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**RESEARCH ARTICLE** / ARAŞTIRMA MAKALESİ

# MIMO PID Control of Rotors with Electromagnetic Bearings

*Elektromanyetik Yataklı Rotorların MIMO PID Kontrolü*

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**Öz**

Elektromanyetik yataklı rotorlar kararsız sistemler oldukları için geribeslemeli kontrol yapılması gerekmektedir. Rotor dinamiği sabit hızda doğrusal ve zamanla değişmeyen özellikte olmakla birlikte, elektromanyetik yatak dinamiği doğrusal değildir. Doğrusal olmayan yatak dinamiği sabit bir sapma akımı kullanılarak bir çalışma noktası civarında doğrusallaştırılabilir. Bu makalede yatay rotor/aktif manyetik yataklı sistemler için merkezi bir MIMO PID kontrolör tasarlayarak çoklu SISO PID kontrol ile performans karşılaştırması yapmaktayız. Rotor dinamiğindeki dinamik eşleşme birbirine dik yönlerde jiroskopik kuvvetler oluşturmaktadır. Eşleşik (dinamik) dengesizlik kuvvetler nedeniyle oluşabilecek yanal yönlerdeki açısal hareketlerin meydana getirdiği jiroskopik kuvvetlerin kompanzasyonu için SISO PID kontrol yeterli performansa sahip değildir.

**Anahtar kelimeler:** Konveks optimizasyon, MIMO PID kontrol, dinamik balanssızlık giderme, jiroskopik rotor dinamiği, elektromanyetik yataklar.

**Abstract**

Rotors with electromagnetic bearings are inherently unstable systems; hence feedback control is an integral part of their operation. While the rotor dynamics is linear and time-invariant at constant operation speed, electromagnetic bearing model is non-linear. Non-linear bearing dynamics can be linearized at an operating point using a constant bias current. In this paper we design a MIMO PID controller for horizontal rotor/active magnetic bearing systems and compare its performance with respect to SISO decentralized PID control. Dynamic coupling in rotor dynamics causes gyroscopic forces to act at orthogonal directions on the rotor. SISO PID control lacks sufficient performance as it has limited capability to compensate for the gyroscopic effects due to angular motions in transverse directions which can be caused by couple (dynamic) unbalance forces.

**Keywords:** Convex optimization, MIMO PID control, dynamic unbalance suppression, gyroscopic rotor dynamics, electromagnetic bearings.

## I. INTRODUCTION

Due to their non-contact and lubricant-free operation, utilization of magnetic bearings in rotating machinery is attracting increasing interest both from industry and academia. Their dynamic equations can be derived in LTI form for small deviations from the operating point. Dynamics depends on the operational speed due to electromagnetic and gyroscopic cross-coupling. Disturbance acting on the rotor/electromagnetic bearing system is often the unbalance force, which is speed-synchronous and sinusoidal.

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Using appropriate control methods, the desired bearing characteristics can be optimized for the particular application. In linear control, compensators are designed using a linear time-invariant model after the linearization of the system. A common strategy is to oppose the unbalance forces by speed synchronous forces through electromagnetic bearings. The amplitude and phase of compensation forces is computed on-line based on the model of the system and the displacements of the rotor due to unbalance in a rotating frame. Burrows et al. [1] describe this method. When the speed of the rotor is constant, linear time-invariant controllers can be designed to reject disturbances (mainly unbalance) at that particular synchronous frequency. This is accomplished by incorporating notch filters in the loop [2]. Perfect disturbance rejection at a particular frequency is also possible by incorporating a transmission-zero into the closed loop system. Decentralized SISO PID controllers are used extensively in commercial systems as they are simple to use and provide sufficient performance. To enhance PID controllers' performance to achieve minimum displacements, especially at critical speeds, much research has been accomplished [3].

An alternative to using decentralized SISO PID control is to design a central MIMO PID controller that uses all sensors to drive all actuators. MIMO PID controllers can achieve satisfactory performance even when the system dynamics are quite coupled. Motivated by this capability of MIMO PID controllers, their implementation in rotor/AMB systems having coupled angular motion deserves attention. However, MIMO PID design is more complex and challenging. General analytical solutions are not available and numerical methods using algorithms based on iterative linear matrix inequalities (ILMI's) are developed for this purpose [4].

PID control has inherent limitations due to limitations on defining the performance criteria for the controllers. Hence, current research is concerned with model based controllers and, especially, the application of modern robust control techniques. The loop shaping design procedure has been used to adjust the open-loop singular values based on the requirements for the sensitivity of the closed-loop system [5]. Design by loop shaping exhibits limited performance when the rotor operates at different operating speeds due to the change in the frequency of the unbalance force. In robust and optimal control theory controller design can be cast in the form of an optimization problem, which has further advantages over conventional PID control. $H_\infty$ control problem was set in linear matrix inequality (LMI) formulation in [6]. This procedure is applicable to a wider range of plants and convex optimization algorithms are utilized for the solution efficiently via semi-definite programming [7]. Optimal controller design methods such as $H_\infty$ and $\mu$ synthesis provide better performance over decentralized PID controller design techniques as shown in [8]. Noshadi et al. have determined the parameters of an active magnetic bearing system before implementing MIMO robust control techniques [9].

LMI frame work also allows for the synthesis of gain-scheduled controllers, which increase robustness and/or performance for the feedback control of LPV plants. In robust LTI $H_\infty$ design, time varying plant characteristics are treated as uncertainty, which often leads to overly conservative controllers. If the system matrices of the plant are functions of time varying parameters, which can be continuously measured, LMI synthesis methods allow the computation of gain scheduled controllers as functions of measured parameters. The derivations of LMI controller synthesis for parameter varying plants are shown in the literature[10].

