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RESEARCH ARTICLE

Backward Shift Operators on Bergman-Besov Spaces as Bergman Projections

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ABSTRACT

We express backward shift operators on all Bergman-Besov spaces in terms of Bergman projections in one and several variables including the Banach function spaces and the special Hilbert spaces such as Drury-Arveson and Dirichlet spaces. These operators are adjoints of the shift operators and their definitions for the case $p = 1$ and proper Besov spaces require the use of nontrivial imbeddings of the spaces into Lebesgue classes. Our results indicate that the backward shifts are compositions of imbeddings into Lebesgue classes followed by multiplication operators by the conjugates of the coordinate variables followed by Bergman projections on appropriate spaces. We apply our results to the wandering subspace property of invariant subspaces of the shift operators on certain of our Hilbert spaces.

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1. INTRODUCTION

Shift operators and their adjoints the backward shift operators have a central position in operator theory. Forward shift operators on holomorphic function spaces have simple representations as operators of multiplication by the coordinate variables. This is true also for the adjoint of the shift operator $f(z) \mapsto zf(z)$ on the Hardy space H^2 on the unit disc. This operator is the backward shift operator with the explicit formula

$$f(z) \mapsto \frac{f(z) - f(0)}{z} \qquad (z \in \mathbb{D},\ f \in H^2). \tag{1}$$

Backward shift operators on other holomorphic function spaces such as the weighted Bergman spaces can be written in terms of the Taylor series of the functions in the spaces, but simple explicit expressions in the spirit of (1) have been lacking until recently.

In Gu and Luo (2024), for weighted Bergman Hilbert spaces A^2_n on the unit disc with nonnegative integer weight parameter n, explicit expressions akin to (1) have been obtained. In the same paper, another formula on the same spaces have been obtained using the Bergman projections again with integer parameters.

It turns out that, by judicious use of dual representations, it is possible to extend the Bergman projection formulas considerably. We obtain expressions for the backward shift operators on weighted Bergman and Bergman-Besov spaces B^p_q on the unit disc and the ball using Bergman projections. The spaces on which our formulas work include weighted Bergman spaces with non-integer weight parameter $q > -1$, Besov spaces which correspond to parameter values $q \leq -1$, Banach Bergman-Besov with parameters $1 \leq p < \infty$, the same spaces of holomorphic functions of several complex variables on the unit ball of $\mathbb{C}^N$, and in particular the Drury-Arveson and Dirichlet spaces.

To place our results in context, we introduce some notation. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^N$ with respect to the norm $|z| = \sqrt{\langle z, z\rangle}$ induced by the usual Hermitian inner product $\langle w, z\rangle = w_1\bar{z}_1 + \cdots + w_N\bar{z}_N$, which is the unit disc $\mathbb{D}$ for $N = 1$. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$, respectively.

We let ν be the Lebesgue measure on $\mathbb{B}$ normalized so that $\nu(\mathbb{B}) = 1$. For $q \in \mathbb{R}$, we also define on $\mathbb{B}$ the measures

$$d\nu_q(z) := (1 - |z|^2)^q \, d\nu(z).$$

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These measures are finite for $q > -1$ and σ-finite otherwise. For $0 < p < \infty$, we denote the Lebesgue classes with respect to ν_q by L^p_q, writing also $L^p = L^p_0$.

The standard *weighted Bergman spaces* on $\mathbb{B}$ are $A^p_q = L^p_q \cap H(\mathbb{B})$ for $q > -1$ normed by $\|f\|_{A^p_q} := \|f\|_{L^p_q}$. Equivalently, the Bergman space A^p_q is imbedded isometrically in L^p_q by the *inclusion map* i. We again write $A^p = A^p_0$ for the unweighted Bergman spaces.

Bergman spaces are generalized to two-parameter Besov spaces B^p_q for $q \le -1$. We defer the precise definition of Besov spaces to a later section; see Definition 2.1. It suffices now to note that the Besov space B^p_q is imbedded isometrically in the Lebesgue space L^p_q with the same parameters p, q by a map that is a combination of a derivative of the function and the product of a power of $1 - |z|^2$, where the order of the derivative and the power are the same. It is also true that an $f \in H(\mathbb{B})$ belongs to B^p_q whenever sufficiently high-order derivatives of f lie in a Bergman space.

