Istanbul Journal of Mathematics
ijmath 2 (2), 55–69, 2024
DOI: 10.26650/ijmath.2024.00016

RESEARCH ARTICLE

Trace formula for finite groups and the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$*

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*A part of this work was presented at the Selçuk University Mathematics Colloquium in April 27, 2022 and completed while visiting the Abdus Salam ICTP in June 2024.

ABSTRACT
Let G be a finite group. The trace formula for G, which is the trivial case of the Arthur trace formula, is well known with many applications. In this note, we further consider a subgroup Γ of G and a representation $\rho : \Gamma \to \mathsf{GL}(V_\rho)$ of Γ on a finite dimensional $\mathbb{C}$-vector space V_ρ, and compute the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G\rho(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ for any function $f : G \to \mathbb{C}$ in two different ways. The expressions for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ denoted by $J(\rho, f)$ and $I(\rho, f)$ are the spectral side and the geometric side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, respectively. The identity $J(\rho, f) = \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = I(\rho, f)$ is a generalization of the trace formula for the finite group G. This theory is then applied to the "automorphic side" of the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$; namely, to the "automorphic side" of the local 0-dimensional Langlands correspondence for $\mathsf{GL}(n)$, where new identities are obtained for the ϵ-factors of representations of $\mathsf{GL}_n(\mathbb{F}_q)$.

Mathematics Subject Classification (2020): 20C15, 11F72

Keywords: Trace formula, finite group, Macdonald correspondence

1. INTRODUCTION

This short note which is the revised version of our colloquium talk notes that we delivered at Selçuk University, Konya in 2022, concerns the trivial case of the Arthur trace formula, namely the trace formula for finite groups. Let G be a finite group. The trace formula for G is an identity involving characters of the finite group G, and although simple, this formula is of central importance with deep applications in the theory of linear representations of G over $\mathbb{C}$. For instance, using the trace formula for G, it is well known that major theorems like the Frobenius reciprocity law (Example 6.1) and the Plancherel formula (Example 6.2) for G follow directly.

In this note, we further consider a subgroup Γ of G and a representation

$$\rho : \Gamma \to \mathsf{GL}(V_\rho)$$

of Γ on a finite dimensional vector space V_ρ over $\mathbb{C}$, and for any function $f : G \to \mathbb{C}$, we compute the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator

$$\mathsf{Ind}_\Gamma^G\rho(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$$

which is defined by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f))(\varphi) = \sum_{g \in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi), \ {}^\forall\varphi \in \mathsf{Ind}_\Gamma^G(V_\rho)$$

in two different ways. The first expression for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ denoted by $J(\rho, f)$ involves multiplicities of irreducible representations appearing in $\mathsf{Ind}_\Gamma^G(\rho)$ and called the spectral side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$. The second expression $I(\rho, f)$ for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, which is called the geometric side of the trace formula for $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$, is

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Submitted: 25.06.2024 • **Last Revision Received:** 12.12.2024 • **Accepted:** 17.12.2024

constructed by the conjugacy classes of Γ and G. The identity (Theorem 3.1 and Theorem 5.1)

$$J(\rho, f) = \mathrm{Tr}(\mathsf{Ind}_{\Gamma}^{G}(\rho)(f)) = I(\rho, f)$$

that we derive and call the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ in this note, is a generalization of the well known trace formula for the finite group G which corresponds to the case $\Gamma = \langle 1 \rangle$ and $\rho = \mathbb{1}_{\langle 1 \rangle} : \langle 1 \rangle \to \mathrm{GL}(\mathbb{C})$. In the remainder of this work, that is in Section 7, we apply this theory to the "automorphic side" of the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$; namely, to the "automorphic side" of the local 0-dimensional Langlands correspondence for $\mathsf{GL}(n)$, where new identities are obtained for the ϵ-factors of representations of $\mathsf{GL}_n(\mathbb{F}_q)$.

The main references that we follow closely in this work are Terras Terras (1999) and Yang Yang (2006). However, we will deal with the trace formula for finite groups in full generality. The trace formula stated and proved in this note should be considered "folklore", videlicet, well-known to researchers in the area, and seems only treated recently in the M.Sc. thesis of Chasek Chasek (2023) and in a note of Lee Lee (2022). Therefore, the only contribution of this note is the last section on the Macdonald correspondence for $\mathsf{GL}_n(\mathbb{F}_q)$, where the main references that we follow are Macdonald (1980); Piatetski-Shapiro (1983); Silberger and Zink (2008); Ye and Zelinger (2021)

2. THE REPRESENTATION $\mathsf{IND}_{\Gamma}^{G}(\rho)$ OF G ON $\mathsf{IND}_{\Gamma}^{G}(V_\rho)$ OVER $\mathbb{C}$ INDUCED FROM $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ UP TO G

To fix the very basic notation, let G be a finite group, Γ a fixed subgroup of G of index $(G : \Gamma) = \iota$, and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ a representation of Γ on a d-dimensional vector space V_ρ over $\mathbb{C}$ whose character is denoted by $\chi_\rho : \Gamma \to \mathbb{C}$ as usual. Set $\chi_\rho(1) = d_\rho$ the dimension $\dim_{\mathbb{C}}(V_\rho)$ of V_ρ called the degree of $\rho : \Gamma \to \mathrm{GL}(V_\rho)$). Introduce further the map

$$\widetilde{\rho} : G \to \left\langle \mathrm{GL}(V_\rho) \cup \{\mathbf{0} : V_\rho \to \{0_{V_\rho}\}\} \right\rangle$$

by defining

$$\widetilde{\rho}(x) = \begin{cases} \rho(x), & x \in \Gamma; \\ \mathbf{0}, & x \in G - \Gamma, \end{cases} \quad {}^\forall x \in G.$$

Recall that Piatetski-Shapiro (1983); Serre (1972), the $\mathbb{C}[G]$-module $\mathrm{Ind}_{\Gamma}^{G}(V_\rho)$ induced from the $\mathbb{C}[\Gamma]$-module V_ρ is defined by

$$\mathrm{Ind}_{\Gamma}^{G}(V_\rho) := \{\varphi : G \to V_\rho \mid \varphi(\gamma x) = \rho(\gamma)(\varphi(x)), {}^\forall x \in G, {}^\forall \gamma \in \Gamma\},$$

which defines a representation

$$\mathsf{Ind}_{\Gamma}^{G}(\rho) : G \to \mathrm{GL}(\mathsf{Ind}_{\Gamma}^{G}(V_\rho))$$

of the finite group G on the induced $\mathbb{C}$-linear space $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ by

$$[\mathsf{Ind}_{\Gamma}^{G}(\rho)(g)](\varphi)(x) = \varphi(xg), \ {}^\forall g \in G, {}^\forall \varphi \in \mathsf{Ind}_{\Gamma}^{G} V_\rho, {}^\forall x \in G,$$

called the representation induced from $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ up to G.

Recall that $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ is an inner product space under the inner product

$$\langle \bullet \mid \bullet \rangle : \mathsf{Ind}_{\Gamma}^{G}(V_\rho) \times \mathsf{Ind}_{\Gamma}^{G}(V_\rho) \to \mathbb{C}$$

on $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ defined by

$$\langle \varphi_1 \mid \varphi_2 \rangle := \sum_{i=1}^{\iota} {}^{\mathsf{H}}[\varphi_2(g_i)]_{\mathcal{B}_{V_\rho}} [\varphi_1(g_i)]_{\mathcal{B}_{V_\rho}}, \ {}^\forall \varphi_1, \varphi_2 \in \mathsf{Ind}_{\Gamma}^{G}(V_\rho),$$

which neither depends on a choice of a complete set of representatives $\mathcal{R}_{\Gamma \backslash G} = \{g_1, \cdots, g_\iota\} \subset G$ of the coset space $\Gamma \backslash G$ nor a choice of an ordered basis $\mathcal{B}_{V_\rho} = \{v_1, \cdots, v_d\}$ of V_ρ over $\mathbb{C}$ (in particular, nor a choice of an ordered orthonormal basis $\mathcal{N}_{V_\rho} = \{e_1, \cdots, e_d\}$ of V_ρ over $\mathbb{C}$), and

$$\mathcal{N}_{\mathsf{Ind}_{\Gamma}^{G}(V_\rho)} := \{\varphi_{ij}\}_{\substack{i=1,\cdots,\iota \\ j=1,\cdots,d}}$$

is an orthonormal basis of $\mathsf{Ind}_{\Gamma}^{G}(V_\rho)$ over $\mathbb{C}$, where for $i = 1, \cdots, \iota$ and $j = 1, \cdots, d$

$$\varphi_{ij} : G \to V_\rho$$

is defined by

$$\varphi_{ij}(x) = \widetilde{\rho}(x g_i^{-1}) e_j, \ {}^\forall x \in G.$$

The $(G:\Gamma)\dim(V_\rho) = \iota d$-dimensional representation $\mathsf{Ind}_\Gamma^G(\rho)$ of G on $\mathsf{Ind}_\Gamma^G(V_\rho)$ over $\mathbb{C}$ decomposes into the direct sum

$$\mathsf{Ind}_\Gamma^G(\rho) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho))\pi_o \tag{1}$$

of non-isomorphic irreducible representations π_o of G over $\mathbb{C}$ with $m(\pi_o, \mathsf{Ind}_\Gamma^G(\rho)) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\Pi(G)$ denotes the set of all isomorphism classes $[\pi]$ of irreducible representations π of G over $\mathbb{C}$.

