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ON THE WEAK SUBDIFFERENTIAL, AUGMENTED NORMAL CONES AND DUALITY IN NONCONVEX OPTIMIZATION

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Abstract

This article studies the properties of the weak subdifferential for nonsmooth and nonconvex analysis studied. This study presents a formulation that is directly involved in convex analysis carried out in the nonconvex case. In this work, we present a theory that applies epigraphs to obtain augmented normal cones.

The perturbation function plays a crucial role in establishing optimality conditions. This study demonstrates that positively homogeneous and lower semicontinuous functions are weakly subdifferentiable. Moreover, under specific conditions related to the objective function, the constraint function, and the feasible set, we show that the perturbation function is positively homogeneous. Thus we obtain a zero duality gap condition by implementing conditions on the objective function, constraint functions, and the set S.

Keywords

Operations Research,
Nonconvex Optimization,
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1. INTRODUCTION

The concept of subgradient marked the real beginning of the convex analysis in the way it is seen now. It is associated with a convex function and provides many useful properties of the derivative from an optimization perspective [3,10]. At boundary points, a convex set has a supporting hyperplane, which gives rise to the notion of the subdifferential, denoted by ∂f. This concept forms the foundation of convex analysis and was introduced by R.T. Rockafellar in his 1963 thesis [11] for convex functions. Later, F.H. Clarke, in his 1973 thesis [4], extended this definition to Lipschitz continuous functions by introducing the Clarke subdifferential $\partial_o f$. Indeed, there is a drawback to this subdifferential notion. The function must be convex to be able to use many nice consequences of this concept.

When dealing with nonconvex functions, $\boldsymbol{h}: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ was addressed, the original definition of subgradients through the affine support inequality applicable to convex functions had to be replaced with an alternative approach.

Clarke introduced the use of distance functions d_C to get a new concept of normal cones to nonconvex set C. Then he applied this concept to epigraphs to obtain normal cones $(v, -1)$ whose v component could be interpreted as a subgradient. This innovation appeared in Clarke [5] and it sparked years of

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efforts by many researchers to advance the idea in various areas and apply then to a range of topics, with one of the most notable one is to optimal control.

Another alternative approach to the subgradient concept to nonconvex functions, known as the weak subdifferential $\partial^w f$ was presented by R.N. Gasimov in his 1992 thesis [7]. The weak subdifferential generalizes the subdifferential concept from convex analysis to nonconvex functions. This concept is founded on the idea of using supporting cones for the epigraph of a given function, which serves as a substitute for the supporting hyperplanes typically used in convex analysis. The idea behind the supporting cones is extremely helpful for nonconvex separation theorems and investigating nonconvex optimality problems.

Azimov and Gasimov well established a necessary and sufficient condition in nonconvex optimization based on the zero duality gap property [1,8] by relating the definition of the weak subdifferential and the perturbation function. Some of the weak subdifferential properties are investigated in [2,9].

The zero duality gap condition defined by an augmented Lagrangian dual function is well studied in [10]. It has been applied to zero duality gap conditions for optimality and approaches for finding solutions in nonconvex mathematical programming.

Motivation:

The subgradient can be viewed as a special case of the normal cone when the set is the epigraph of a convex function. In this study, we have presented a theorem that establishes a connection between the augmented normal cone of the epigraph and the weak subdifferential. Our findings demonstrate the existence of this theorem within the framework of nonconvex analysis.

Furthermore, we derive a zero duality gap condition by imposing conditions on the objective function, constraint functions, and the set S.

2. PROPERTIES OF THE WEAK SUBGRADIENT AND AUGMENTED NORMAL CONE

In convex duality theory, the definitions of conjugate functions and subdifferentials are as follows:

Let $\boldsymbol{X}$ be a normed space and $\boldsymbol{X}^*$ be the topological dual of $\boldsymbol{X}$. Suppose that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We will denote the norm of $\boldsymbol{X}$ by $\|\cdot\|$, the norm of $\boldsymbol{X}^*$ by $\|\cdot\|_*$, and the value of a linear functional $\boldsymbol{x}^* \in \boldsymbol{X}^*$ at a point $\boldsymbol{x} \in \boldsymbol{X}$ by $\boldsymbol{x}^*, \boldsymbol{x}\rangle$. Let $\boldsymbol{h}: \mathbb{R}^{\boldsymbol{n}} \to \overline{\mathbb{R}}$ be a given function.