We use the notation B^p_q for the full collection of Bergman-Besov spaces for $q \in \mathbb{R}$. For all $q \in \mathbb{R}$ and $1 \le p < \infty$, Bergman-Besov projections P_s exist from the Lebesgue class L^p_q onto the Bergman-Besov space B^p_q for s satisfying a certain well-known inequality; see Theorem 3.1.

The *shift operator* S on a holomorphic function space on the unit disc $\mathbb{D}$ is simply the operator of multiplication by the coordinate variable z; so $S(f)(z) = zf(z)$. When the space on which S acts matters, we attach the parameters of the space to S and write $S^p_q : B^p_q \to B^p_q$.

For function spaces on the unit ball $\mathbb{B}$, we have N coordinate variables $z_1, \dots, z_N$ and hence N shift operators; so $S_j(f)(z) = z_j f(z)$ for $j = 1, \dots, N$. But we can also indicate the spaces on which the shifts act and write $(S^p_q)_j : B^p_q \to B^p_q$.

The adjoints $(S^p_q)^*$ and $(S^p_q)^*_j$ of the shift operators on B^p_q act on the dual spaces $(B^p_q)^*$ and are called the *backward shift operators*. For $1 < p < \infty$, we have $(B^p_q)^* = B^{p'}_q$, where p' is the exponent conjugate to p, that is, $p' = p/(p-1)$. For $p = 1$, the corresponding dual spaces are the weighted Bloch spaces $\mathcal{B}^\infty_\alpha$, whose definitions are also deferred to a later section; see Definition 2.2. For the dual spaces, see Theorem 3.3.

The main purpose of this paper is to establish expressions for the backward shift operators on A^p_q and B^p_q in terms of Bergman projections. We use the fact that the adjoint of a shift operator on a complex Lebesgue space is merely the operator of multiplication by the *conjugate* of the coordinate variable used in the shift operator. Our formulas have the following form: We take a function in B^p_q and imbed it in the associated Lebesgue space L^p_q as explained above, then we multiply the imbedded function by the conjugate of the corresponding coordinate variable, and lastly we project back to the Bergman or Besov space by a suitable Bergman projection.

Our main results in this direction are Theorems 4.2, 4.4, 6.2, and 6.4. They and their proofs occupy Sections 4 and 6. In Sections 2 and 3 following, we summarize all relevant information on the spaces we work on and the projections used in the theorems. In Section 5, we show that the invariant subspaces of the backward shifts in certain Besov spaces on the disc have the wandering subspace property.

2. PRELIMINARIES

In this section, we give the remaining notation and the necessary details on the function spaces. We use multi-index notation in which $\alpha = (\alpha_1, \dots, \alpha_N)$ is an N-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, $0^0 = 1$, and $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ for $z \in \mathbb{C}^n$. A star $(\ )^*$ indicates adjoints for operators and duals for spaces. We show an integral inner product on a function space X by $[\,\cdot\,,\cdot\,]_X$.

The *Pochhammer symbol* $(a)_b$ is defined by

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}$$

when a and $a+b$ are off the pole set $-\mathbb{N}$ of the gamma function Γ. This is a shifted rising factorial since $(a)_k = a(a+1)\cdots(a+k-1)$ for positive integer k. In particular, $(1)_k = k!$ and $(a)_0 = 1$. A very useful identity is

$$(a)_{n+m} = (a)_n (a+n)_m \tag{2}$$

for $n, m \in \mathbb{N}$. Stirling formula gives

$$\frac{\Gamma(c+a)}{\Gamma(c+b)} \sim c^{a-b}, \qquad \frac{(a)_c}{(b)_c} \sim c^{a-b}, \qquad \frac{(c)_a}{(c)_b} \sim c^{a-b} \qquad (\operatorname{Re} c \to \infty), \tag{3}$$

where $A \sim B$ means that $|A/B|$ is bounded above and below by two strictly positive constants, that is, $A = O(B)$ and $B = O(A)$ for all A, B of interest. For $A = O(B)$, we also write $A \lesssim B$.