3. THE TRACE FORMULA FOR G WITH RESPECT TO THE SUBGROUP Γ AND $\rho : \Gamma \to \mathrm{GL}(V_\rho)$

Let $\mathsf{r} : G \to \mathrm{GL}(V_\mathsf{r})$ be a representation of the finite group G on a d-dimensional vector space V_r over $\mathbb{C}$. For any function $f : G \to \mathbb{C}$, there exists a $\mathbb{C}$-linear operator

$$\mathsf{r}(f) : V_\mathsf{r} \to V_\mathsf{r}$$

on V_r defined by

$$(\mathsf{r}(f))(v) = \sum_{g\in G} f(g)\mathsf{r}(g)(v), \ ^\forall v \in V_\mathsf{r}.$$

We can compute the trace $\mathrm{Tr}(\mathsf{r}(f))$ of the operator $\mathsf{r}(f) : V_\mathsf{r} \to V_\mathsf{r}$ on V_r.

In this note, in particular, we are interested in computing the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the linear operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$. Observe that, the operator

$$\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_\Gamma^G(V_\rho)$ given by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f))(\varphi) = \sum_{g\in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi), \ ^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho)$$

is defined explicitly by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{g\in G} f(g)[\mathsf{Ind}_\Gamma^G(\rho)(g)](\varphi)(x) = \sum_{g\in G} f(g)\varphi(xg), \ ^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ ^\forall x \in G.$$

Therefore changing the G-variables $xg \rightsquigarrow y$ and partitioning G as $G = \bigsqcup_{i=1}^{\iota} \Gamma g_i$, the equalities

$$\begin{aligned}(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) &= \sum_{y\in G} f(x^{-1}y)\varphi(y) \\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\varphi(\gamma g_i) \qquad ^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ ^\forall x \in G, \\ &= \sum_{i=1}^{\iota}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma g_i)\rho(\gamma)(\varphi(g_i))\end{aligned} \tag{2}$$

follow immediately. Define now an $\mathsf{End}_\mathbb{C}(V_\rho)$-valued function

$$K_f : G \times G \to \mathsf{End}_\mathbb{C}(V_\rho)$$

on $G \times G$ by

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\rho(\gamma), \ ^\forall x, y \in G.$$

The operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is then an "integral operator" on $\mathsf{Ind}_\Gamma^G(V_\rho)$ with "kernel" $K_f : G \times G \to \mathsf{End}_\mathbb{C}(V_\rho)$, given by

$$(\mathsf{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{i=1}^{\iota} K_f(x, g_i)(\varphi(g_i)), \qquad ^\forall \varphi \in \mathsf{Ind}_\Gamma^G(V_\rho), \ ^\forall x \in G. \tag{3}$$

The trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ is an identity that computes the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ in two different ways:

$$\text{"Spectral side"} = \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \text{"Geometric side"}.$$

More precisely, the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ states the following:

Theorem 3.1. (Trace formula for finite groups – Version 1) *For any function $f : G \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of the operator*

$$\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ satisfies the identity

$$J(\rho, f) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f)) = \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma)\frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{t \in G_\gamma \backslash G} f(t^{-1}\gamma t) = I(\rho, f), \tag{4}$$

where

- $\{\Gamma\}$ = *a set consisting of all representatives for the conjugacy classes in* Γ*;*
- $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ *for* $\gamma \in \{\Gamma\}$*;*
- $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma \in \{\Gamma\}$*.*

Here, $J(\rho, f)$ and $I(\rho, f)$ are called the spectral side and the geometric side of the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, respectively.

In the next section we sketch a proof of this theorem.

4. PROOF OF THE TRACE FORMULA FOR G WITH RESPECT TO THE SUBGROUP Γ AND $\rho : \Gamma \to \mathrm{GL}(V_\rho)$

To establish the spectral side $J(\rho, f)$ of the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, observe that the irreducible decomposition

$$\mathrm{Ind}_\Gamma^G(\rho) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o$$

of the representation $\mathrm{Ind}_\Gamma^G(\rho)$ of the finite group G on the vector space $\mathrm{Ind}_\Gamma^G(V_\rho)$ over $\mathbb{C}$ induced from $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ up to G, stated in (1) yields, for any function $f : G \to \mathbb{C}$, the decomposition

$$\mathrm{Ind}_\Gamma^G(\rho)(f) = \bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o(f)$$

of the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$. Therefore, the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ is given by

$$\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \mathrm{Tr}\left(\bigoplus_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\pi_o(f)\right) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho))\mathrm{Tr}(\pi_o(f)) = J(\rho, f),$$

which is the spectral side of the formula stated in (4).

For the geometric side $I(\rho, f)$ of the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, recall that, by (3), the operator

$$\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ has an expression of the form

$$(\mathrm{Ind}_\Gamma^G(\rho)(f)\varphi)(x) = \sum_{i=1}^{\iota} K_f(x, g_i)(\varphi(g_i)), \qquad {}^\forall\varphi \in \mathrm{Ind}_\Gamma^G(V_\rho),\ {}^\forall x \in G.$$

On the other hand, the matrix representation $[\mathrm{Ind}_\Gamma^G(\rho)(f)]_{\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)}} \in \mathbb{C}^{\iota d \times \iota d}$ of the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$ with respect to the "lexicographically" ordered orthonormal basis $\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)} = \{\varphi_{ij}\}_{\substack{i=1,\cdots,\iota \\ j=1,\cdots,d}}$ of $\mathrm{Ind}_\Gamma^G(V_\rho)$ is given by

$$[\mathrm{Ind}_\Gamma^G(\rho)(f)]_{\mathcal{N}_{\mathrm{Ind}_\Gamma^G(V_\rho)}} = \begin{bmatrix} \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{11} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{11} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{11} \right\rangle \\ \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{12} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{12} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{12} \right\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{11}) \mid \varphi_{\iota d} \right\rangle & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{12}) \mid \varphi_{\iota d} \right\rangle & \cdots & \left\langle \mathrm{Ind}_\Gamma^G(\rho)(f)(\varphi_{\iota d}) \mid \varphi_{\iota d} \right\rangle \end{bmatrix}.$$

Therefore, the trace $\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f))$ of the operator $\mathsf{Ind}_\Gamma^G(\rho)(f) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$ is given by

$$\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle,$$

which is in explicit form given by

$$\begin{aligned}
\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} [(\mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}))(g_{i_o})]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} K_f(g_{i_o}, g_i)(\varphi_{st}(g_i)) \right]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} {}^{\mathsf{H}}[\varphi_{st}(g_{i_o})]_{\mathcal{N}_{V_\rho}} \left[\sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i)\rho(\gamma)(\varphi_{st}(g_i)) \right]_{\mathcal{N}_{V_\rho}} \\
&= \sum_{(s,t)=(1,1)}^{(\iota,d)} \sum_{i_o=1}^{\iota} \sum_{j_o=1}^{d} \overline{\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle} \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle.
\end{aligned}$$

Note that, for $s, i_o = 1, \cdots, \iota$; $t, j_o = 1, \cdots, d$

$$\varphi_{st}(g_{i_o}) = \widetilde{\rho}(g_{i_o} g_s^{-1}) e_t = \begin{cases} \rho(g_{i_o} g_s^{-1}) e_t = e_t, & \Gamma g_{i_o} = \Gamma g_s \Leftrightarrow i_o = s; \\ 0_{V_\rho}, & \Gamma g_{i_o} \neq \Gamma g_s \Leftrightarrow i_o \neq s \end{cases}$$

and

$$\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle = \begin{cases} 1, & i_o = s \text{ and } j_o = t; \\ 0, & i_o = s \text{ and } j_o \neq t; \\ 0, & i_o \neq s. \end{cases}$$

Also, for $\gamma \in \Gamma$; $s, i, i_o = 1, \cdots, \iota$; $t, j_o = 1, \cdots, d$,

$$\rho(\gamma)(\varphi_{st}(g_i)) = \varphi_{st}(\gamma g_i) = \widetilde{\rho}(\gamma g_i g_s^{-1}) e_t = \begin{cases} \rho(\gamma g_i g_s^{-1}) e_t = \rho(\gamma) e_t, & \Gamma\gamma g_i = \Gamma g_i = \Gamma g_s \Leftrightarrow i = s; \\ 0_{V_\rho}, & \Gamma\gamma g_i = \Gamma g_i \neq \Gamma g_s \Leftrightarrow i \neq s \end{cases}$$

and

$$f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle = \begin{cases} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma) e_t \mid e_{j_o} \right\rangle, & \Gamma\gamma g_i = \Gamma g_i = \Gamma g_s \Leftrightarrow i = s; \\ 0, & \Gamma\gamma g_i = \Gamma g_i \neq \Gamma g_s \Leftrightarrow i \neq s. \end{cases}$$

