Applications, 74, 2011, pp. 2534–2547.

[9] Kasimbeyli R, Mammadov M. On weak subdifferentials, directional derivatives, and radial epiderivatives for nonconvex functions, SIAM J. on Optimization, 20, 2009, pp. 841–855

[10] Kasimbeyli R. A nonlinear cone separation theorem and scalarization in nonconvex vector optimization, SIAM J. on Optimization, 20, 2010, pp. 1591–1619.

[11] Kasimbeyli R, Karimi M. Separation theorems for nonconvex sets and application in optimization, Operations Research Letters, 47, 2019, pp. 569–573.

[12] Rockafellar RT. Convex analysis, Princeton University Press, Princeton, 1970.

[13] Rockafellar RT, Wets R. J-B. Variational analysis, Springer-Verlag Berlin Heidelberg, 2009.