For $q \in \mathbb{R}$ and $w, z \in \mathbb{B}$, the *Bergman-Besov kernels* are

$$K_q(z,w) := \begin{cases} \dfrac{1}{(1-\langle z,w\rangle)^{1+N+q}} = \displaystyle\sum_{k=0}^{\infty} \frac{(1+N+q)_k}{k!} \langle z,w\rangle^k, & q > -(1+N), \\ {}_2F_1(1,1;1-(N+q);\langle z,w\rangle) = \displaystyle\sum_{k=0}^{\infty} \frac{k!\ \langle z,w\rangle^k}{(1-(N+q))_k}, & q \le -(1+N), \end{cases}$$

where ${}_2F_1 \in H(\mathbb{D})$ is the Gauss hypergeometric function. They are the reproducing kernels of Hilbert Bergman-Besov spaces. Notice that

$$K_{-(1+N)}(z,w) = \frac{1}{\langle z,w\rangle} \log \frac{1}{1-\langle z,w\rangle}.$$

These kernels are positive definite and sesquiholomorphic, and hence give rise to reproducing kernel Hilbert spaces of holomorphic functions which are the Bergman-Besov Hilbert spaces B^2_q. In particular, for $q > -1$, the B^2_q are weighted Bergman spaces A^2_q, B^2_{-1} is the Hardy space H^2, B^2_{-N} is the Drury-Arveson space, and $B^2_{-(1+N)}$ is the Dirichlet space. When $N = 1$, the Hardy and the Drury-Arveson spaces coincide.

To define the Bergman-Besov spaces for $p \neq 2$, we proceed as follows. Let the coefficient of $\langle z,w\rangle^k$ in the series expansion of $K_q(z,w)$ be $c_k(q)$ for any $q \in \mathbb{R}$. Note that $c_0(q) = 1$, $c_k(q) > 0$ for any k, and by (3),

$$c_k(q) \sim k^{N+q} \qquad (k \to \infty), \tag{4}$$

for every q. This explains the choice of the parameters of the hypergeometric function in K_q.

Let $f \in H(\mathbb{B})$ be given on $\mathbb{B}$ by its convergent *homogeneous expansion* $f = \sum_{k=0}^{\infty} f_k$ in which f_k is a homogeneous polynomial in $z_1, \dots, z_N$ of degree k. For $N = 1$, f_k is simply the kth term in the Taylor series of $f \in H(\mathbb{D})$. For any $s, t \in \mathbb{R}$, we define the *radial fractional differential operator* D^t_s on $H(\mathbb{B})$ by

$$D^t_s f := \sum_{k=0}^{\infty} d_k(s,t) f_k := \sum_{k=0}^{\infty} \frac{c_k(s+t)}{c_k(s)} f_k. \tag{5}$$

Note that $d_0(s,t) = 1$ so that $D^t_s(1) = 1$, $d_k(s,t) > 0$ for any k, and

$$d_k(s,t) \sim k^t \qquad (k \to \infty),$$

for any s, t by (4). So D^t_s is a continuous operator on $H(\mathbb{B})$ and is of order t. In particular, $D^t_s z^\gamma = d_{|\gamma|}(s,t) z^\gamma$ for any multi-index γ. More importantly,

$$D^0_s = I, \qquad D^u_{s+t} D^t_s = D^{t+u}_s, \qquad \text{and} \qquad (D^t_s)^{-1} = D^{-t}_{s+t}$$

for any s, t, u, where the inverse is two-sided. Thus any D^t_s maps $H(\mathbb{B})$ onto itself. The coefficients $d_k(s,t)$ are chosen in such a way that

$$D^t_q K_q(z,w) = K_{q+t}(z,w) \tag{6}$$

for any $q, t \in \mathbb{R}$, where differentiation is performed in the holomorphic variable z.

Consider now the linear transformation I^t_s defined for $f \in H(\mathbb{B})$ by

$$I^t_s f(z) := (1-|z|^2)^t\, D^t_s f(z).$$

When $t = 0$, s is irrelevant and I^0_s is just the inclusion i.