Therefore, for a fixed $s = 1, \cdots, \iota$ and a fixed $t = 1, \cdots, d$,

$$\begin{aligned}
\left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle &= \sum_{i_o=1}^{\iota} \sum_{j_o=1}^{d} \overline{\left\langle \varphi_{st}(g_{i_o}) \mid e_{j_o} \right\rangle} \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_{i_o}^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_{j_o} \right\rangle \\
&= \sum_{i=1}^{\iota} \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_i) \left\langle \rho(\gamma)(\varphi_{st}(g_i)) \mid e_t \right\rangle \\
&= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma)(\varphi_{st}(g_s)) \mid e_t \right\rangle \\
&= \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma) e_t \mid e_t \right\rangle,
\end{aligned}$$

proving that

$$\mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) = \sum_{(s,t)=(1,1)}^{(\iota,d)} \left\langle \mathsf{Ind}_\Gamma^G(\rho)(f)(\varphi_{st}) \mid \varphi_{st} \right\rangle = \sum_{\substack{1\le s\le\iota \\ 1\le t\le d}} \sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s) \left\langle \rho(\gamma) e_t \mid e_t \right\rangle. \tag{5}$$

Introduce:

– $\{\Gamma\}$ = a set consisting of all representatives γ for the conjugacy classes C_γ^Γ in Γ;

– $\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\Gamma\}$;
– $G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\Gamma\}$.

Then the subgroup Γ of G decomposes as

$$\Gamma = \bigsqcup_{\gamma\in\{\Gamma\}} C_\gamma = \bigsqcup_{\gamma\in\{\Gamma\}} \{\delta^{-1}\gamma\delta \mid \delta \in \mathcal{R}_{\Gamma_\gamma\backslash\Gamma}\},$$

where $\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}$ denotes any fixed complete set of representatives of $\Gamma_\gamma\backslash\Gamma$. Therefore, for $s = 1, \cdots, \iota$ and $t = 1, \cdots, d$, the following identities hold

$$\begin{aligned}\sum_{\gamma\in\Gamma} f(g_s^{-1}\gamma g_s)\, \langle \rho(\gamma)e_t \mid e_t\rangle &= \sum_{\gamma\in\{\Gamma\}} \sum_{\omega\in C_\gamma} f(g_s^{-1}\omega g_s)\, \langle \rho(\omega)e_t \mid e_t\rangle \\ &= \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\, \left\langle \rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle .\end{aligned} \tag{6}$$

Now, substituting eq. (6) into eq. (5),

$$\begin{aligned}\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\substack{1\le s\le\iota\\ 1\le t\le d}} \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\, \left\langle \rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle \\ &= \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \sum_{\substack{1\le s\le\iota\\ 1\le t\le d}} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\, \left\langle \rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle \\ &= \sum_{\gamma\in\{\Gamma\}} \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \left(\sum_{1\le s\le\iota} f(g_s^{-1}\delta^{-1}\gamma\delta g_s)\right)\left(\sum_{1\le t\le d} \left\langle \rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle\right),\end{aligned}$$

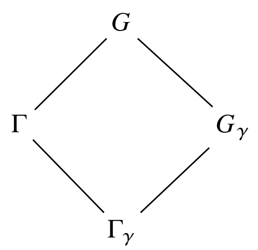
where for $\gamma \in \{\Gamma\}$ and $\delta \in \mathcal{R}_{\Gamma_\gamma\backslash\Gamma}$,

$$\sum_{1\le t\le d} \left\langle \rho(\delta^{-1}\gamma\delta)e_t \mid e_t\right\rangle = \mathrm{Tr}\rho(\delta^{-1}\gamma\delta) = \mathrm{Tr}\rho(\gamma) = \chi_\rho(\gamma).$$

Therefore,

$$\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) \sum_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \sum_{1\le s\le\iota} f(g_s^{-1}\delta^{-1}\gamma\delta g_s).$$

Observe that, for $\gamma \in \{\Gamma\}$, the group G has the following subgroups as seen in the Hasse diagram:



Each chain of subgroups

$$\Gamma_\gamma \le \Gamma \le G$$

and

$$\Gamma_\gamma \le G_\gamma \le G$$

of G produces different partitions of G into cosets modulo Γ_γ, as will be discussed in the next two observations.

Observation 4.1. For $\gamma \in \{\Gamma\}$, the group G partitions into cosets modulo Γ_γ as

$$G = \bigsqcup_{s=1}^{\iota} \bigsqcup_{\delta\in\mathcal{R}_{\Gamma_\gamma\backslash\Gamma}} \Gamma_\gamma\delta g_s,$$

where $\mathcal{R}_{\Gamma_\gamma\backslash\Gamma} \subset \Gamma$ is any fixed complete set of representatives of the coset space $\Gamma_\gamma\backslash\Gamma$.

Proof. We have already fixed a complete set of representatives $\mathcal{R}_{\Gamma\backslash G} = \{g_1, \cdots, g_\iota\} \subset G$ of the coset space $\Gamma\backslash G$. So, there is a decomposition of the group G into cosets modulo Γ as $G = \bigsqcup_{s=1}^{\iota} \Gamma g_s$.

There is also the decomposition of the subgroup Γ of G into cosets modulo Γ_γ as

$$\Gamma = \bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta.$$

Therefore the group G decomposes into cosets modulo Γ_γ as:

$$G = \Gamma g_1 \sqcup \cdots \sqcup \Gamma g_\iota = (\bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta) g_1 \sqcup \cdots \sqcup (\bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta) g_\iota = \bigsqcup_{s=1}^{\iota} \bigsqcup_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \Gamma_\gamma \delta g_s.$$

Observation 4.2. For $\gamma \in \{\Gamma\}$, there is another partitioning of the group G into cosets modulo Γ_γ given by

$$G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y t,$$

where $\mathcal{R}_{G_\gamma \backslash G} \subset G$ and $\mathcal{R}_{\Gamma_\gamma \backslash G_\gamma} \subset G_\gamma$ are any fixed complete set of representatives of the coset spaces $G_\gamma \backslash G$ and $\Gamma_\gamma \backslash G_\gamma$, respectively.

Proof. We have already fixed a complete system of representatives $\mathcal{R}_{G_\gamma \backslash G} \subset G$ of the coset space $G_\gamma \backslash G$. So, there is a decomposition of the group G into cosets modulo G_γ as $G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} G_\gamma t$.

There is also the decomposition of the subgroup G_γ of G into cosets modulo Γ_γ as

$$G_\gamma = \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y.$$

Therefore the group G decomposes into cosets modulo Γ_γ as:

$$G = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} G_\gamma t = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \left(\bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y \right) t = \bigsqcup_{t \in \mathcal{R}_{G_\gamma \backslash G}} \bigsqcup_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} \Gamma_\gamma y t.$$

So by Observation 4.1,

$$\begin{aligned} \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{\delta \in \mathcal{R}_{\Gamma_\gamma \backslash \Gamma}} \sum_{1 \le s \le \iota} f(g_s^{-1} \delta^{-1} \gamma \delta g_s) \\ &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{x \in \mathcal{R}_{\Gamma_\gamma \backslash G}} f(x^{-1} \gamma x), \end{aligned}$$

and by Observation 4.2,

$$\begin{aligned} \mathrm{Tr}(\mathsf{Ind}_\Gamma^G(\rho)(f)) &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{x \in \mathcal{R}_{\Gamma_\gamma \backslash G}} f(x^{-1} \gamma x) \\ &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \sum_{t \in \mathcal{R}_{G_\gamma \backslash G}} \sum_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} f(t^{-1} y^{-1} \gamma y t) \\ &= \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) \frac{|G_\gamma|}{|\Gamma_\gamma|} \sum_{t \in \mathcal{R}_{G_\gamma \backslash G}} f(t^{-1} \gamma t) = I(\rho, f), \end{aligned}$$

which is the geometric side of the formula stated in (4), completing the proof of Theorem 3.1.

5. TRACE FORMULA FOR G IN "ARTHUR FORM"

Finite groups are (locally) compact groups under the discrete topology. By compactness of G, there is a *unique* (left and right invariant) Haar measure $d\mu_G^{\mathrm{Haar}}$ on G, which in this case is nothing but the counting measure $d\mu_G^{\mathrm{Count}}$ on G. The same holds true for the subgroup Γ of G as well as the subgroups Γ_γ and G_γ of Γ and G, respectively, for $\gamma \in \{\Gamma\}$. So the invariant measures on the coset spaces $\Gamma_\gamma \backslash G_\gamma$ and $G_\gamma \backslash G$, which are defined by the products $d\mu_{G_\gamma}^{\mathrm{Haar}} = d\mu_{\Gamma_\gamma}^{\mathrm{Haar}} \times d\mu_{\Gamma_\gamma \backslash G_\gamma}^{\mathrm{Haar}}$ and $d\mu_G^{\mathrm{Haar}} = d\mu_{G_\gamma}^{\mathrm{Haar}} \times d\mu_{G_\gamma \backslash G}^{\mathrm{Haar}}$ for $\gamma \in \{\Gamma\}$ are all uniquely defined and are the counting measures on $\Gamma_\gamma \backslash G_\gamma$ and $G_\gamma \backslash G$, respectively.