Definition 1: (a) A function $\boldsymbol{h^w}: \boldsymbol{X} \times \boldsymbol{X}^* \times \mathbb{R}_+ \to \overline{\mathbb{R}}$ defined by

$$\boldsymbol{h^w}(\boldsymbol{x_0}, \boldsymbol{x}^*, \boldsymbol{c}) = \sup_{\mathbf{x} \in \mathbf{X}} \{-\boldsymbol{c}\|\boldsymbol{x} - \boldsymbol{x_0}\| + \boldsymbol{c}\|\boldsymbol{x_0}\| + \langle \boldsymbol{x}, \boldsymbol{x}^* \rangle - \boldsymbol{h}(\boldsymbol{x})\}$$

is called the weak conjugate of $\boldsymbol{h}$.

(b) A function $\boldsymbol{h^{ww}}: \boldsymbol{X} \to \overline{\mathbb{R}}$ defined by

$$\boldsymbol{h^{ww}}(\boldsymbol{x}) = \sup_{(\boldsymbol{x}^*, \boldsymbol{c}) \in \boldsymbol{X}^* \times \mathbb{R}_+} \{-\boldsymbol{c}\|\boldsymbol{x} - \boldsymbol{x_0}\| + \boldsymbol{c}\|\boldsymbol{x_0}\| + \langle \boldsymbol{x}, \boldsymbol{x}^* \rangle - \boldsymbol{h^w}(\boldsymbol{x_0}, \boldsymbol{x}^*, \boldsymbol{c})\}$$

is called the weak biconjugate of $\boldsymbol{h}$.

For $\boldsymbol{c=0}$, $\boldsymbol{h^w(x_0, x^*, c) = h^*(x^*)}$, where $\boldsymbol{h}^*$ is the ordinary conjugate function in convex analysis.

Azimov and Gasimov introduced the following weak subdifferential notion, which is the generalization of the classic subdifferential from convex analysis [1].

Definition 2: A pair $(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$ is referred to as a weak subgradient of h at x_0 on S provided that

$$\langle x^*, x - x_0\rangle - c\|x - x_0\| \leq h(x) - h(x_0) \text{ for all } x \in S \tag{1}$$

The set of all weak subgradients of h at $x_0$is referred to as the weak subdifferential of h at x_0 and is denoted as $\partial_S^w h(x)$:

$$\partial_S^w h(\bar{x}) = \{(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+ : (1) \text{ is satisfied}\}.$$

If $\partial_S^w h(x_0) \neq \emptyset$, then h is called the weakly subdifferentiable at x_0. If we let $S = \mathbb{R}^n$ then we ignore the subscript S in $\partial_S^w f(x_0)$, and denote it by $\partial^w h(x_0) = \partial_{\mathbb{R}^n}^w h(x_0)$. It is obvious that if function h is subdifferentiable at x_0 then h is also weakly subdifferentiable at x_0 . One can check if $x^* \in \partial h(x_0)$ then by definition $(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$ for every $c \geq 0$. The weak subgradient of a function h is geometrically interpreted as:
$(x^*, c) \in \mathbb{R}^n \times \mathbb{R}_+$ is a weak subgradient of h at $x \in X$ if one can found a function

$$f(x) = \langle x^*, x - x_0\rangle - c\|x - x_0\| + h(x_0)$$

which is a continuous, concave, and satisfies $h(x) \leq f(x), \forall x \in X$ and $h(x_0) = f(x_0)$. The hypograph of the function f is defined as hypo $(f) = \{(x, a) \in X \times \mathbb{R} \mid f(x) \geq a\}$ and it is a closed cone in $X \times \mathbb{R}$ with its vertex at $(x_0, f(x_0))$. To verify:

$$\begin{aligned}\text{hypo } (f) - \big(x_0, h(x_0)\big) &= \big\{\big(x - x_0, a - h(x_0)\big) \in X \times \mathbb{R} \mid \langle x^*, x - x_0\rangle - c\|x - x_0\| \geq a - h(x_0)\big\} \\ &= \{(u, b) \in X \times \mathbb{R} \mid \langle x^*, u\rangle - c\|u\| \geq b\}.\end{aligned}$$

Thus, from (2.1) and (2.2) that hypo (f) is a supporting cone of the set

$$epi(h) = \{(x, a) \in X \times \mathbb{R} \mid h(x) \leq a\}$$

at the point $(x_0, h(x_0))$ in the way that epi(h) $\subset (X \times \mathbb{R}) \setminus hypo(f)$ and $cl(epi(h)) \cap graph(f) \neq \emptyset$ where $graph\ (f) = \{(x, a) \in X \times \mathbb{R} \mid f(x) = a\}$.
Azimov and Gasimov obtained the weak subdifferential for subclasses of lower Lipschitz functions [1]. Lower Lipschitz function definition is given as follows:

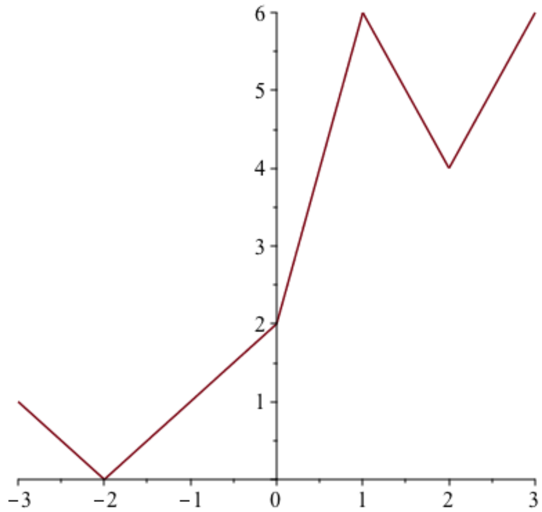
Definition 3: A $g: X \to \mathbb{R}$ is said to be "lower locally Lipschitz" at $x_0 \in X$ if there exists a positive constant L and a neighborhood $\mathcal{N}(x_0)$ around x_0 such that

$$-L\|x - x_0\| \leq g(x) - g(x_0), \ \forall x \in \mathcal{N}(x_0). \tag{2}$$

g is lower Lipschitz at x_0 with the Lipschitz constant L if the inequality (2) holds for all $x \in X$.
An example of the weak subdifferential is presented.
Example 4: Let $h: \mathbb{R} \to \mathbb{R}$ be given as

$$h(x) = \begin{cases} |x+2| & if\ x \le 0 \\ 4x+2 & if\ 0 < x \le 1 \\ 2|x-2|+4 & if\ x > 1 \end{cases}$$

The graph of function h is given below.



We want to calculate the weak subdifferentiable of h at $x_0 = 1$.

First, we consider the case $x < -2$. It follows from definition (2.1) that:

$$\langle y, x - x_0\rangle - c\|x - x_0\| \le h(x) - h(x_0)$$
$$\langle y, x - 1\rangle + c(x-1) \le -(x+2) - h(1)$$
$$y(x-1) + c(x-1) \le (-x-2) - 6$$
$$(x-1)(y+c) \le -x - 8$$

Then $\partial^w h(1)$ for the case $x < -2$ obtained as:

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : y + c > 2\}$$

Then we consider the case $-2 \le x \le 0$. It follows from the definition (2.1),

$$\langle y, x - x_0\rangle - c\|x - x_0\| \le h(x) - h(x_0)$$
$$\langle y, x - 1\rangle + c(x-1) \le (x+2) - h(1)$$
$$y(x-1) - c(1-x) \le (x+2) - 6$$
$$(x-1)(w+c) \le x - 4$$

Then $\partial^w h(1)$ for the case $-2 \le x \le 0$ obtained as:

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : y + c \ge 4\}$$

We consider the case $0 < x \le 1$. We have the following from the definition (2.1) that,

$$\langle y, x - x_0\rangle - c\|x - x_0\| \le h(x) - h(x_0)$$
$$\langle y, x - 1\rangle - c|x-1| \le h(x) - h(1)$$
$$y(x-1) - c(1-x) \le 4x + 2 - 6$$
$$(x-1)(y+c) \le 4(x-1)$$

Then $\partial^w h(1)$ for the case $0 < x \le 1$ obtained as:

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : y + c \geq 4\}$$

Now we consider the case when $1 < x < 2$. It follows from the definition (2.1) that,

$$\langle y, x-1 \rangle - c|x-1| \leq h(x) - h(1)$$
$$y(x-1) - c(x-1) \leq 2|x-2| + 4 - 6,$$
$$(y-c)(x-1) \leq -2(x-1),$$

$\partial^w h(1)$ for the case $1 < x < 2$ is obtained as:

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : y - c \leq -2\}$$

The last case is when $x \geq 2$. Then we have,

$$\langle y, x-1 \rangle - c|x-1| \leq h(x) - h(1)$$
$$y(x-1) - c(x-1) \leq 2|x-2| + 4 - 6,$$
$$(y-c)(x-1) \leq 2(x-3),$$

$\partial^w h(1)$ for the case $x \geq 2$ is obtained as:

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : y - c \leq -2\}$$

Then finally we obtained that

$$\partial^w h(1) = \{(y,c) \in \mathbb{R} \times \mathbb{R}_+ : 4 - c \leq y \leq c - 2\}$$

We give the normal cone and augmented normal cone definitions below. The augmented normal cone notion is introduced to the literature by Kasimbeyli and Mammadov and it is quite helpful for obtaining the optimality condition in nonconvex optimization [8].