Definition 2.1. For $q \in \mathbb{R}$ and $0 < p < \infty$, we define the *Bergman-Besov space* B^p_q to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to L^p_q for some s, t satisfying

$$q + pt > -1. \tag{7}$$

It is well-known that under (7), Definition 2.1 is independent of s, t and the norms $\|f\|_{B^p_q} := \|I^t_s f\|_{L^p_q}$ are all equivalent. For $p = 2$, these norms are also equivalent to the norms obtained from the reproducing kernels. Explicitly,

$$\|f\|^p_{B^p_q} = \int_{\mathbb{B}} |D^t_s f(z)|^p (1-|z|^2)^{q+pt}\, d\nu(z) \qquad (q + pt > -1). \tag{8}$$

When $q > -1$, we can take $t = 0$ in (7) and obtain the weighted Bergman spaces $A^p_q = B^p_q$. When $q \le -1$, we call the spaces B^p_q *proper* Besov spaces. For $0 < p < 1$, what we call norms are actually quasinorms.

The Lebesgue class of all essentially bounded functions on $\mathbb{B}$ with respect to any ν_q is the same, so we denote them all by $\mathcal{L}^\infty$.

For $\alpha \in \mathbb{R}$, we also define the weighted Lebesgue spaces $\mathcal{L}^\infty_\alpha$ to consist of all measurable φ defined on $\mathbb{B}$ for which $(1-|z|^2)^\alpha \varphi(z)$ belongs to $\mathcal{L}^\infty$ normed by

$$\|\varphi\|_{\mathcal{L}^\infty_\alpha} := \operatorname*{ess\,sup}_{z\in\mathbb{B}} (1-|z|^2)^\alpha\, |\varphi(z)|.$$

Definition 2.2. For $\alpha \in \mathbb{R}$, we define the *Bloch-Lipschitz space* $\mathcal{B}^\infty_\alpha$ to consist of all $f \in H(\mathbb{B})$ for which $I^t_s f$ belongs to $\mathcal{L}^\infty_\alpha$ for some s, t satisfying

$$\alpha + t > 0. \tag{9}$$

It is well-known that under (9), Definition 2.2 is independent of s, t and the norms $\|f\|_{\mathcal{B}^\infty_\alpha} := \|I^t_s f\|_{\mathcal{L}^\infty_\alpha}$ are all equivalent. Explicitly,

$$\|f\|_{\mathcal{B}^\infty_\alpha} = \sup_{z\in\mathbb{B}} |D^t_s f(z)|(1-|z|^2)^{\alpha+t} \qquad (\alpha + t > 0). \tag{10}$$

If $\alpha > 0$, we can take $t = 0$ in (9) and obtain the weighted Bloch spaces. When $\alpha < 0$, these spaces are the holomorphic Lipschitz spaces $\Lambda_{-\alpha} = \mathcal{B}^\infty_\alpha$. The *usual Bloch space* $\mathcal{B}^\infty_0 = \mathcal{B}^\infty$ corresponds to $\alpha = 0$. There is no mention of the little Bloch space in this paper.

Remark 2.3. Definitions 2.1 and 2.2 imply that I^t_s imbeds B^p_q isometrically into L^p_q if and only if (7) holds, and I^t_s imbeds $\mathcal{B}^\infty_\alpha$ isometrically into $\mathcal{L}^\infty_\alpha$ if and only if (9) holds. By (8) and (10), $f \in B^p_q$ if and only if $D^t_s f \in A^p$ for $q + pt = 0$, and $f \in \mathcal{B}^\infty_\alpha$ if and only if $D^t_s f \in \mathcal{B}^\infty$ for $\alpha + t = 0$.

The reproducing property in the B^2_q is this: Given a $q \in \mathbb{R}$, there are $t, s_1, s_2 \in \mathbb{R}$ such that for any $f \in B^2_q$ and $z \in \mathbb{B}$, we have

$$f(z) = [f(\cdot), K_q(z,\cdot)]_{B^2_q} = C_q \int_{\mathbb{B}} D^t_{s_1} f(w) D^t_{s_2} K_q(z,w)(1-|w|^2)^{q+2t}\, d\nu(w),$$

where $[\cdot,\cdot]$ are the inner products associated to the norms in (8) and the C_q are normalizing constants. For Bergman spaces, $q > -1$ and naturally $t = 0$.