Therefore, in this setting, for $\gamma \in \{\Gamma\}$,

$$a_\Gamma^G(\gamma) := \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) = \int_{\Gamma_\gamma \backslash G_\gamma} d\mu_{G_\gamma}^{\mathrm{Haar}} = \int_{\Gamma_\gamma \backslash G_\gamma} d\mu_{G_\gamma}^{\mathrm{Count}} = \sum_{y \in \mathcal{R}_{\Gamma_\gamma \backslash G_\gamma}} 1 = \frac{|G_\gamma|}{|\Gamma_\gamma|}.$$

Define the orbital integral $\mathrm{O}(\gamma, f)$ of $f : G \to \mathbb{C}$ over the conjugacy class C_γ^G of $\gamma \in \{\Gamma\}$ in G by

$$\mathrm{O}(\gamma, f) := \int_{G_\gamma \backslash G} f(t^{-1}\gamma t) \frac{d\mu_G^{\mathrm{Haar}}}{d\mu_{G_\gamma}^{\mathrm{Haar}}} = \int_{G_\gamma \backslash G} f(t^{-1}\gamma t) \frac{d\mu_G^{\mathrm{Count}}}{d\mu_{G_\gamma}^{\mathrm{Count}}} = \sum_{t \in \mathcal{R}_{G_\gamma \backslash G}} f(t^{-1}\gamma t).$$

So, the trace formula for G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$ stated in (4) becomes:

Theorem 5.1. (Trace formula for finite groups – Arthur form) *For any function $f : G \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of the operator*

$$\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$$

on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ satisfies the identity

$$J(\rho, f) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho)) \mathrm{Tr}(\pi_o(f)) = \mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f)) = \sum_{\gamma \in \{\Gamma\}} \chi_\rho(\gamma) a_\Gamma^G(\gamma) \mathrm{O}(\gamma, f) = I(\rho, f), \quad (7)$$

where

- *$\{\Gamma\}$ = a set consisting of all representatives for the conjugacy classes in Γ;*
- *$\Gamma_\gamma = \{\delta \in \Gamma \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\Gamma\}$;*
- *$G_\gamma = \{g \in G \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\Gamma\}$.*

Here, $J(\rho, f)$ and $I(\rho, f)$ are the spectral side and the geometric side of the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, respectively.

Theorem 5.1 is exactly the trace formula given in Arthur (Arthur 2005, eq (1.3)).

6. TEST FUNCTIONS $F : G \to \mathbb{C}$

Clearly, the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on the $\mathbb{C}$-linear space $\mathrm{Ind}_\Gamma^G(V_\rho)$ depends on the choice of $f : G \to \mathbb{C}$. Therefore the trace formula for the finite group G with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$, namely, the spectral side $J(\rho, f)$ and the geometric side $I(\rho, f)$ of the formula corresponding to the trace $\mathrm{Tr}(\mathrm{Ind}_\Gamma^G(\rho)(f))$ of $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ depend on the function $f : G \to \mathbb{C}$ as well. The function $f : G \to \mathbb{C}$ is called a "test function" of the trace formula for G (with respect to the subgroup Γ and $\rho : \Gamma \to \mathrm{GL}(V_\rho)$), and choosing $f : G \to \mathbb{C}$ carefully, the trace identities for the operator $\mathrm{Ind}_\Gamma^G(\rho)(f) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$ yield extremely deep results.

The first example is the Frobenius reciprocity law for the finite group G.

Example 6.1. (Frobenius reciprocity law for G) For any $\sigma_o \in [\sigma] \in \Pi(G)$,

$$\left\langle \mathrm{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G = \left\langle \rho, \mathrm{Res}_\Gamma^G(\sigma_o) \right\rangle_\Gamma .$$

Proof. For $\sigma_o \in [\sigma] \in \Pi(G)$, define a test function $f_{\sigma_o} : G \to \mathbb{C}$ by $f_{\sigma_o}(g) = \overline{\chi}_{\sigma_o}(g) = \overline{\chi_{\sigma_o}(g)}$ for $g \in G$. The spectral side $J(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathrm{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$ is then given by

$$\begin{aligned} J(\rho, f_{\sigma_o}) &= \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho)) \mathrm{Tr}(\pi_o(f_{\sigma_o})) \\ &= \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho)) \mathrm{Tr}(\sum_{g \in G} f_{\sigma_o}(g)\pi_o(g)) \\ &= \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho)) \sum_{g \in G} f_{\sigma_o}(g) \mathrm{Tr}(\pi_o(g)). \end{aligned}$$

Now, by the orthogonality of irreducible characters of G, for $\pi_o \in [\pi] \in \Pi(G)$,

$$\sum_{g \in G} f_{\sigma_o}(g) \mathrm{Tr}(\pi_o(g)) = \sum_{g \in G} \overline{\chi}_{\sigma_o}(g) \chi_{\pi_o}(g) = \begin{cases} |G|, & \pi_o = \sigma_o, \\ 0, & [\pi_o] \cap [\sigma_o] = \varnothing. \end{cases}$$

Therefore, the spectral side $J(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathrm{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathrm{Ind}_\Gamma^G(V_\rho) \to \mathrm{Ind}_\Gamma^G(V_\rho)$ on $\mathrm{Ind}_\Gamma^G(V_\rho)$ is

$$J(\rho, f_{\sigma_o}) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathrm{Ind}_\Gamma^G(\rho)) \sum_{g \in G} f_{\sigma_o}(g) \mathrm{Tr}(\pi_o(g)) = |G| m(\sigma_o, \mathrm{Ind}_\Gamma^G(\rho)) = |G| \left\langle \mathrm{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G ,$$

that is,

$$\frac{1}{|G|}J(\rho, f_{\sigma_o}) = \left\langle \mathsf{Ind}_\Gamma^G(\rho), \sigma_o \right\rangle_G .$$

Next, computing the "$|G|^{-1}$ multiple" of the geometric side $I(\rho, f_{\sigma_o})$ of the trace formula for the operator $\mathsf{Ind}_\Gamma^G(\rho)(f_{\sigma_o}) : \mathsf{Ind}_\Gamma^G(V_\rho) \to \mathsf{Ind}_\Gamma^G(V_\rho)$ on $\mathsf{Ind}_\Gamma^G(V_\rho)$,

$$\begin{aligned}
\frac{1}{|G|}I(\rho, f_{\sigma_o}) &= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma) a_\Gamma^G(\gamma) \mathrm{O}(\gamma, f_{\sigma_o}) \\
&= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \sum_{t\in\mathcal{R}_{G_\gamma\backslash G}} f_{\sigma_o}(t^{-1}\gamma t) \\
&= \frac{1}{|G|}\sum_{\gamma\in\{\Gamma\}} \chi_\rho(\gamma)\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathrm{vol}(G_\gamma\backslash G)\overline{\chi}_{\sigma_o}(\gamma) \\
&= \sum_{\gamma\in\{\Gamma\}} \frac{\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma)}{|\Gamma_\gamma|} \\
&= \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}} \frac{|\Gamma|}{|\Gamma_\gamma|}\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma).
\end{aligned}$$

Observe that, for $\gamma \in \{\Gamma\}$, the coset space $\Gamma_\gamma\backslash\Gamma$ is in bijective correspondence with the conjugacy class C_γ^Γ of $\gamma \in \{\Gamma\}$ in Γ under the bijection $\Gamma_\gamma\backslash\Gamma \to C_\gamma^\Gamma$ defined by $\Gamma_\gamma\delta \mapsto x^{-1}\gamma\delta$ for all $\delta \in \Gamma$. Therefore, $\frac{|\Gamma|}{|\Gamma_\gamma|} = |C_\gamma^\Gamma|$, and the following identities

$$\frac{1}{|G|}I(\rho, f_{\sigma_o}) = \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}} \frac{|\Gamma|}{|\Gamma_\gamma|}\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\{\Gamma\}} |C_\gamma^\Gamma|\chi_\rho(\gamma)\overline{\chi}_{\sigma_o}(\gamma) = \frac{1}{|\Gamma|}\sum_{\delta\in\Gamma}\chi_\rho(\delta)\overline{\chi}_{\sigma_o}(\delta) = \left\langle \rho, \mathsf{Res}_\Gamma^G(\sigma_o)\right\rangle_H$$

follow at once, completing the proof of the Frobenius reciprocity law.