Although advanced control techniques such as $H_\infty$, $\mu$ synthesis, or adaptive control techniques lead to superior performance, controllers designed by these methods are of high order and difficult, if not impossible, to realize. Thus we have decided to study and develop a MIMO centralized PID controller, which has not been used before for rotor/electromagnetic bearing systems in the literature, and can be implemented using simple controllers commercially available. We compare the performance of the MIMO PID control with respect to decentralized SISO PID controlled system.

The paper is organized as follows: In Section 2 we develop a state-space model of horizontal rotor/electromagnetic bearing systems. In Section 3 we contemplate MIMO PID control design and present the synthesis algorithms. In Section 4 numerical results and simulations for a horizontal rotor/electromagnetic bearing systems with SISO and MIMO PID controllers are provided and we compare their performance. Finally in Section 5 we conclude with our comments on the results.

## II. ROTOR/ELECTROMAGNETİC BEARING MODEL

Dynamics of a rigid rotor can be derived using Newton-Euler equations

$$\mathcal{F} = \dot{\jmath} = \frac{d}{dt}(M_r v) \quad \text{and} \quad \mathcal{M} = \dot{\mathcal{H}} = \frac{d}{dt}(I\omega) \tag{1}$$

Where $M_r$ is rotor mass, v is mass center velocity, $I$ is the inertia dyadic, $\omega$ is the angular velocity of the rotor. Linear and angular momentum are denoted by $\dot{\jmath}$ and $\dot{\mathcal{H}}$ respectively. Hence $\mathcal{F}$ and $\mathcal{M}$ are the total external force and the total external moment about the mass center of the rotor respectively.

External moment in (1) can be obtained in body fixed rotating coordinates $\zeta\eta\xi$ to read

$$\left(\frac{d}{dt}\right)_{\zeta\eta\xi}(I\omega) + \omega \times I\omega = \mathcal{M}_{\zeta\eta\xi} \tag{2}$$

Assuming that the rotor mass is distributed evenly around the rotation axes, inertia dyadic and angular velocity of the rotor in rotating coordinates are given by

$$I_{\zeta\omega\xi} = \begin{pmatrix} I_t & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & I_r \end{pmatrix} \text{and} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\sin\phi + \dot{\psi}\cos\phi \\ \dot{\theta}\cos\phi - \dot{\psi}\sin\phi \\ \dot{\phi} + \dot{\psi}\sin\theta \end{pmatrix} \tag{3}$$

Where $I_t$ and $I_r$ are the principle moment of inertia of the rotor in transverse and radial directions respectively. The angular displacements $\phi$, $\theta$ and $\psi$ are reflecting the orientations of body fixed rotating frame $\zeta\eta\xi$ about the inertial $zyx$ frame (321 Euler angles). However, since the angles ψ and θ during the motion of the rotor are negligible, we have cosθ≈1 and sinθ≈0 and angular velocities in equations (3) simplifies to

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \tag{4}$$

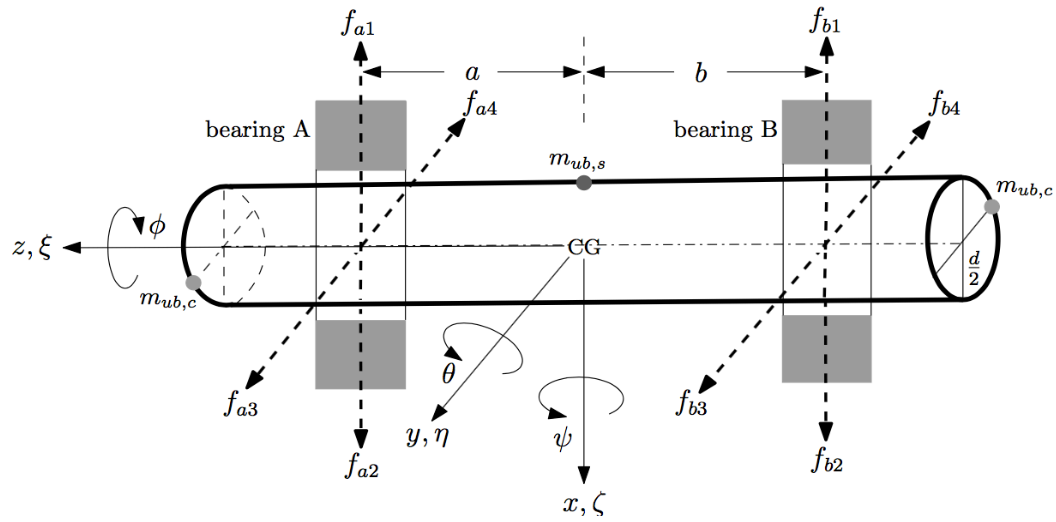


**Figure1.** Horizontal rigid rotor

Similarly, moment vector in rotating coordinates is

$$\mathcal{M}_{\zeta\eta\xi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathcal{M}_{xyz} \tag{5}$$