Definition 5: Let $A \subseteq \mathbb{R}^n$ and $x_0 \in S$. The normal cone to A at x_0 defined as

$$N_A(x_0) = \{v \in \mathbb{R}^n : \langle v, x - x_0 \rangle \leq 0 \text{ for all } x \in A\}.$$

It is clear that if $x_0 \in int(S)$ then the set $N_A(x_0)$ consists of a single element, that is the zero element $0 \in \mathbb{R}^n$. Such a normal cone is called trivial. If $x_0 \notin int\, S$ and S is convex then normal cone is called nontrivial, the zero of $\mathbb{R}^n$ may be the only element in this cone. A normal cone is called nontrivial if it contains non-zero elements.

Definition 6:
Let $x_0 \in A$ and $A \backslash \{x_0\} \neq \emptyset$. The augmented normal cone to A at x_0 is defined as:

$$N_A^a(x_0) = \{(v,c) \in \mathbb{R}^n \times \mathbb{R} : \langle v, x - x_0 \rangle - c\|x - x_0\| \leq 0 \text{ for all } x \in A\}. \quad (3)$$

Below, we recall definition of the augmented normal cone (for nonconvex sets) earlier introduced in (3). Here we will use a slightly different but an equivalent formulation (in the definition we use $\langle v, x - x_0 \rangle + c\|x - x_0\|$ instead of $\langle v, x - x_0 \rangle + c\|x - x_0\|$).

Since, for pairs (v,c) with $\|v\| \leq c$, the inequality $\{(v,c) \in \mathbb{R}^n \times \mathbb{R} : \langle v, x - x_0 \rangle - c\|x - x_0\| \leq 0$ is obviously satisfied for all x$\in \mathbb{R}^n$, an augmented normal cone consisting of only such elements is called trivial. The trivial augmented normal cone will be denoted by $N_A^{triv}(x_0)$ and defined as

$$N_A^a(x_0) = \{(v,c) \in \mathbb{R}^n \times \mathbb{R} : \|v\| \leq c\}$$

Remark 7: It follows from the definition of normal and augmented normal cone that, for a given set $A \subset \mathbb{R}^n$, if the normal cone $N_A(x_0)$ is not empty, then for the pair $v \in N_A(x_0)$ with $c \geq 0$ belongs to the augmented normal cone $N_A^a(x_0)$. Conversely, if $(v, c) \in N_A^a(x_0)$ with $c \leq 0$, then $v \in N_A(x_0)$. Therefore, we will use the following definition for the augmented normal cone in this paper:

$$N_A^a(x_0) = \{(v, c) \in \mathbb{R}^n \times \mathbb{R}: c \leq \|v\|,\ \langle v, x - x_0 \rangle - c\|x - x_0\| \leq 0 \text{ for all } x \in A\}.$$

The following lemma was proved in [8, Lemma 4].

Lemma 8: If $x_0 \in int\ A$ then $N_A^a(x_0) = N_A^{triv}(x_0)$

Now we will establish a relationship between the augmented normal cone to the epigraph of a given function f at some point $(x_0, f(x_0))$ and the weak subdifferential of f at x_0.

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a given function. The epigraph $epi(f)$ of f is defined as follows:

$$epi(f) = \{(x, a) \in \mathbb{R}^n \times \mathbb{R}: f(x) \leq a\}.$$

Then the augmented normal cone to the set $epi(f) \in \mathbb{R}^n \times \mathbb{R}$ at the point $(x_0, f(x_0))$ can be rewritten as follows:

$$N_{epi(f)}^a\big(x_0, \mathrm{f}(x_0)\big) = \big\{\big((v, \beta), c\big) \in \big(\mathbb{R}^n \times \mathbb{R}\big) \times \mathbb{R}_+ :$$
$$\langle v, x - x_0 \rangle + \beta\big(a - f(x_0)\big) - c\|(x - x_0), (a - f(x_0))\| \leq 0 \text{ for all } (x, a) \in epi(f).$$

where the set $\|(x - x_0), (a - f(x_0))\|_{\mathbb{R}^n \times \mathbb{R}} = \|x - x_0\|_{\mathbb{R}^n} + |a - f(x_0)|_{\mathbb{R}}$. Obviously, such a setting satisfies the definition of the norm, such that:

- $\|(x, a)\| = \|x\| + |\alpha| \geq 0$ for all $(x, a) \in \mathbb{R}^n \times \mathbb{R}$ and obviously $\|(x, a)\| = 0$ if and only if $(x, a) = (0_{\mathbb{R}^n}, 0)$;
- $\|\lambda(x, a)\| = \|(\lambda x, \lambda a)\| = \|\lambda x\| + |\lambda a| = |\lambda|(\|x\| + |a|) = |\lambda|\|(x, a)\|$ for all $(x, a) \in \mathbb{R}^n \times \mathbb{R}$ and $\lambda \in \mathbb{R}$;
- $\|(x_1, a_1) + (x_2, a_2)\| = \|(x_1, x_2) + (a_1 + a_2)\| = \|(x_1, x_2)\| + |a_1 + a_2| \leq \|x_1\| + \|x_2\| + |a_1| + |a_2| = \|(x_1, a_1)\| + \|(x_2, a_2)\|$ for all $(x_1, a_1) \in \mathbb{R}^n \times \mathbb{R}$ and $(x_2, a_2 \in \mathbb{R}^n \times \mathbb{R}$.

3. DUALITY IN NONCONVEX OPTIMIZATION

Suppose that X and Y are normed spaces and assume that X^* and Y^* are their dual spaces, respectively.

Taking the function $\boldsymbol{h}: \boldsymbol{X} \to \overline{\mathbb{R}}$ into account we consider a nonlinear problem in the following form:

$$(P) \quad \Big\{\inf_{x \in X} f(x).$$

The problem *(P)* is referred to as the primal problem. Its infimum is denoted by *inf (P)*, and any $\boldsymbol{x} \in \boldsymbol{X}$ that satisfies $\boldsymbol{f(x) = inf(P)}$ is called an optimal solution of *(P)*. Problem *(P)* is considered nontrivial if there exists $\boldsymbol{x} \in \boldsymbol{X}$ such that $\boldsymbol{f(x)} < +\infty$. .

$$\boldsymbol{X = \{x \in S: g_j(x) \leq 0, j = 1, \dots, m\}} \neq \emptyset.$$

For the problem (P) the associated dualizing parameterization function

$$\boldsymbol{\phi(x, y)} = \begin{cases} \boldsymbol{f(x)} & \boldsymbol{if\ x \in S\ and\ g_j(x) \leq 0}, \quad \boldsymbol{j = 1, \dots, m}, \\ +\infty & \boldsymbol{otherwise}. \end{cases}$$

The function $\boldsymbol{\phi}: \boldsymbol{X} \times \boldsymbol{Y} \to \mathbb{R} \cup \{+\infty\}$ defined above satisfies that $\boldsymbol{\phi}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{f}(\boldsymbol{x})$. It is easy to check

$$\boldsymbol{inf}_{\boldsymbol{x}\in\boldsymbol{X}}\boldsymbol{\phi}(\boldsymbol{x}, \mathbf{0}) = \boldsymbol{inf}\ (\boldsymbol{P}).$$

Utilizing the classical method for constructing the dual of a minimization problem [6, 10] we can now define the corresponding dual problem. To formulate the dual problem based on the function $\boldsymbol{\phi}$, the weak conjugate function $\boldsymbol{\phi}^{\boldsymbol{w}}$ must be calculated. By definition 2.1, the weak conjugate function $\boldsymbol{\phi}^{\boldsymbol{w}}$ is from $(\boldsymbol{X}^* \times \mathbb{R}_+ \times \boldsymbol{X}) \times (\boldsymbol{Y}^* \times \mathbb{R}_+ \times \boldsymbol{Y})$ into $\overline{\mathbb{R}}$ and is given by the following definition:

$$\boldsymbol{\phi}^{\boldsymbol{w}}((\boldsymbol{x}^*, \boldsymbol{c}, \boldsymbol{x_0}), (\boldsymbol{y}^*, \boldsymbol{\alpha}, \boldsymbol{y_0})) = \underset{(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{X}\times\boldsymbol{Y}}{\boldsymbol{sup}} \begin{cases} -\boldsymbol{c}\|\boldsymbol{x} - \boldsymbol{x_0}\| + \boldsymbol{c}\|\boldsymbol{x_0}\| + \langle \boldsymbol{x}^*, \boldsymbol{x}\rangle \\ -\boldsymbol{\alpha}\|\boldsymbol{y} - \boldsymbol{y_0}\| + \boldsymbol{\alpha}\|\boldsymbol{y_0}\| + \langle\ \boldsymbol{y}^*, \boldsymbol{y}\rangle - \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})\} \end{cases}$$

is referred to as the dual problem of $(\boldsymbol{P})$ based on the dualizing parameterization $\boldsymbol{\phi}$.