For the second example, define an inner product

$$(\bullet \mid \bullet) : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$$

on the group algebra $\mathbb{C}[G]$ of G over $\mathbb{C}$ by

$$(f \mid h) := |G|\sum_{g\in G} f(g^{-1})h(g), \quad {}^\forall f, h \in \mathbb{C}[G].$$

The Fourier transform

$$\mathcal{F}_\mathsf{r} : \mathbb{C}[G] \to \mathsf{End}_\mathbb{C}(V_\mathsf{r})$$

on G coupled to a representation $\mathsf{r} : G \to \mathsf{GL}(V_\mathsf{r})$ of G on a d-dimensional vector space V_r over $\mathbb{C}$ is defined by

$$\mathcal{F}_\mathsf{r} : f \mapsto \mathcal{F}_\mathsf{r} f =: \widehat{f}(\mathsf{r}) \overset{def}{=} \sum_{g\in G} f(g)\mathsf{r}(g) = \mathsf{r}(f), \quad {}^\forall f \in \mathbb{C}[G].$$

It is well known and easy to derive that

$$\widehat{f * h}(\mathsf{r}) = \widehat{f}(\mathsf{r}) \circ \widehat{h}(\mathsf{r}), \quad {}^\forall f, h \in \mathbb{C}[G]. \tag{8}$$

Here, for $f, h \in \mathbb{C}[G]$, their convolution product $f * h \in \mathbb{C}[G]$ is defined by $(f * h)(x) = \sum_{g\in G} f(xg^{-1})h(g)$ for all $x \in G$. So $(f * h)(1) = \sum_{g\in G} f(g^{-1})h(g)$.

Now, having set the stage, we can now state and prove the Plancherel formula for G in the following example.

Example 6.2. (Plancherel formula for G) For any two functions $f, h : G \to \mathbb{C}$,

$$(f \mid h) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)).$$

Proof. For any two functions $f, h : G \to \mathbb{C}$ and for any $\pi_o \in [\pi] \in \Pi(G)$, by (8) the identity $\widehat{f * h}(\pi_o) = \widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)$ holds true. Therefore, the right-hand side of the Plancherel formula becomes

$$\sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f}(\pi_o) \circ \widehat{h}(\pi_o)) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\widehat{f * h}(\pi_o)) = \sum_{\pi_o\in[\pi]\in\Pi(G)} \mathrm{d}_{\pi_o}\mathrm{Tr}(\pi_o(f * h)).$$

Now, we consider the standard case of the trace formula for the finite group G; that is, the trace formula for G with respect to the subgroup $\Gamma = \langle 1 \rangle$ and $\rho = \not\Vdash_\Gamma : \Gamma \to \mathrm{GL}(\mathbb{C})$. Then, by (7), the spectral side $J(\not\Vdash_{\langle 1 \rangle}, f * h)$ of the trace formula for G with respect to the subgroup $\langle 1 \rangle$ and $\not\Vdash_{\langle 1 \rangle} : \langle 1 \rangle \to \mathrm{GL}(\mathbb{C})$ reads as

$$J(\not\Vdash_{\langle 1 \rangle}, f * h) = \sum_{\pi_o \in [\pi] \in \Pi(G)} m(\pi_o, \mathsf{Ind}_{\langle 1 \rangle}^G(\not\Vdash_{\langle 1 \rangle}))\mathrm{Tr}(\pi_o(f * h)) = \sum_{\pi_o \in [\pi] \in \Pi(G)} \mathrm{d}_{\pi_o} \mathrm{Tr}(\pi_o(f * h)),$$

and the geometric side of $I(\not\Vdash_{\langle 1 \rangle}, f * h)$ of the trace formula for G with respect to the subgroup $\langle 1 \rangle$ and $\not\Vdash_{\langle 1 \rangle} : \langle 1 \rangle \to \mathrm{GL}(\mathbb{C})$ reduces to

$$I(\not\Vdash_{\langle 1 \rangle}, f * h) = \sum_{\gamma \in \{\langle 1 \rangle\}} \chi_{\not\Vdash_{\langle 1 \rangle}}(\gamma) a_{\langle 1 \rangle}^G(\gamma) \mathrm{O}(\gamma, f * h) = |G|(f * h)(1) = |G| \sum_{g \in G} f(g^{-1})h(g) = (f \mid h),$$

completing the proof of the Plancherel formula for G.

Our third example needs some preliminaries. Hence deserves discussion in a new section.

7. THE "AUTOMORPHIC SIDE" OF THE MACDONALD CORRESPONDENCE FOR $\mathrm{GL}_N(\mathbb{F}_Q)$

Macdonald proved Macdonald (1980) an analogue of the local Langlands correspondence for $\mathsf{GL}_n(F)$ where F is a local field with finite residue class field $\kappa_F = \mathbb{F}_q$ for the finite group $\mathsf{GL}_n(\mathbb{F}_q)$; namely, following the reformulation of Vogan Vogan (2020), there exists a bijective correspondence

$$\mathcal{M}_n(\mathbb{F}_q) : \mathcal{X}_n(\mathsf{WM}_{\mathbb{F}_q}) \rightleftarrows \Pi(\mathsf{GL}_n(\mathbb{F}_q)) \tag{9}$$

between the set $\mathcal{X}_n(\mathsf{WM}_{\mathbb{F}_q})$ of "isomorphism classes of complex n-dimensional admissible, that is Frob_q equivariant and semisimple representations of the absolute Weil-Macdonald group $\mathsf{WM}_{\mathbb{F}_q}$ of $\mathbb{F}_q$" and the set $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$ of "isomorphism classes of irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$" (but we can also assume over $\overline{\mathbb{Q}}_\ell, \Omega_\ell, ...$) where $q = p^f$ with p a prime number and $0 < f \in \mathbb{Z}$. Here, $\mathsf{WM}_{\mathbb{F}_q}$ is defined by $\mathsf{WM}_{\mathbb{F}_q} = (\varprojlim_m \mathbb{F}_{q^m}^\times) \ltimes \mathbb{C}^+$ where the inverse limit $\varprojlim_m \mathbb{F}_{q^m}^\times$ is with respect to the connecting maps given by the norm maps $\mathbb{F}_{q^m}^\times \xleftarrow{\mathrm{N}_{md/m}} \mathbb{F}_{q^{md}}^\times$ for all $0 < m, d \in \mathbb{Z}$. The action of Frob_q on $\mathsf{WM}_{\mathbb{F}_q}$ is defined by the q-th power map and the multiplication by q on the components of $\mathsf{WM}_{\mathbb{F}_q}$, and the action of $\varprojlim_m \mathbb{F}_{q^m}^\times$ on the additive group $\mathbb{C}^+$ is defined via the canonical isomorphism $\varprojlim_m \mathbb{F}_{q^m}^\times \approx I_F/P_F$ where F is a local field with $\kappa_F = \mathbb{F}_q$, and I_F, P_F are the inertia and the wild inertia subgroups of the Weil group W_F of F together with the homomorphism $|.|_F \circ \mathsf{Art}_F : W_F \xrightarrow{\mathsf{Art}_F} F^\times \xrightarrow{|.|_F} \mathbb{R}_{>0}^\times$, where $\mathsf{Art}_F : W_F \to F^\times$ is the local Artin reciprocity law of F. This correspondence satisfies the "naturality" properties; that is, matching of corresponding local ϵ-factors and corresponding local ζ- and L-factors, and corresponding conductors. The bijection (9) is called the Macdonald correspondence for $\mathrm{GL}_n(\mathbb{F}_q)$, which needs further discussion and postponed to a future study

In this note we are interested in $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$, namely the "automorphic side" of the Macdonald correspondence (9) for $\mathrm{GL}_n(\mathbb{F}_q)$, and the main references that we follow closely are Carter (1993); Green (1955) and Macdonald (1980); Ye and Zelingher (2021).

7.1. Parabolic induction and cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$

To describe the "automorphic side" of the Macdonald correspondence (9) for $\mathrm{GL}_n(\mathbb{F}_q)$ precisely, let us fix :

- $\mathcal{J} = \{j_1, \cdots, j_s\}$ an ordered partition of n; that is, $0 < j_1, \cdots, j_s \in \mathbb{Z}$ such that the ordered sum $j_1 + \cdots + j_s = n$.
- $\mathsf{P}_\mathcal{J}$ the standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the partition $\mathcal{J}$ defined by

$$\mathsf{P}_\mathcal{J} := \begin{bmatrix} \mathsf{GL}_{j_1}(\mathbb{F}_q) & \times & \times & \times & \times \\ & \mathsf{GL}_{j_2}(\mathbb{F}_q) & \times & \times & \times \\ & & \ddots & \times & \times \\ & 0 & & \ddots & \times \\ & & & & \mathsf{GL}_{j_s}(\mathbb{F}_q) \end{bmatrix}_{n \times n};$$

– $\mathsf{M}_{\mathcal{J}}$ the Levi factor of $\mathsf{P}_{\mathcal{J}}$ by

$$\mathsf{M}_{\mathcal{J}} := \begin{bmatrix} \mathsf{GL}_{j_1}(\mathbb{F}_q) & & & & \\ & \mathsf{GL}_{j_2}(\mathbb{F}_q) & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \mathsf{GL}_{j_s}(\mathbb{F}_q) \end{bmatrix}_{n\times n};$$