The force acting on the rotor caused by two electromagnets is opposing directions is [11]

$$F_r = F_+ - F_- = k_m \left( \left( \frac{i_+}{s_0 - r} \right)^2 - \left( \frac{i_-}{s_0 + r} \right)^2 \right)$$

where $k_m$ is the magnetic bearing constant $k_m := \frac{\mu_0 A_A n_c^2}{4} \cos\alpha_M$ with $\alpha_M$ denoting the angle between the magnet centerline and a pole), $s_0$ is the nominal air gap between the rotor and the bearing, and $r$ is the displacement of the rotor. Nonlinearity of the magnetic force can be reduced by introducing a bias current $i_0$ added to the control currents $\pm i_c$ in each control axis. Then electromagnetic force can be linearized around the operating point to read

$$F_r \cong F_{rOP} + \frac{\partial F_r}{\partial i}\Big|_{OP} (i_c - i_{cOP}) + \frac{\partial F_r}{\partial r}\Big|_{OP} (r - r_{OP})$$

Therefore, with $i_{cOP} = 0$ and $r_{OP} = 0$, the linearized magnetic bearing force for small currents and displacements around the operating point is given by

$$F_{r,lin} = k_i i_c - k_s r \tag{6}$$

with current gain of the actuator $k_i$ and the bearing (negative) stiffness $k_s$ defined to be

$$k_i := 4k_m \frac{i_0}{s_0^2} \text{ and } k_s := -4k_m \frac{i_0^2}{s_0^3}.$$

Total displacement of the rotor at the bearing positions are the superposition of translation and rotation motions, which are

$$r_{A,x} = x + a\theta, \quad r_{B,x} = x - b\theta, \quad r_{A,y} = y - a\psi, r_{B,y} = y + b\psi \tag{7}$$

Substituting (4-7) into (1) leads to the equations of motion

$$\ddot{x} = \frac{1}{M_r}\left(f_{A,x} + f_{B,x} + \frac{M_r}{\sqrt{2}}g\right) \tag{8a}$$

$$\ddot{y} = \frac{1}{M_r}\left(f_{A,y} + f_{B,y} + \frac{M_r}{\sqrt{2}}g\right) \tag{8b}$$

$$\ddot{\psi} = \frac{1}{I_t}\left(-\Omega I_p\dot{\theta} + bf_{B,y} - af_{A,y}\right) \tag{8c}$$

$$\ddot{\theta} = \frac{1}{I_t}\left(\Omega I_p\dot{\psi} + af_{A,y} - bf_{B,x}\right) \tag{8d}$$

with bearing forces

$$f_{A,x} = f_{a2} - f_{a1} = k_{i,A}i_{A,x} - k_{s,A}\left(x + a\theta\right) \; f_{B,x} = f_{b2} - f_{b1} = k_{i,B}i_{B,x} - k_{s,B}\left(x + b\theta\right)$$

$$f_{A,y} = f_{a3} - f_{a4} = k_{i,A}i_{A,y} - k_{s,A}\left(y + a\psi\right) \; f_{B,y} = f_{b3} - f_{b4} = k_{i,B}i_{B,y} - k_{s,A}\left(y + b\psi\right)$$

Equations of motion (8) can be given in state-space form

$$\dot{x}_r = Ax_r + Bu + \overline{g} \;, \qquad y = Cx_r \tag{9}$$

where $x_r := \left(x \; y \; \psi \; \theta \; \dot{x} \; \dot{y} \; \dot{\psi} \; \dot{\theta}\right)^T$, and $u := \left(i_{cA,x} \; i_{cA,y} \; i_{cB,x} \; i_{cB,x}\right)^T$.

Matrices A, B, and C are

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_{s,A}+k_{s,B}}{M_r} & 0 & 0 & \frac{ak_{s,B}+bk_{s,A}}{M_r} & 0 & 0 & 0 & 0 \\ 0 & -\frac{k_{s,A}+k_{s,B}}{M_r} & \frac{bk_{s,A}+ak_{s,B}}{M_r} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{bk_{s,A}+ak_{s,B}}{I_t} & -\frac{b^2k_{s,B}+a^2k_{s,A}}{I_t} & 0 & 0 & 0 & 0 & \frac{-\Omega I_r}{I_t} \\ \frac{ak_{s,B}+bk_{s,A}}{I_t} & 0 & 0 & -\frac{b^2k_{s,B}+a^2k_{s,A}}{I_t} & 0 & 0 & \frac{\Omega I_r}{I_t} & 0 \end{pmatrix}$$

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k_{i,A}}{M_r} & 0 & \frac{k_{i,B}}{M_r} & 0 \\ 0 & \frac{k_{i,A}}{M_r} & 0 & \frac{k_{i,B}}{M_r} \\ 0 & -\frac{ak_{i,A}}{I_t} & 0 & \frac{bk_{i,B}}{I_t} \\ \frac{ak_{i,A}}{I_t} & 0 & -\frac{bk_{i,B}}{I_t} & 0 \end{pmatrix} \quad C=\begin{pmatrix} 1 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 1 & -a & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The gravity term $\bar{g}$ in (9) can be neglected because the bias current can be adjusted larger than enough to compensate for the additional current required to suspend the rotor against gravity.

## III. CONTROLLER DESIGN

For many years single input-single output (SISO) PID controllers have been used with acceptable performance for multi input-multi output (MIMO) plants. Generally this is accomplished by matching inputs and outputs in pairs. As such, SISO PID controllers can be designed for each channel using standard tuning rules for PID control. For decoupled MIMO plants multi-loop decentralized SISO PID design can be satisfactory. An alternative to decoupled SISO PID control is to synthesize a central MIMO PID controller that combines all sensors to drive all actuators. MIMO PID controllers can achieve much better performance even when the plant dynamics are predominantly coupled. However, MIMO PID design is more challenging and complex. Iterative optimization algorithms based on linear matrix inequalities (LMI's) are developed for this purpose. However the solution they provide is either overly conservative [12] or may converge to a local minima instead of the global minimum [13].