When $(\boldsymbol{x}^*, \boldsymbol{c}) = (\mathbf{0}, \mathbf{0}), \boldsymbol{x_0} = \mathbf{0}, \boldsymbol{y_0} = \mathbf{0}$, the value of $\boldsymbol{\phi}^{\boldsymbol{w}}$ will be referred to simply as:

$$\boldsymbol{\phi}^{\boldsymbol{w}}(\mathbf{0}, \boldsymbol{y}^*, \boldsymbol{\alpha}) = \underset{(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{X}\times\boldsymbol{Y}}{\boldsymbol{sup}} \{-\boldsymbol{\alpha}\|\boldsymbol{y}\| + \langle \boldsymbol{y}, \boldsymbol{y}^*\rangle - \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})\}$$

$$(\boldsymbol{P}^{\boldsymbol{w}}) \qquad \underset{(\boldsymbol{y}^*,\boldsymbol{\alpha})\in\boldsymbol{Y}^*\times\mathbb{R}_+}{\boldsymbol{sup}} \{-\boldsymbol{\phi}^{\boldsymbol{w}}(\mathbf{0}, \boldsymbol{y}^*, \boldsymbol{\alpha})\}$$

The supremum of problem *$(\boldsymbol{P}^{\boldsymbol{w}})$* is represented by *sup $(\boldsymbol{P}^{\boldsymbol{w}})$*. Any element $(\boldsymbol{y}^*, \boldsymbol{\alpha}) \in \boldsymbol{Y}^* \times \mathbb{R}_+$ that satisfies $\boldsymbol{\phi}^{\boldsymbol{w}}(\mathbf{0}, \boldsymbol{y}^*, \boldsymbol{\alpha})$= *sup $(\boldsymbol{P}^{\boldsymbol{w}})$* is referred to as an optimal solution of *$(\boldsymbol{P}^{\boldsymbol{w}})$*.

Assume that $\boldsymbol{h}$ is a function from $\boldsymbol{Y}$ into $\overline{\mathbb{R}}$. We define the perturbation function associated with problem *(P)* as follows:

$$\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{inf}_{\boldsymbol{x}\in\boldsymbol{X}}\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})$$

The perturbation function definition implies that

$$\boldsymbol{h}(\boldsymbol{y}) = \boldsymbol{inf}\ (\boldsymbol{P}).$$

The theorem below provides conditions for strong duality, based on the properties of the perturbation function $\boldsymbol{h}$.

Theorem 9: [1, Lemma 4.2] (i) $\boldsymbol{h}^{\boldsymbol{w}}(\boldsymbol{y}^*, \boldsymbol{\alpha}, \mathbf{0}) = \boldsymbol{\phi}^{\boldsymbol{w}}(\mathbf{0}, \boldsymbol{y}^*, \boldsymbol{\alpha})$;

(ii) $\boldsymbol{sup}(\boldsymbol{P}^{\boldsymbol{w}}) = \boldsymbol{h}^{\boldsymbol{ww}}$;

(iii) Suppose the perturbation function $\boldsymbol{h}$, defined by (10), is proper and weakly subdifferentiable at $\mathbf{0} \in \boldsymbol{Y}$. Then $\boldsymbol{inf}(\boldsymbol{P}) = \boldsymbol{sup}\ (\boldsymbol{P}^{\boldsymbol{w}})$, and any weak subgradient of $\boldsymbol{h}$ at $\mathbf{0} \in \boldsymbol{Y}$ is an optimal solution of $(\boldsymbol{P}^{\boldsymbol{w}})$.

Theorem 10: [10, Theorem 2.7] Let $\boldsymbol{h}$ be bounded from below on some neighborhood of zero and positively homogeneous function from $\boldsymbol{X}$ into $\mathbb{R}$. Then $\boldsymbol{h}$ is weakly subdifferentiable at $\mathbf{0}_{\boldsymbol{X}}$.

In the following theorem, we show that if the objective and constraint functions defined on the conic set are positively homogeneous then the perturbation function is also positively homogeneous.

Theorem 11: Assume that $\boldsymbol{S}$ is a cone in $\mathbb{R}^{\boldsymbol{n}}$. Let $\boldsymbol{f}: \boldsymbol{S} \to \mathbb{R}$ be positively homogeneous and the mappings $\boldsymbol{g} = (\boldsymbol{g_1}, \boldsymbol{g_2}, \dots, \boldsymbol{g_m}): \boldsymbol{S} \to \mathbb{R}$ are positively homogeneous. Then the perturbation function $\boldsymbol{h}(\boldsymbol{y})$ is positively homogeneous.