– $\mathsf{N}_{\mathcal{J}}$ the unipotent radical of $\mathsf{P}_{\mathcal{J}}$ by

$$\mathsf{N}_{\mathcal{J}} := \begin{bmatrix} 1_{j_1} & \times & \times & \times & \times \\ & 1_{j_2} & \times & \times & \times \\ & & \ddots & \times & \times \\ & 0 & & \ddots & \times \\ & & & & 1_{j_s} \end{bmatrix}_{n\times n}.$$

Note that, $\mathsf{M}_{\mathcal{J}}$ acts on $\mathsf{N}_{\mathcal{J}}$ by conjugation, and

$$\mathsf{P}_{\mathcal{J}} = \mathsf{M}_{\mathcal{J}} \ltimes \mathsf{N}_{\mathcal{J}}.$$

For $1 \le i \le s$, let $\pi_i : \mathsf{GL}_{j_i}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_i})$ be a representation of $\mathsf{GL}_{j_i}(\mathbb{F}_q)$ on a d_i-dimensional vector space V_{π_i} over $\mathbb{C}$. Then, the natural isomorphism $\mathsf{M}_{\mathcal{J}} \xrightarrow{\sim} \prod_{1\le i\le s} \mathsf{GL}_{j_i}(\mathbb{F}_q)$ yields a representation $\mathsf{M}_{\mathcal{J}} \xrightarrow{\sim} \prod_{1\le i\le s} \mathsf{GL}_{j_i}(\mathbb{F}_q) \xrightarrow{\otimes_{1\le i\le s}\pi_i} \mathsf{GL}(\bigotimes_{1\le i\le s} V_{\pi_i})$ of the Levi factor $\mathsf{M}_{\mathcal{J}}$ of $\mathsf{P}_{\mathcal{J}}$ on the $\mathbb{C}$-linear space $\bigotimes_{1\le i\le s} V_{\pi_i}$, which in return defines via inflation to $\mathsf{P}_{\mathcal{J}}$ a representation $\widetilde{\otimes}_{1\le i\le s}\pi_i : \mathsf{P}_{\mathcal{J}} \longrightarrow \mathsf{GL}(\bigotimes_{1\le i\le s} V_{\pi_i})$ of $\mathsf{P}_{\mathcal{J}}$ on $\bigotimes_{1\le i\le s} V_{\pi_i}$ by letting the matrices in $\mathsf{N}_{\mathcal{J}}$ act as the identity on $\bigotimes_{1\le i\le s} V_{\pi_i}$. Consider the representation

$$\pi_1 \circ \cdots \circ \pi_s := \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\widetilde{\otimes}_{1\le i\le s}\pi_i) : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\bigotimes_{1\le i\le s} V_{\pi_i}))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\bigotimes_{1\le i\le s} V_{\pi_i})$ constructed by this parabolic induction process. For $1 < n \in \mathbb{Z}$, an irreducible representation $\pi : \mathsf{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathsf{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$ is called cuspidal, if it does not occur in any representation of $\mathsf{GL}_n(\mathbb{F}_q)$ of the form $\pi_1 \circ \pi_2$, where $\pi_1 : \mathsf{GL}_{j_1}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_1})$ and $\pi_2 : \mathsf{GL}_{j_2}(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_{\pi_2})$ are both irreducible and $0 < j_1, j_2 \in \mathbb{Z}$ such that $j_1 + j_2 = n$. For $n = 1$, all irreducible representations of $\mathsf{GL}_1(\mathbb{F}_q) = \mathbb{F}_q^\times$ over $\mathbb{C}$ are by definition cuspidal. The "Philosophy of Cusp Forms" of Harish-Chandra (for details Bump (2013)) states that the cuspidal representations of $\mathsf{GL}_n(\mathbb{F}_q)$ are the basic building blocks of all irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ in the sense that for $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$, there exists an ordered partition $\mathcal{J} = \{j_1, \cdots, j_s\}$ of n and cuspidal representations $\pi_1, \cdots, \pi_s$ of $\mathsf{GL}_{j_1}(\mathbb{F}_q), \cdots, \mathsf{GL}_{j_s}(\mathbb{F}_q)$, respecively, such that π_o is an irreducible constituent of $\pi_1 \circ \cdots \circ \pi_s$. This completes the description of the set $\Pi(\mathsf{GL}_n(\mathbb{F}_q))$ of isomorphism classes of irreducible representations of $\mathsf{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$ via the "Philosophy of Cusp Forms".

7.2. ζ-functions (integrals) of $\mathsf{GL}_n(\mathbb{F}_q)$

For a representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathsf{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$, denoting the set of all $n \times n$ matrices over $\mathbb{F}_q$ by $\mathsf{M}_n(\mathbb{F}_q)$, Macdonald attached a function

$$\mathsf{Z}(\bullet, \pi) : \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)] \longrightarrow \mathsf{End}_{\mathbb{C}}(V_\pi)$$

defined by

$$\mathsf{Z}(\Phi, \pi) := \pi(\Phi) = \sum_{g\in\mathsf{GL}_n(\mathbb{F}_q)} \Phi(g)\pi(g), \quad {}^\forall\Phi \in \mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)],$$

called the ζ-function of the representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \longrightarrow \mathsf{GL}(V_\pi)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on V_π over $\mathbb{C}$ (Macdonald 1980, Section 2).

Now, the trace formula for $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the parabolic subgroup $\mathsf{P}_{\mathcal{J}}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ corresponding to an ordered partition $\mathcal{J}$ of n and $\rho : \mathsf{P}_{\mathcal{J}} \longrightarrow \mathrm{GL}(V_\rho)$ computes the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)))$ of the operator $\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)) : \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho) \longrightarrow \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$. More precisely, we have the following theorem, which is essentially a reformulation of Theorem 5.1 in this setting.

Theorem 7.1. *For any function* $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$*, the trace* $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)))$ *of the operator*

$$\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho)) : \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho) \to \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$$

on the $\mathbb{C}$*-linear space* $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ *satisfies the identity*

$$\sum_{\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))\mathrm{Tr}(\mathsf{Z}(\Phi, \pi_o)) = \mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))) = \sum_{\gamma \in \{\mathsf{P}_{\mathcal{J}}\}} \chi_\rho(\gamma) a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) \mathsf{O}(\gamma, \Phi),$$

where

- $\{\mathsf{P}_{\mathcal{J}}\}$ *= a set consisting of all representatives for the conjugacy classes in* $\mathsf{P}_{\mathcal{J}}$*;*
- $\mathsf{P}_{\mathcal{J}_\gamma} = \{\delta \in \mathsf{P}_{\mathcal{J}} \mid \delta^{-1}\gamma\delta = \gamma\}$ *for* $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$*;*
- $\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g = \gamma\}$ *for* $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$*.*

Furthermore, for $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$*,*

$$a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{P}_{\mathcal{J}_\gamma}|}$$

and the orbital integral $\mathsf{O}(\gamma, \Phi)$ *of* $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ *over the conjugacy class* $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ *of* γ *in* $\mathrm{GL}_n(\mathbb{F}_q)$ *is given by*

$$\mathsf{O}(\gamma, \Phi) = \sum_{t \in \mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma \backslash \mathrm{GL}_n(\mathbb{F}_q)}} \Phi(t^{-1}\gamma t).$$

There is an important special case of Theorem 7.1. Let $\mathcal{J}_1 = \{1, \cdots, 1\}$ be the ordered partition of n given by $\overbrace{1 + \cdots + 1}^{n-\text{copies}} = n$. Then $\mathsf{N}_{\mathcal{J}_1}$ becomes the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of the upper traingular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. Now, define a 1-dimensional representation

$$\theta_\psi : \mathsf{N}_{\mathcal{J}_1} \to \mathbb{C}^\times$$

of $\mathsf{N}_{\mathcal{J}_1}$ over $\mathbb{C}$ by

$$\theta_\psi : [x_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}} \mapsto \psi(x_{12} + x_{23} + \cdots + x_{(n-1)n}), \quad {}^\forall [x_{ij}]_{\substack{1 \le i \le n \\ 1 \le j \le n}} \in \mathsf{N}_{\mathcal{J}_1}, \tag{10}$$

where $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ is a non-trivial additive character of $\mathbb{F}_q$. Set $\mathbb{C} = V_{\theta_\psi}$. The representation

$$\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi) : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ is multiplicity free; that is, it has multiplicity one property, which states:

- If $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ then $m(\pi_o, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) \le 1$.

Now, $\pi_o \in [\pi] \in \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ is said to have a Whittaker model if there exists a non-trivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ of $\mathbb{F}_q$ such that $m(\pi_o, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) = 1$. Moreover,

- If π_o is a cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, then π_o has a Whittaker model.

The trace formula for $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the subgroup $\mathsf{N}_{\mathcal{J}_1}$ of upper triangular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$ and $\theta_\psi : \mathsf{N}_{\mathcal{J}_1} \to \mathbb{C}^\times$ given by (10) computes the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)))$ of the operator $\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) : \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}) \to \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$, which has a simpler form thanks to the multiplicity one property of the representation $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi) : \mathsf{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}))$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$, which follows as a corollary of Theorem 7.1.