### 3.1. MIMO PID Control in State-Space

In MIMO PID control rotors suspended by magnetic bearings a single four input-four output PID controller can be used for both bearings with the rotor position measurements $y=\left(r_{A,x}r_{A,y}r_{B,x}r_{B,y}\right)^T$ fed back into the controller and the system input (controller output) $u=\left(i_{cA,x}i_{cA,y}i_{cB,x}i_{cB,y}\right)^T$ sent to electromagnetic bearings/actuators. Linearized system derived in Section 2 can be represented by the state-space equations

$$\dot{x}_r = Ax_r + Bu \quad y = Cx_r \tag{10}$$

with the following PID controller

$$u = K_1 y + K_2 \int_0^t t d\tau + K_3 \frac{dy}{dt} \tag{11}$$

where $x_r \in \mathbb{R}^n$ is the rotor state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and $K_1, K_2, K_3 \in \mathbb{R}^{p\times m}$ are matrices to be designed.

Let $v_1 = x_r$ and $v_2 = \int_0^t y d\tau$. We define $v := \left(v_1{}^T v_2{}^T\right)^T$. Hence the dynamics of the augmented system can be represented by

$$\dot{v} = \tilde{A}x_r + \tilde{B}u \quad y = \left(C\ 0\right)v \tag{12}$$

where $\tilde{A} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$

Also note that $\dot{y} = C\dot{x}_r = CAx_r + CBu = \begin{pmatrix} CA & 0 \end{pmatrix} v + CBu$.

Denoting $\tilde{C}_1 = \begin{pmatrix} C & 0 \end{pmatrix}$, $\tilde{C}_2 = \begin{pmatrix} 0 & I \end{pmatrix}$, $\tilde{C}_3 = \begin{pmatrix} CA & 0 \end{pmatrix}$, and $\tilde{y}_i = \tilde{C}_i v$, for $i = 1,2,3;$ equation (11) transforms to

$$u = K_1\tilde{y}_1 + K_2\tilde{y}_2 + K_3\tilde{y}_3 + K_3CBu \tag{13}$$

Assuming that $\left(I - K_3CB\right)$ is invertible, multivariable PID control design reduces to the following static output-feedback problem

$$\dot{v} = \tilde{A}v + \tilde{B}u, \quad \tilde{y} = \tilde{C}v, \quad u = \tilde{K}\tilde{y}, \tag{14}$$

where $\tilde{y} := \left(\tilde{y}_1^T\ \tilde{y}_2^T\ \tilde{y}_3^T\right)^T$, $\tilde{K} := \left(\tilde{K}_1\ \tilde{K}_2\ \tilde{K}_3\right) = \left(\left(I - K_3CB\right)^{-1} K_1\ \left(I - K_3CB\right)^{-1} K_2\ \left(I - K_3CB\right)^{-1} K_3\right)$

and $\tilde{C} := \left(\tilde{C}_1^T\ \tilde{C}_1^T\ \tilde{C}_1^T\right)^T$.

Once the composite matrix $\tilde{K} := \left(\tilde{K}_1\ \tilde{K}_2\ \tilde{K}_3\right)$ is computed, PID gain matrices can be recovered from the equations

$$K_3 = \tilde{K}_3\left(I + CB\tilde{K}_3\right)^{-1}, \quad K_2 = \left(I - K_3CB\right)\tilde{K}_2, K_1 = \left(I - K_3CB\right)\tilde{K}_1 \tag{15}$$

It is easy to show that matrices $\left(I - K_3CB\right)$ and $\left(I + CB\tilde{K}_3\right)$ are both invertible [12].

### 3.2. Static Output Feedback Stabilization

As is well known, static output feedback stabilization (SOFS) problem is among the important open questions in control theory. See the survey in [14] and a recent paper [15]. Common approach to solve the SOFS problems is to exploit the special structure of a particular problem as there is no general analytic solution. However numerical solutions may also be possible to a SOFS problem. This approach is rejected in [4], where linear matrix inequalities (LMIs) are used, which are very effective if the associated problem has a numerical solution. The objective of SOFS problem is to find a static output feedback controller $u = Ky$ where $K \in \mathbb{R}^{p \times m}$, such that the closed loop system $\dot{x} = \left(A + BKC\right)x$ is asymptotically stable.

**Lemma**[4] System (10) is stabilizable via static output feedback, if and only if, there exist matrices $P \succ 0$ and $K$ satisfying the following matrix inequality:

$$A^T P + PA - PBB^T P + \left(B^T P + KC\right)^T \left(B^T P + KC\right) \prec 0 \tag{16}$$

*Remark*: It can be shown that that matrix inequality (16) is valid for all state-space realizations of the system under consideration [12]. In general industrial PID controllers are designed by modeling in frequency domain. Hence we can use any state-space realization of the system to investigate a possible solution for static output feedback stabilization provided that the model of the system is available in frequency domain.

Solution of the matrix inequality (16) is very complicated due the negative sign nonlinear term $-PBB^T P$. Suppose that it is possible to find a matrix $\Gamma$ depending affinely on P and satisfies

$$\Gamma \preceq PBB^T P \tag{17}$$

Then it is can be shown that it is possible to stabilize the system (10) by $u = Ky$ provided that the following inequality

$$A^T P + PA - \Gamma + \left(B^T P + KC\right)^T \left(B^T P + KC\right) \succ 0 \tag{18}$$

can lead to a solution for matrices $(P,K)$. Using Schur complement formula [16], the inequality (17) is equivalent to

$$\begin{pmatrix} A^T P + PA - \Gamma & \left(B^T P + KC\right)^T \\ B^T P + KC & -I \end{pmatrix} \prec 0 \tag{19}$$

Once other parameters in $\Gamma$ are given, matrix inequality (19) depends affinely on the pair $(P,K)$. In [4], $\Gamma$ is given by

$$\Gamma = X^T BB^T P + P^T BB^T X - X^T BB^T X \tag{20}$$

where $X \succ 0$. In this case inequality (17) is always satisfied and the equality holds if and only if $X^T B = P^T B$. Once $X$ is known, matrix inequality (19) can be solved by LMI solvers very efficiently. Based upon the facts above, the following iterative linear matrix inequality (ILMI) algorithm is developed to solve SOFS problem.