Proof: We must show that $\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{\lambda h}(\boldsymbol{y})$. Now the definition of the perturbation function implies that:

$$\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{f}(\boldsymbol{x}) \text{ if } \boldsymbol{x} \in \boldsymbol{S} \; \boldsymbol{and} \; \boldsymbol{g_j}(\boldsymbol{x}) \leq \boldsymbol{\lambda y}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}.$$

Since $\boldsymbol{g_j's}$ are positively homogeneous we have that

$$\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{f}(\boldsymbol{x}) \text{ if } \boldsymbol{x} \in \boldsymbol{S} \; \boldsymbol{and} \; \boldsymbol{g_j}\left(\frac{\boldsymbol{1}}{\lambda}\boldsymbol{x}\right) = \frac{\boldsymbol{1}}{\lambda}\boldsymbol{g_j}(\boldsymbol{x}) \leq \boldsymbol{y}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}.$$

Now if we denote $\frac{\boldsymbol{1}}{\lambda}\boldsymbol{x} = \boldsymbol{z}$ then we obtain:

$$\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{f}(\boldsymbol{\lambda z}) \text{ if } \boldsymbol{\lambda z} \in \boldsymbol{S} \; \boldsymbol{and} \; \boldsymbol{g_j}(\boldsymbol{z}) \leq \boldsymbol{y}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}.$$

By the assumption that $\boldsymbol{f}$ is positively homogeneous and $\boldsymbol{S}$ is a cone, then it yields that:

$$\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{\lambda f}(\boldsymbol{z}) \text{ if } \boldsymbol{z} \in \boldsymbol{S} \; \boldsymbol{and} \; \boldsymbol{g_j}(\boldsymbol{z}) \leq \boldsymbol{y}, \boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}.$$

Hence

$$\boldsymbol{h}(\boldsymbol{\lambda y}) = \boldsymbol{\lambda h}(\boldsymbol{y}).$$

The proof is completed.

Based on the results of Theorem 9, Theorem 10 and Theorem 11 the following theorem can be stated.

Theorem 12: Assume that S is a cone in $\mathbb{R}^n$. Let $f: S \to \mathbb{R}$ be positively homogeneous and lower semicontinuous and the mappings $g = (g_1, g_2, \dots, g_m): S \to \mathbb{R}$ are positively homogeneous and lower semicontinuous. Then the zero duality gap satisfies.

Proof. If S is a cone in $\mathbb{R}^n$, the functions $f: S \to \mathbb{R}$ and $g = (g_1, g_2, \dots, g_m): S \to \mathbb{R}$ are positively homogeneous and lower semicontinuos then by Theorem 9 we know that the perturbation function h is positively homogeneous. Also, under the condition of lower semicontinuity of f and g, h is also lower semicontinuous. Thus, from Theorem 10 we obtain that h is weakly subdifferentiable at $\mathbf{0}$. Finally, Theorem 9 implies that the zero duality gap is satisfied. The proof is completed.

In convex analysis, the following proposition is well known that relates to normal cone and subdifferential.

Proposition 13: Let h be convex, proper function then

$$\partial f(\bar{x}) = \left\{v \in \mathbb{R}^n : (v, -1) \in N\left(\left(\bar{x}, f(\bar{x})\right), epi\, f\right)\right\}.$$

Theorem 14: Assume that $\boldsymbol{f}: \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ is a proper function.

i) If $\boldsymbol{f}$ is weakly subdifferentiable function at $\overline{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ and $(\boldsymbol{v}, \boldsymbol{c}) \in \boldsymbol{\partial^w f}(\overline{\boldsymbol{x}})$ then $((\boldsymbol{v}, \boldsymbol{-1}), \boldsymbol{c}) \in \boldsymbol{N^a_{epi(f)}}(\overline{\boldsymbol{x}}, \mathbf{f}(\overline{\boldsymbol{x}}))$.

ii) If $\boldsymbol{f}$ is Lipschitz function at $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ with Lipschitz constant $\boldsymbol{L}$ and $((\boldsymbol{v}, -\mathbf{1}), \boldsymbol{c}) \in \boldsymbol{N}^{\boldsymbol{a}}_{\boldsymbol{epi(f)}}\,(\bar{\boldsymbol{x}}, \mathbf{f}(\bar{\boldsymbol{x}}))$ with $\boldsymbol{c} \geq \mathbf{0}$ then $\boldsymbol{f}$ is weakly subdifferentiable function at $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ and $(\boldsymbol{v}, \boldsymbol{c} + \boldsymbol{cL}) \in \ \boldsymbol{\partial^w f}(\bar{\boldsymbol{x}})$.