Corollary 7.2. For any function $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$, the trace $\mathrm{Tr}(\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)))$ of the operator

$$\mathsf{Z}(\Phi, \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi)) : \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi}) \to \mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$$

on the $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_{\theta_\psi})$ satisfies the identity

$$\underbrace{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o,\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))\mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))}_{\displaystyle\sum_{\substack{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))\\ m(\pi_o,\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))=1}}\mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))} = \mathrm{Tr}(\mathsf{Z}(\Phi,\mathsf{Ind}_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\theta_\psi))) = \underbrace{\sum_{\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}}\chi_{\theta_\psi}(\gamma)a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)}_{\displaystyle\sum_{\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}}\theta_\psi(\gamma)a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)},$$

where

- $\{\mathsf{N}_{\mathcal{J}_1}\}$ = a set consisting of all representatives for the conjugacy classes in $\mathsf{N}_{\mathcal{J}_1}$;
- $\mathsf{N}_{\mathcal{J}_1\gamma} = \{\delta\in\mathsf{N}_{\mathcal{J}_1} \mid \delta^{-1}\gamma\delta=\gamma\}$ for $\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}$;
- $\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g\in\mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g=\gamma\}$ for $\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}$.

Furthermore, for $\gamma\in\{\mathsf{N}_{\mathcal{J}_1}\}$,

$$a_{\mathsf{N}_{\mathcal{J}_1}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{N}_{\mathcal{J}_1\gamma}|}$$

and the orbital integral $\mathrm{O}(\gamma,\Phi)$ of $\Phi:\mathsf{M}_n(\mathbb{F}_q)\to\mathbb{C}$ over the conjugacy class $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ of γ in $\mathrm{GL}_n(\mathbb{F}_q)$ is given by

$$\mathrm{O}(\gamma,\Phi) = \sum_{t\in\mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma\backslash\mathrm{GL}_n(\mathbb{F}_q)}}\Phi(t^{-1}\gamma t).$$

7.3. ϵ-factors of representations of $\mathrm{GL}_n(\mathbb{F}_q)$

Let $\psi:\mathbb{F}_q^+\to\mathbb{C}^\times$ be a non-trivial additive character of $\mathbb{F}_q$. For $x\in\mathsf{M}_n(\mathbb{F}_q)$, let ${}_x\mu:\mathsf{M}_n(\mathbb{F}_q)\to\mathsf{M}_n(\mathbb{F}_q)$ be the additive homomorphism defined by ${}_x\mu: y\mapsto xy$ for all $y\in\mathsf{M}_n(\mathbb{F}_q)$. For $x\in\mathsf{M}_n(\mathbb{F}_q)$, we consider the Fourier transform

$$\mathcal{F}_{\mathsf{r}_x}:\mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)]\to\mathsf{End}_{\mathbb{C}}(V_{\mathsf{r}_x})$$

on $\mathsf{M}_n(\mathbb{F}_q)$ coupled to the representation $\mathsf{r}_x:\mathsf{M}_n(\mathbb{F}_q)\xrightarrow{{}_x\mu}\mathsf{M}_n(\mathbb{F}_q)\xrightarrow{\mathrm{Tr}}\mathbb{F}_q\xrightarrow{\psi}\mathbb{C}^\times$ of the additive group $\mathsf{M}_n(\mathbb{F}_q)$ on the 1-dimensional vector space $V_{\mathsf{r}_x}=\mathbb{C}$ over $\mathbb{C}$ defined by

$$\mathcal{F}_{\mathsf{r}_x}:\Phi\mapsto\mathcal{F}_{\mathsf{r}_x}\Phi=:\widehat{\Phi}(\mathsf{r}_x)=:\widehat{\Phi}(x)\overset{\text{def}}{=}|\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}}\sum_{g\in\mathsf{M}_n(\mathbb{F}_q)}\Phi(g)\mathsf{r}_x(g)=\mathsf{r}_x(\Phi),\quad {}^\forall\Phi\in\mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)].$$

Let $\pi:\mathrm{GL}_n(\mathbb{F}_q)\to\mathsf{GL}(V_\pi)$ be a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on a d-dimensional vector space V_π over $\mathbb{C}$. For each $\Phi\in\mathbb{C}[\mathsf{M}_n(\mathbb{F}_q)]$, by Macdonald (Macdonald 1980, eq. (2.3)),

$$\mathsf{Z}(\Phi,\pi)=\sum_{g\in\mathsf{GL}_n(\mathbb{F}_q)}\Phi(g)\pi(g)=\sum_{x\in\mathsf{M}_n(\mathbb{F}_q)}\widehat{\Phi}(-x)W(\pi,\psi;x),$$

where

$$W(\pi,\psi;x)=|\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}}\sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)}\psi(\mathsf{Tr}(hx))\pi(h),\quad {}^\forall x\in\mathsf{M}_n(\mathbb{F}_q). \tag{11}$$

Now choosing $x=1$ for $x\in\mathsf{M}_n(\mathbb{F}_q)$, Macdonald proved (Macdonald 1980, eq. (2.4)) that

$$W(\pi,\psi;1)\pi(g)=\pi(g)W(\pi,\psi;1),\quad {}^\forall g\in\mathsf{GL}_n(\mathbb{F}_q).$$

Therefore, there exists a constant $(\pi,\psi)\in\mathbb{C}$ such that

$$W(\pi,\psi;1)=(\pi,\psi)\pi(1).$$

The epsilon factor $\epsilon(\pi,\psi)$ of the representation $\pi:\mathrm{GL}_n(\mathbb{F}_q)\to\mathsf{GL}(V_\pi)$ of $\mathrm{GL}_n(\mathbb{F}_q)$ on the d-dimensional vector space V_π over $\mathbb{C}$ with respect to the choice of a non-trivial additive character $\psi:\mathbb{F}_q^+\to\mathbb{C}^\times$ is defined by

$$\epsilon(\pi,\psi):=(\check{\pi},\psi),$$

where $\check{\pi}:\mathsf{GL}_n(\mathbb{F}_q)\to\mathsf{GL}(V_{\check{\pi}})$ denotes the contragradient of $\pi:\mathsf{GL}_n(\mathbb{F}_q)\to\mathsf{GL}(V_\pi)$; that is, the representation of $\mathsf{GL}_n(\mathbb{F}_q)$ on the dual space $V_{\check{\pi}}$ of V_π defined by $\check{\pi}(g)={}^t\pi(g^{-1})$.

Thus, it follows from (11) that

$$W(\check{\pi},\psi;1)=|\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}}\sum_{h\in\mathsf{GL}_n(\mathbb{F}_q)}\psi(\mathsf{Tr}(h))\check{\pi}(h)=\epsilon(\pi,\psi)\check{\pi}(1).$$

Therefore,

$$\mathsf{Tr}(W(\check{\pi},\psi;1)) = |\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}} \sum_{h\in \mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathsf{Tr}(h))\mathsf{Tr}(\check{\pi}(h)) = \epsilon(\pi,\psi)\mathsf{Tr}(\check{\pi}(1)),$$

proving that

$$\epsilon(\pi,\psi) = \frac{|\mathsf{M}_n(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi)} \sum_{h\in \mathsf{GL}_n(\mathbb{F}_q)} \psi(\mathsf{Tr}(h))\chi_{\check{\pi}}(h), \tag{12}$$

as $\mathsf{Tr}(\check{\pi}(1)) = \dim(\pi)$.

Moreover, Ye and Zelingher studied the effect of the linear algebraic operations $\boxplus$ and $\boxtimes$ on $\Pi(\mathsf{GL}_*(\mathbb{F}_q)) = \bigsqcup_{0\le n\in\mathbb{Z}} \Pi(\mathsf{GL}_n(\mathbb{F}_q))$ to ϵ-factors in Ye and Zelingher (2021). More precisely, for $\pi_1 \in \Pi(\mathsf{GL}_{n_1}(\mathbb{F}_q))$ and $\pi_2 \in \Pi(\mathsf{GL}_{n_2}(\mathbb{F}_q))$, there exist $\pi_1 \boxplus \pi_2 \in \Pi(\mathsf{GL}_{n_1+n_2}(\mathbb{F}_q))$ and $\pi_1 \boxtimes \pi_2 \in \Pi(\mathsf{GL}_{n_1 n_2}(\mathbb{F}_q))$, whose ϵ-factor, and ζ- and L-factors are known instead of $\pi_1 \boxplus \pi_2$ and $\pi_1 \boxtimes \pi_2$ themselves, and it is proved Ye and Zelingher (2021) by Ye and Zelingher that

$$\epsilon(\pi_1 \boxplus \pi_2,\psi) = \epsilon(\pi_1,\psi)\epsilon(\pi_2,\psi). \tag{13}$$