**Optimization algorithm**[4]

*Initial data: A state-space realization (A,B,C) of the system.*

Step 1: *Choose* $Q_0 \succ 0$ *and solve P for the Riccati equation*

$$A^T P + PA - PBB^T P + Q_0 = 0, \quad P \succ 0 .$$

*Set* $i = 1$ *and* $X_1 = P$.

Step 2: *Solve the optimization problem OP1 for* $P_i, K$ *and* $\alpha_i$ *. given below.*

**OP1:** *Minimize* $\alpha_i$ *subject to the following LMI constraints*

$$\begin{pmatrix} \sum_{1i} & \left(B^T P_i + KC\right)^T \\ B^T P_i + KC & -I \end{pmatrix} \prec 0, \; P_i \prec 0 \tag{21}$$

where

$$\sum_{1i} = A^T P_i + P_i A - X_i BB^T P_i - P_i BB^T X_i + X_i BB^T X_i - \alpha_i P_i$$

*Denote by* $\alpha_i^*$ *the minimized value of* $\alpha_i$ *.*

*Step 3: If* $\alpha_i^* \le 0$*,the matrices* $\left(P_i, K\right)$ *solves the problem. Stop. Otherwise go to* Step 4.

Step 4: *Solve the optimization problem OP2 for* $P_i$ *and K given below.*

**OP2:**Minimize $tr\left(P_i\right)$subject to LMI constraints (21) with $\alpha_i = \alpha_i^*$, where tr represents the trace of a matrix. Denote by $P_i^*$ the optimal $P_i$.

Step 5: If $\left\|X_i B - P_i^* B\right\| < \varepsilon$ , where $\varepsilon$ is the tolerance prescribed, go to Step 6; otherwise set i:=i+1, Xi = $P_i^*$, and go to Step 2.

Step 6: Algorithm cannot decide whether this SOFS problem is solvable or not. Stop

### 3.3. Feedback Stabilization with MIMO PID Controllers

Consider system (10) again, but now we use PID controller (11) instead of static output feedback controller $u = \tilde{K}y$ such that the closed-loop system $\dot{v} = \left(\tilde{A} + \tilde{B}\tilde{K}\tilde{C}\right)v$ is asymptotically stable. Our objective here is to design the feedback matrices $K_1, K_2, K_3$ of the MIMO PID controller.

The stabilizing feedback matrices $K_1, K_2, K_3$ in (14) can be found by solving $\tilde{K}$ through the application of the optimization algorithm given in Section 3.2 to system $\left(\tilde{A}, \tilde{B}, \tilde{C}\right)$. In order to guarantee the invertability of the matrix $\left(I + CB\tilde{K}_3\right)$, we can add the following LMI

$$I + \left(CB\tilde{K}_3\right) + CB\tilde{K}_3 \succ 0 \tag{22}$$

Hence Steps 2 and 4 in the algorithm are changed to read:
Step 2': *Solve the optimization problem for* $\left(P_i, \tilde{K}\right)$ *and* $\alpha_i$ *: Minimize* $\alpha_i$ *subject to the constraints LMI (21) and LMI (22) (OP1).*Step 4': *Solve the optimization problem for* $P_i$ *and* $\tilde{K}$ *: Minimize* $tr\left(P_i\right)$ *subject to the constraints LMIs (21) and (22) with* $\alpha_i = \alpha_i^*$ *(OP2).*

Note that the inequality (22) stems from the fact that if (22) holds, we have

$$\left(I+CB\tilde{K}_3\right)^T\left(I+CB\tilde{K}_3\right)=\left(I+CB\tilde{K}_3\right)^T+CB\tilde{K}_3+\left(CB\tilde{K}_3\right)^T\succ 0$$

Hence it follows that $I+CB\tilde{K}_3$ is invertible. However, note that LMI (22) is a very conservative condition. It is suggested to firs check whether $I+CB\tilde{K}_3$ is invertible before using this constraint.

## IV. RESULTS AND SIMULATIONS

In this section we synthesise SISO and MIMO PID controllers for a horizontal rotor/electromagnetic bearing system and compare their performance.

### 4.1. Numerical Example

Rotor with its electromagnetic bearings and the associated data is shown in Figure 2 and Table 1.