Proof. (i)
Assume that $\boldsymbol{f}$ is Lipschitz at $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$. Then clearly $\boldsymbol{f}$ is lower Lipschitz and hence weakly subdifferentiable at $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$. Let $(\boldsymbol{v}, \boldsymbol{c}) \in \ \boldsymbol{\partial^w f}(\bar{\boldsymbol{x}})$. Then by definition of the weak subdifferential, we have:

$$f(x) - \mathrm{f}(\bar{x}) \geq \langle v, x - \bar{x} \rangle - c\|x - \bar{x}\| \text{ for all } x \in \mathbb{R}^{\boldsymbol{n}}.$$

Hence

$$\alpha - \mathrm{f}(\bar{x}) \geq \langle v, x - \bar{x} \rangle - c\|x - \bar{x}\| - c|\alpha - \mathrm{f}(\bar{x})| \text{ for all } (x, a) \in epi\ (f).$$

This can be written in the form:

$$\langle v, x - \bar{x} \rangle + (-1)(\alpha - \mathrm{f}(\bar{x})) - c\|(x - \bar{x}), (\alpha - \mathrm{f}(\bar{x}))\| \text{ for all } (x, a) \in epi\ (f).$$

which implies that $((\boldsymbol{v}, -\mathbf{1}), \boldsymbol{c}) \in \ \boldsymbol{N}^{\boldsymbol{a}}_{\boldsymbol{epi(f)}}\,(\bar{\boldsymbol{x}}, \mathbf{f}(\bar{\boldsymbol{x}}))$.

(ii) Now assume that $((\boldsymbol{v}, -\mathbf{1}), \boldsymbol{c}) \in \ \boldsymbol{N}^{\boldsymbol{a}}_{\boldsymbol{epi(f)}}\,(\bar{\boldsymbol{x}}, \mathbf{f}(\bar{\boldsymbol{x}}))$. This means that

$$\langle \boldsymbol{v}, \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + (-\mathbf{1})(\boldsymbol{\alpha} - \mathbf{f}(\bar{\boldsymbol{x}})) - \boldsymbol{c}\|(\boldsymbol{x} - \bar{\boldsymbol{x}}), (\boldsymbol{\alpha} - \mathbf{f}(\bar{\boldsymbol{x}}))\| \text{ for all } (\boldsymbol{x}, \boldsymbol{a}) \in \boldsymbol{epi}\ (\boldsymbol{f}). \tag{4}$$

Now by letting $\boldsymbol{\alpha} = \boldsymbol{f}(\boldsymbol{x})$ we obtain:

$$\langle \boldsymbol{v}, \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle + (-\mathbf{1})(\boldsymbol{f}(\boldsymbol{x}) - \mathbf{f}(\bar{\boldsymbol{x}})) - \boldsymbol{c}\|(\boldsymbol{x} - \bar{\boldsymbol{x}}), (\boldsymbol{f}(\boldsymbol{x}) - \mathbf{f}(\bar{\boldsymbol{x}}))\| \text{ for all } \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}.$$

By the hypothesis, $\boldsymbol{f}$ is Lipschitz function at $\bar{\boldsymbol{x}}$ with Lipschitz constant $\boldsymbol{L}$. Hence,

$$\boldsymbol{f}(\boldsymbol{x}) - \mathbf{f}(\bar{\boldsymbol{x}}) \leq \boldsymbol{L}\|\boldsymbol{x} - \bar{\boldsymbol{x}}\| \text{ for all } \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$$

Then it follows from (4) that

$$\boldsymbol{f}(\boldsymbol{x}) - \mathbf{f}(\bar{\boldsymbol{x}}) \geq \langle \boldsymbol{v}, \boldsymbol{x} - \bar{\boldsymbol{x}} \rangle - (\boldsymbol{c} + \boldsymbol{cL})\|\boldsymbol{x} - \bar{\boldsymbol{x}}\| \text{ for all } \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}.$$

which shows that the proof is completed.

4. CONCLUSION

In this work, we have presented a theorem relates the augmented normal cone when the set is epigraph and the weak subdifferential. We showed the existence of such a theorem in the nonconvex analysis. We additionally obtain a zero duality gap condition by imposing conditions on the objective function, constraint functions and the set S.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

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