Proposition 2.4. *(Kaptanoğlu and Üreyen 2008, Proposition 3.1) (Kaptanoğlu and Tülü 2011, Proposition 2.1) For any $p > 0$ and $q, \alpha, s, t \in \mathbb{R}$, the maps $D^t_s : B^p_q \to B^p_{q+pt}$ and $D^t_s : \mathcal{B}^\infty_\alpha \to \mathcal{B}^\infty_{\alpha+t}$ are Banach space isomorphisms. They are also isometries when the parameters of the imbeddings I in the norms of the spaces are chosen as s, u for the domain and as $s+t, u-t$ for the target space.*

3. BERGMAN-BESOV PROJECTIONS

Bergman-Besov projections are the linear transformations P_s defined for $s \in \mathbb{R}$ and suitable φ by

$$P_s\varphi(z) = \int_{\mathbb{B}} \varphi(w) K_s(z,w)\, d\nu_s(w) \qquad (z \in \mathbb{B}).$$

Theorem 3.1. *(Kaptanoğlu 2005, Theorem 1.2) (Kaptanoğlu and Tülü 2011, Theorem 1.3) For $1 \le p < \infty$, the map $P_s : L^p_q \to B^p_q$ is bounded if and only if*

$$q + 1 < p(s+1). \tag{11}$$

Given an s satisfying (11), if t satisfies (7), then

$$P_s I^t_s f = \frac{N!}{(1+s+t)_N} f \tag{12}$$

holds for $f \in B^p_q$. Further, $P_s : \mathcal{L}^\infty_\alpha \to \mathcal{B}^\infty_\alpha$ is bounded if and only if

$$\alpha < s + 1. \tag{13}$$

Given an s satisfying (13), if t satisfies (9), then (12) holds for $f \in \mathcal{B}^\infty_\alpha$.

Remark 3.2. Note that $1 + s + t > 0$ if either (7) and (11), or (9) and (13) hold, thus in all cases considered in Theorem 3.1.

The *dual* of a Banach (or Hilbert) space is the space of all bounded linear functionals on the space. For $1 < p < \infty$, the dual of L^p_q is $L^{p'}_q$ under the pairing $[\cdot,\cdot]_q$, where

$$[\varphi, \psi]_q := \int_{\mathbb{B}} \varphi\overline{\psi}\, d\nu_q. \tag{14}$$

The dual of any L^1_q can be realized as any one of $\mathcal{L}^\infty_\alpha$ under the pairing $[\cdot,\cdot]_{q+\alpha}$.

Theorem 3.3. *(Kaptanoğlu 2005, Remark 7.3) (Kaptanoğlu and Tülü 2011, Theorem 6.2) For $1 < p < \infty$, the dual space of B_q^p can be identified with $B_q^{p'}$ under each of the pairings*

$$[f, g]_{q,s,t} := \int_{\mathbb{B}} I_s^t f \, \overline{I_{q+t}^{-q+s} g} \, d\nu_q \tag{15}$$

for s, t satisfying (11) and (7), that is, for every bounded linear functional Φ on B_q^p, there is a unique $g \in B_q^{p'}$ such that $\Phi f = [f, g]_{q,s,t}$ for $f \in B_q^p$. The dual space of any B_q^1 can be identified with any $\mathcal{B}_\alpha^\infty$ under each of the pairings $[f, g]_{q+\alpha,s,t}$ for s, t satisfying (11) and (7) with $p = 1$, that is, for every bounded linear functional Φ on B_q^1, there is a unique $g \in \mathcal{B}_\alpha^\infty$ such that $\Phi f = [f, g]_{q+\alpha,s,t}$ for $f \in B_q^1$.

Equation (15) takes simpler forms for Bergman spaces for which $q > -1$ and we can take $t = 0$. For $1 < p < \infty$, we can also take $s = q$, but for $p = 1$, we must have $s > q$. Then the pairings are

$$\int_{\mathbb{B}} f \, \overline{g} \, d\nu_q \qquad \text{and} \qquad \int_{\mathbb{B}} f \, \overline{I_{q+\alpha}^{-q-\alpha+s} g} \, d\nu_{q+\alpha} \tag{16}$$

for $1 < p < \infty$ and $p = 1$, respectively.

So we have two pairings (14) and (15) with one and three parameters to use without and with an I_s^t, respectively.