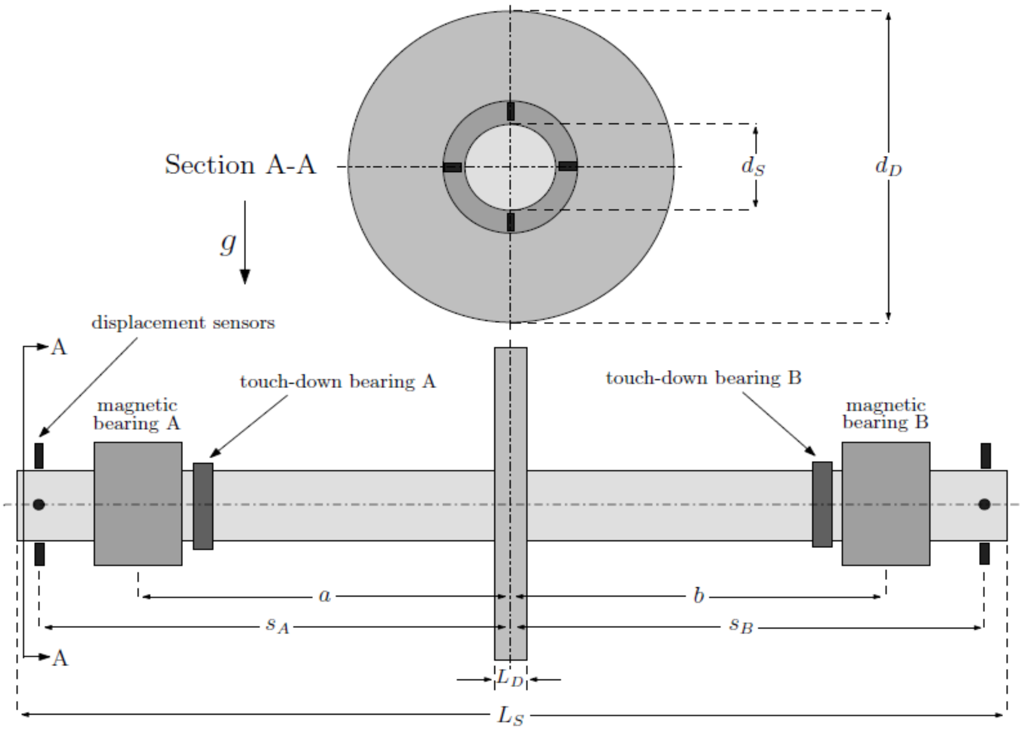


**Figure 2.** Horizontal rotor with radial electromagnetic bearings

**Table 1.**Rotor/electromagnetic bearing data

| Parameter | Value | Unit | Parameter | Value | Unit | Parameter | Value | Unit |
|---|---|---|---|---|---|---|---|---|
| $M_S$ | 85.90 | kg | $L_S$ | 1.50 | m | $s_0$ | 2.0x10$^{-3}$ | m |
| $M_D$ | 77.10 | kg | $L_D$ | 0.05 | m | $s_1$ | 5.0x10$^{-3}$ | m |
| $I_t$ | 17.28 | kg.m$^2$ | $d_S$ | 0.10 | m | $i_0$ | 3.0 | A |
| $I_r$ | 2.41 | kg.m$^2$ | $d_D$ | 0.50 | m | $k_m$ | 7.8455x10$^{-5}$ | N.m$^2$/A$^2$ |
| $a$ | 0.60 | m | $s_A$ | 0.75 | m | $k_s$ | -3.5305x10$^{-5}$ | N/m |
| $b$ | 0.60 | m | $s_B$ | 0.75 | m | $k_i$ | 237.4 | N/A |

5 kW 6000 rpm DC motor drives the rotor shown above. A passive magnetic bearing can be used to support the rotor axially. To calculate the magnetic bearing constant $k_m$, we target the force supplied

by the electromagnets to be three times the rotor weight. Optimal bias current $i_0$ is decided to be 3.0 A by trial and error during the simulations. It has been chosen large enough as a low bias current value would increase the linearization error. Note that the currents supplied to the bearing should not be too large with respect to operating point of the system which is set around the bias current [17].

### 4.2. Simulations

Coupled non-linear dynamics of the rotor/electromagnetic bearing system used in the simulations is shown in Figure 3.

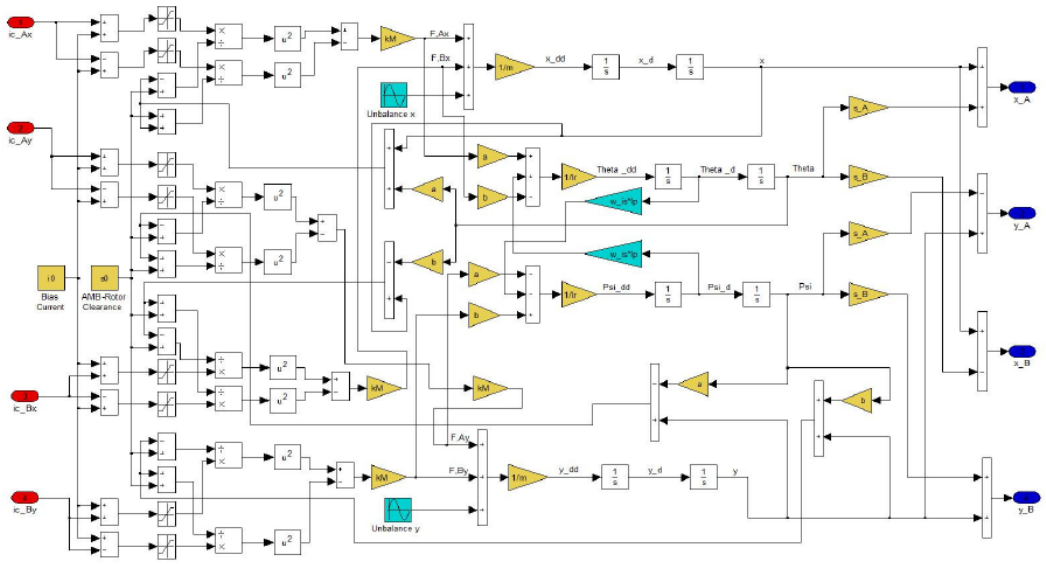


**Figure 3.** Nonlinear rotor/electromagnetic bearing model in Simulink

To activate the gyroscopic coupling of the rotor a couple unbalance force, as shown in Figure 1, is applied to the rotor. With static and couple unbalance forces, equations of motion given in (8) become