Now, combining (13) with the identity (12), for $\pi_1 \in \Pi(\mathsf{GL}_{n_1}(\mathbb{F}_q))$ and $\pi_2 \in \Pi(\mathsf{GL}_{n_2}(\mathbb{F}_q))$, we have

$$\begin{aligned}\epsilon(\pi_1 \boxplus \pi_2,\psi) &= \frac{|\mathsf{M}_{n_1}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_1)} \sum_{h'\in \mathsf{GL}_{n_1}(\mathbb{F}_q)} \psi(\mathsf{Tr}(h'))\chi_{\check{\pi}_1}(h') \frac{|\mathsf{M}_{n_2}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_2)} \sum_{h''\in \mathsf{GL}_{n_2}(\mathbb{F}_q)} \psi(\mathsf{Tr}(h''))\chi_{\check{\pi}_2}(h'') \\ &= \frac{|\mathsf{M}_{n_1}(\mathbb{F}_q)\times \mathsf{M}_{n_1}(\mathbb{F}_q)|^{-\frac{1}{2}}}{\dim(\pi_1)\dim(\pi_2)} \sum_{\substack{h'\in \mathsf{GL}_{n_1}(\mathbb{F}_q)\\ h''\in \mathsf{GL}_{n_2}(\mathbb{F}_q)}} \psi\left(\mathsf{Tr}\begin{bmatrix} h' & 0\\ 0 & h''\end{bmatrix}\right)\chi_{\check{\pi}_1\oplus\check{\pi}_2}\left(\begin{bmatrix} h' & 0\\ 0 & h''\end{bmatrix}\right).\end{aligned} \tag{14}$$

In case the representation $\pi : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(V_\pi)$ has no 1-component; that is, π is not a constituent of $\rho_{n-1} \circ (1)$, where ρ_{n-1} is the regular representation of $\mathsf{GL}_{n-1}(\mathbb{F}_q)$ over $\mathbb{C}$ and (1) is the trivial representation $\mathsf{GL}_1(\mathbb{F}_q) \to \mathbb{C}^\times$, Macdonald further proved (Macdonald 1980, Proposition 2.7) that

$${}^t\mathsf{Z}(\widehat{\Phi},\check{\pi}) = \epsilon(\pi,\psi)\mathsf{Z}(\Phi,\pi), \tag{15}$$

where ${}^t\mathsf{Z}(\widehat{\Phi},\check{\pi})$ is the transpose of $\mathsf{Z}(\widehat{\Phi},\check{\pi})$. Applying Theorem 5.1, the trace formula for finite groups in Arthur form, to the identity (15), the following identities follow immediately.

Theorem 7.3. *Let $\rho : \mathsf{P}_{\mathcal{J}} \to \mathrm{GL}(V_\rho)$ be a representation of the standard parabolic subgroup $\mathsf{P}_{\mathcal{J}}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ with respect to the partition $\mathcal{J}$ on the vector space V_ρ over $\mathbb{C}$. Assume that the representation*

$$\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho) : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho))$$

of $\mathrm{GL}_n(\mathbb{F}_q)$ on the induced $\mathbb{C}$-linear space $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho)$ has no 1-component. The epsilon factor $\epsilon(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho),\psi)$ of the representation $\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho) : \mathrm{GL}_n(\mathbb{F}_q) \to \mathsf{GL}(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(V_\rho))$ with respect to the choice of a non-trivial additive character $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ has then the following description:

$$\frac{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\check{\rho}))\,\overbrace{\mathrm{Tr}({}^t\mathsf{Z}(\widehat{\Phi},\pi_o))}^{\mathrm{Tr}(\mathsf{Z}(\widehat{\Phi},\pi_o))}}{\sum_{\pi_o\in[\pi]\in\Pi(\mathsf{GL}_n(\mathbb{F}_q))} m(\pi_o, \mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho))\mathrm{Tr}(\mathsf{Z}(\Phi,\pi_o))} = \epsilon(\mathsf{Ind}_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\rho),\psi) = \frac{\sum_{\gamma\in\{\mathsf{P}_{\mathcal{J}}\}} \chi_{\check{\rho}}(\gamma)a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\widehat{\Phi})}{\sum_{\gamma\in\{\mathsf{P}_{\mathcal{J}}\}} \chi_{\rho}(\gamma)a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma)\mathrm{O}(\gamma,\Phi)},$$

where

- *$\{\mathsf{P}_{\mathcal{J}}\}$ = a set consisting of all representatives for the conjugacy classes in $\mathsf{P}_{\mathcal{J}}$;*
- *$\mathsf{P}_{\mathcal{J}_\gamma} = \{\delta \in \mathsf{P}_{\mathcal{J}} \mid \delta^{-1}\gamma\delta = \gamma\}$ for $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$;*
- *$\mathrm{GL}_n(\mathbb{F}_q)_\gamma = \{g \in \mathrm{GL}_n(\mathbb{F}_q) \mid g^{-1}\gamma g = \gamma\}$ for $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$.*

Furthermore, for $\gamma \in \{\mathsf{P}_{\mathcal{J}}\}$,

$$a_{\mathsf{P}_{\mathcal{J}}}^{\mathrm{GL}_n(\mathbb{F}_q)}(\gamma) = \frac{|\mathrm{GL}_n(\mathbb{F}_q)_\gamma|}{|\mathsf{P}_{\mathcal{J}_\gamma}|},$$

and the orbital integral $\mathrm{O}(\gamma,\Phi)$ of $\Phi : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ over the conjugacy class $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ of γ in $\mathrm{GL}_n(\mathbb{F}_q)$ is given by

$$\mathrm{O}(\gamma,\Phi) = \sum_{t\in\mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma\backslash\mathrm{GL}_n(\mathbb{F}_q)}} \Phi(t^{-1}\gamma t)$$

and the orbital integral $\mathrm{O}(\gamma, \widehat{\Phi})$ *of* $\widehat{\Phi} : \mathsf{M}_n(\mathbb{F}_q) \to \mathbb{C}$ *over the conjugacy class* $C_\gamma^{\mathrm{GL}_n(\mathbb{F}_q)}$ *of* γ *in* $\mathrm{GL}_n(\mathbb{F}_q)$ *is given by*

$$\mathrm{O}(\gamma, \widehat{\Phi}) = \sum_{t \in \mathcal{R}_{\mathrm{GL}_n(\mathbb{F}_q)_\gamma \backslash \mathrm{GL}_n(\mathbb{F}_q)}} \widehat{\Phi}(t^{-1}\gamma t)$$

The identities given by Theorem 7.3 do not seem to appear in the literature and may have applications in the ϵ-factor analysis of representations of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\mathbb{C}$.

ACKNOWLEDGEMENTS

A part of this work was presented at the Selçuk University Mathematics Colloquium in April 27, 2022 and completed while visiting the Abdus Salam ICTP in June 2024. The author thanks the Abdus Salam ICTP for hospitality and scientific exchange during the School and the Workshop on Number Theory and Physics in June 2024. The author also would like to thank the anonymous reviewers for their helpful comments and feedback, which greatly strengthened the overall manuscript.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Author declared no conflict of interest.
Financial Disclosure: Author declared no financial support.

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REFERENCES

Arthur, J., 2005, An introduction to the trace formula. Clay Math. Proc. 4, 1-263.
Bump, D., 2013, Lie Groups, 2nd edition. Graduate Texts in Mathematics, Springer-Verlag, New York.
Carter, R. W., 1993, Finite Groups of Lie Type: Conjugacy Classes and Complex Characters, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester.
Chasek, M. D., 2023, A Vector-valued Trace Formula for Finite Groups. M.A. Thesis, The University of Maine, US.
Green, J. A., 1955, The characters of the finite general linear groups, Trans. Amer. Math. Soc. 80, 402-447.
Lee, S., 2022, Trace formula for finite groups, https://seewoo5.github.io/jekyll/update/2022/11/18/finite-group-trace-formula.html.
Macdonald, I. G., 1980, Zeta functions attached to finite general linear group. Math. Ann. 249, 1-15.
Piatetski-Shapiro, I., 1983, Complex Representations of $\mathrm{GL}(2, k)$ for Finite Fields K. Contemp. Math. 16.
Serre, J. P., 1972, Linear Representations of Finite Groups. Graduate Texts in Mathematics 42, Springer-Verlag, New York and Heidelberg.
Silberger A. and Zink, E. W., 2008, Explicit Shintani base change and the Macdonald correspondence for characters of $\mathsf{GL}_n(k)$. Journal of Algebra 319, 4147-4176.
Terras, A., 1999, Fourier Analysis on Finite Groups and Applications. London Mathematical Society Student Texts 43, Cambridge University Press, Cambridge.
Vogan, D., 2020, Local Langlands conjecture for finite groups of Lie type. Joint Mathematics Meetings AMS-MAA, Denver, 15-18.
Yang J. H, 2006, Trace formulas on finite groups. Commun. Korean Math. Soc. 21, 17–25.
Ye, R. and Zelingher, E., 2021, Epsilon factors of representations of finite general linear groups. Journal of Number Theory 221, 122-142.