Most useful is the Banach space adjoint of $I_s^t : B_q^p \to L_q^p$ for $1 \leq p < \infty$ under the conditions (7) and (11). We use Theorem 3.3. For $1 < p < \infty$, the adjoint is the operator $(I_s^t)^* : L_q^{p'} \to B_q^{p'}$ such that $[I_s^t f, \psi]_q = [f, (I_s^t)^* \psi]_{q,s,t}$ for $f \in B_q^p$ and $\psi \in L_q^{p'}$, where s satisfies (11). For $p = 1$, it is the operator $(I_s^t)^* : \mathcal{L}_\alpha^\infty \to \mathcal{B}_\alpha^\infty$ such that $[I_s^t f, \psi]_{q+\alpha} = [f, (I_s^t)^* \psi]_{q+\alpha,s,t}$ for $f \in B_q^1$ and $\psi \in \mathcal{L}_\alpha^\infty$.

Theorem 3.4. *Let $q \in \mathbb{R}$. For $1 < p < \infty$, we have $(I_s^t)^* = \dfrac{(1+s+t)_N}{N!} P_{q+t}$, and for $p = 1$, we have $(I_s^t)^* = \dfrac{(1+s+t)_N}{N!} P_{q+\alpha+t}$ for any $\alpha \in \mathbb{R}$, where s, t satisfy (11) and (7). Explicitly, for $f \in B_q^p$ and $\psi \in (L_q^p)^*$,*

$$\int_{\mathbb{B}} I_s^t f \, \overline{\psi} \, d\nu_{q+\alpha} = \frac{(1+s+t)_N}{N!} \int_{\mathbb{B}} I_s^t f \, \overline{I_{q+\alpha+t}^{-q-\alpha+s} P_{q+\alpha+t} \psi} \, d\nu_{q+\alpha},$$

where $\alpha = 0$ for $1 < p < \infty$ and $\alpha \in \mathbb{R}$ is arbitrary for $p = 1$.

This theorem says that the composition of an I-type operator following a P-type operator can be removed under certain integrals.

Proof. We give the proof only for $1 < p < \infty$; the proof for $p = 1$ follows the same lines and is omitted.

Let $f \in B_q^p$, $\psi \in L_q^{p'}$, and put $F = D_s^t f$. By Proposition 2.4, $F \in B_{q+pt}^p$; but since $q + pt > -1$, actually $F \in A_{q+pt}^p$, a Bergman space which can be described without using any derivative. We have $q + pt + 1 < p(s + t + 1)$, so by Theorem 3.1, P_{s+t} maps A_{q+pt}^p onto itself and $P_{s+t} F = \dfrac{N!}{(1+s+t)_N} F$. We compute by first writing the integrals explicitly, next differentiating under the integral sign using (6), then interchanging the order of integration by the Fubini theorem, and finally using the information about F just stated. We obtain

$$\begin{aligned}
[f, P_{q+t}\psi]_{q,s,t} &= \int_{\mathbb{B}} (1-|z|^2)^t D_s^t f(z) \, \overline{(1-|z|^2)^{-q+s}} \\
&\quad \overline{\cdot D_{q+t}^{-q+s} \int_{\mathbb{B}} \psi(w) K_{q+t}(z,w) (1-|w|^2)^{q+t} \, d\nu(w)} \, (1-|z|^2)^q \, d\nu(z) \\
&= \int_{\mathbb{B}} (1-|z|^2)^{s+t} F(z) \int_{\mathbb{B}} \overline{\psi(w) K_{s+t}(z,w)} \, (1-|w|^2)^{q+t} \, d\nu(w) \, d\nu(z) \\
&= \int_{\mathbb{B}} (1-|w|^2)^{q+t} \, \overline{\psi(w)} \int_{\mathbb{B}} F(z) K_{s+t}(w,z) (1-|z|^2)^{s+t} \, d\nu(z) \, d\nu(w) \\
&= \int_{\mathbb{B}} (1-|w|^2)^{q+t} \, \overline{\psi(w)} \, P_{s+t} F(w) \, d\nu(w) \\
&= \int_{\mathbb{B}} (1-|w|^2)^t \frac{N!}{(1+s+t)_N} F(w) \, \overline{\psi(w)} \, d\nu_q(w) \\
&= \int_{\mathbb{B}} \frac{N!}{(1+s+t)_N} I_s^t f(w) \, \overline{\psi(w)} \, d\nu_q(w) = \left[\frac{N!}{(1+s+t)_N} I_s^t f, \psi \right]_q .
\end{aligned}$$

The proof is complete.