$$\ddot{x}=\frac{1}{M_r}\left(f_{A,x}+f_{B,x}+\frac{M_r}{\sqrt{2}}g+m_{ub,s}\Omega^2\frac{d}{2}\cos\left(\Omega t+\varphi_s\right)\right),\ \ddot{y}=\frac{1}{M_r}\left(f_{A,y}+f_{B,y}+\frac{M_r}{\sqrt{2}}g+m_{ub,s}\Omega^2\frac{d}{2}\cos\left(\Omega t+\varphi_s\right)\right),$$

$$\ddot{\psi}=\frac{1}{I_t}\left(-\Omega I_p\dot{\theta}+bf_{B,y}-af_{A,y}+\left(a+b\right)m_{ub,c}\Omega^2\frac{d}{2}\cos\left(\Omega t+\varphi_c\right)\right),\ \ddot{\theta}=\frac{1}{I_r}\left(\Omega I_p\dot{\psi}+af_{A,y}-bf_{B,x}+\left(a+b\right)m_{ub,c}\Omega^2\frac{d}{2}\cos\left(\Omega t+\varphi_c\right)\right).$$

Last two equations show that couple unbalance causes angular acceleration of the rotor in transverse directions, hence the coupled angular dynamics of the system.

Decentralized (decoupled) PID controller has been designed in MATLAB®-Simulink with manual tuning at a rotor speed of 6000 rpm. Disturbance force applied is 300 gm.cm static unbalance Results of the simulations are shown in Figures 4a and 4b for bearing A in y-direction. Sensor calibration for the output $Y_A$ is 0.1 V/mm. Simulations for bearing B give similar results. A 1 V amplitude discrete pulse (vector disturbance for all four channels) with 0.025 seconds duration is injected into the loop at the input of the controller during the simulations. It is verified from the simulations that the closed-loop system is internally stable. However it has been observed that the decentralized PID controller could not stabilize a rotor/AMB even with 50 gm.cm couple (dynamic) unbalance.

MIMO centralized PID controller is designed by semidefinite programming, using the optimization algorithm given in Section 3.2 and LMI (22) given in Section 3.3, with optimization toolboxes YALMIP [18] and SEDUMI [7] for MATLAB®. After the computation of the composite matrix $\tilde{K}$, controller gain matrices $K_1$, $K_2$, and $K_3$ are found by using equations (15). They are given as

$$K_1 = \begin{pmatrix} 8590.3 & 121 & 194.9 & 98.1 \\ 80.9 & 8605.4 & 112.0 & 91.5 \\ 105.7 & 100.1 & 8705.8 & 59.8 \\ 103.0 & 90.6 & 89.7 & 8606.0 \end{pmatrix} K_2 = \begin{pmatrix} 70.3 & 3.0 & 0.9 & 4.1 \\ 1.5 & 75.7 & 2.0 & 3.5 \\ 1.7 & 1.1 & 80.8 & 3.8 \\ 3.0 & 1.6 & 1.7 & 76.3 \end{pmatrix} K_3 = \begin{pmatrix} 10.9 & 0.3 & 0.1 & 0.1 \\ 1.1 & 9.8 & 0.7 & 0 \\ 0.1 & 0.3 & 11.1 & 0.8 \\ 0 & 1.0 & 0.9 & 12.3 \end{pmatrix}$$

Results of the simulations at 6000 rpm operation speed are shown in Figure 5 for bearing A in y-direction. Disturbance force applied is 300 gm.cm static unbalance and 250 gm.cm couple (dynamic) unbalance. A 1 V amplitude discrete pulse (vector disturbance for all four channels) with 0.025 seconds duration is injected into the loop at the input of the controller during the simulations. It is verified from the simulations that the closed-loop system is internally stable. Note that the controller can accommodate the whirling motion due to unbalance of the rotor.

It is also clear from Figure 5bthat vibration has a peak value less than 0.1 V (except the transient due to disturbance injected), corresponding to 1mm of rotor displacement.

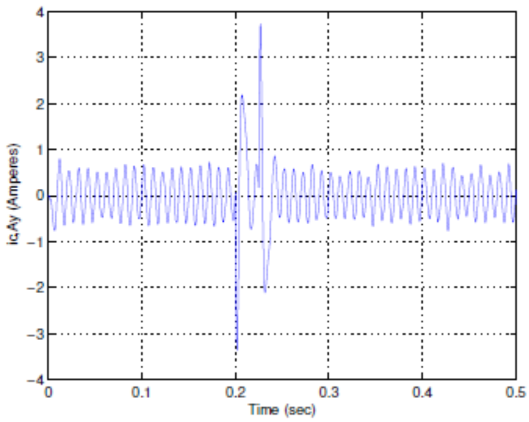


**Figure 4a.** Control current (Amperes) for decentralized PID Control

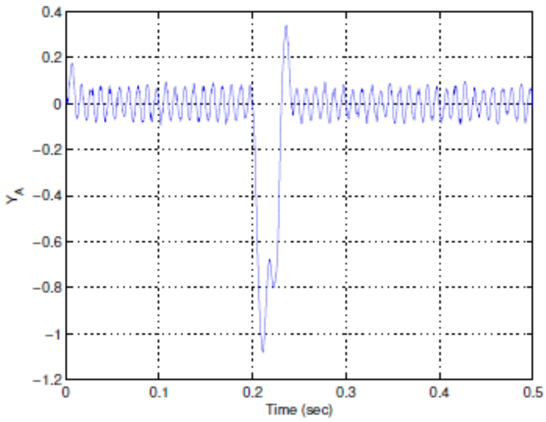


**Figure 4b.** Rotor position (Volts)for decentralized PID Control

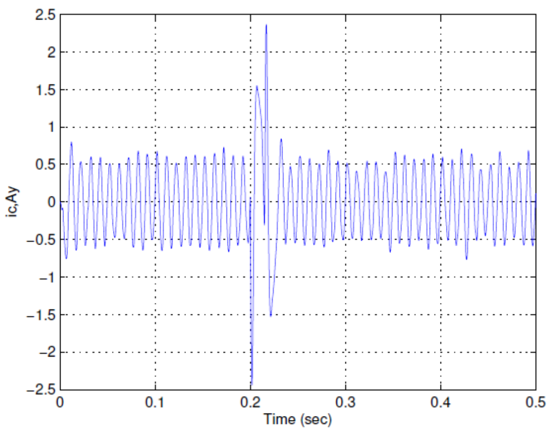


**Figure 5a.** Control current (Amperes) for MIMO PID Control

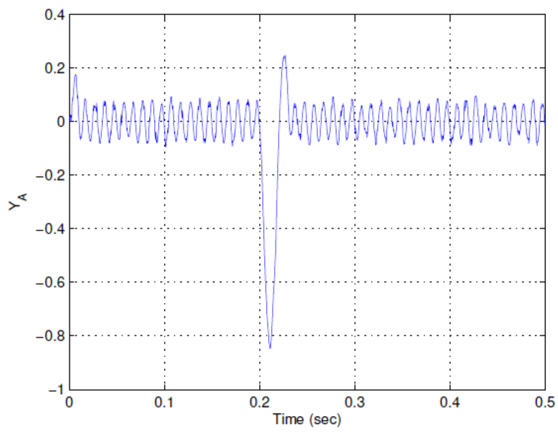


**Figure 5b.** Rotor position (Volts) for MIMO PID Control

It can clearly be noticed from the Figures 5a and 5b that MIMO PID control achieves better performance than decentralized PID control with respect to transient effects as well.

## V. DISCUSSION AND CONCLUSION

In this paper we studied centralized MIMO PID control of rotor/electromagnetic bearing systems and compared its performance relative to the decentralized SISO PID control. In our numerical example we have shown that MIMO PID control can tolerate four times more dynamic unbalance force with respect to SISO PID controlled system. It can be seen in Figure 6 that decentralized control system cannot stabilize the rotor if couple unbalance exceeds 50 gm.cm at 6000 rpm. Simulation results show that rotor touches the roller bearings which are set at 2 mm radial clearance with 300 gm.cm static and 50 gm.cm couple unbalance. However, as shown in Figure 7, the MIMO PID controller can accommodate 300 gm.cm static and 250 gm.cm couple unbalance at this speed with less that 1 mm vibration peak amplitude. This result was expected due to the fact that centralized MIMO PID can also compensate for the cross-coupling gyroscopic terms caused by the angular motion of the rotor.

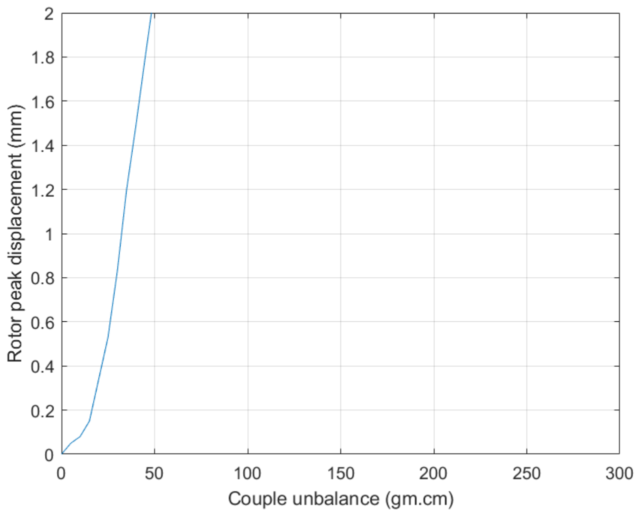


**Figure 6.** Rotor peak displacement versus couple unbalance with SISO PID Control

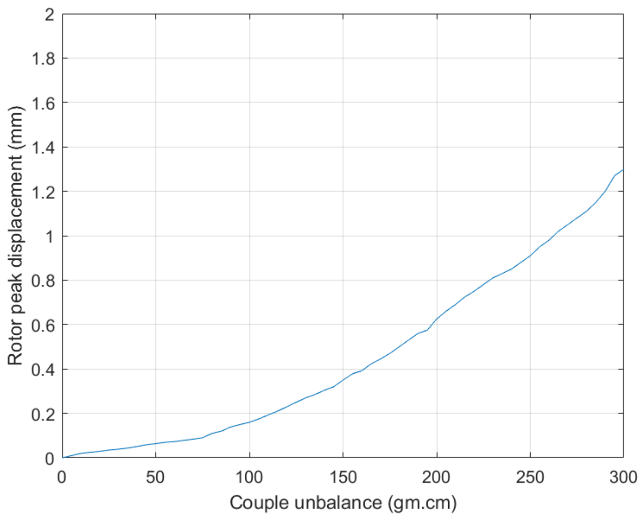


**Figure 7.** Rotor peak displacement versus couple unbalance with MIMO PID Control

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