Theorem 3.4 takes simpler forms for Bergman spaces for which $q > -1$ and we can take $t = 0$. For $1 < p < \infty$, we can further take $s = q$ and then $i^* = \dfrac{(1+q)_N}{N!} P_q$, where i is the inclusion from A_q^p into L_q^p. For $p = 1$, we must have $q < s$, and then

$i^* = \dfrac{(1+s)_N}{N!} P_q$ upon taking $\alpha = 0$. The simpler explicit forms for $f \in A_q^p$ and $\psi \in (L_q^p)^*$ for general α are

$$\int_{\mathbb{B}} f\,\overline{\psi}\,d\nu_q = \frac{(1+q)_N}{N!} \int_{\mathbb{B}} f\,\overline{P_q \psi}\,d\nu_q \qquad (1 < p < \infty), \tag{17}$$

$$\int_{\mathbb{B}} f\,\overline{\psi}\,d\nu_{q+\alpha} = \frac{(1+s)_N}{N!} \int_{\mathbb{B}} f\,\overline{I_{q+\alpha}^{-q-\alpha+s} P_{q+\alpha}\psi}\,d\nu_{q+\alpha} \qquad (p = 1). \tag{18}$$

We need the following results when we compute the backward shifts explicitly on specific Bergman-Besov spaces. First, using multi-index notation,

$$\langle z, w\rangle^k = \sum_{|\gamma|=k} \frac{k!}{\gamma!}\, z^\gamma \overline{w}^\gamma. \tag{19}$$

Lemma 3.5. *Let α, β be two multi-indices, $|\alpha| = n$, and $s > -1$. Then*

$$\int_{\mathbb{B}} z^\alpha \overline{z}^\beta (1-|z|^2)^s\,d\nu(z) = \begin{cases} 0, & \text{if } \beta \neq \alpha, \\ \dfrac{N!\,\alpha!}{(1+s)_{N+n}}, & \text{if } \beta = \alpha. \end{cases}$$

Proof. This is (Alpay and Kaptanoğlu 2001, Lemma 1).

Lemma 3.6. *Let α, β be two multi-indices, $n = |\alpha|$, $m = |\beta|$, and $r, s \in \mathbb{R}$ with $r + s > -1$. Let $J = P_s\big(z^\alpha \overline{z}^\beta (1-|z|^2)^r\big)$. Then*

$$J = \begin{cases} \dfrac{N!\,(1+N+s)_{n-m}}{(1+r+s)_{N+n}}\, \dfrac{\alpha!}{(\alpha-\beta)!}\, z^{\alpha-\beta}, & \text{if } s > -(1+N), \\ \dfrac{N!\,((n-m)!)^2}{(1-(N+s))_{n-m}\,(1+r+s)_{N+n}}\, \dfrac{\alpha!}{(\alpha-\beta)!}\, z^{\alpha-\beta}, & \text{if } s \leq -(1+N), \end{cases}$$

for $\alpha \geq \beta$, and $J = 0$ otherwise, where $\alpha \geq \beta$ means $\alpha_j \geq \beta_j$ for all $j = 1, \ldots, N$.

Proof. In what follows, by Lemma 3.5, the only value of γ that gives a nonzero integral is $\gamma = \alpha - \beta \geq 0$, which also explains why the integral is 0 for $\alpha < \beta$.

For $s > -(1+N)$, by the way Bergman-Besov projections and kernels are defined, (19), and Lemma 3.5,

$$\begin{aligned} J &= \int_{\mathbb{B}} w^\alpha \overline{w}^\beta (1-|w|^2)^{r+s} \sum_{k=0}^\infty \frac{(1+N+s)_k}{k!} \sum_{|\gamma|=k} \frac{k!}{\gamma!}\, z^\gamma \overline{w}^\gamma\,d\nu(w) \\ &= \frac{(1+N+s)_{n-m}}{(\alpha-\beta)!}\, z^{\alpha-\beta} \int_{\mathbb{B}} |w^\alpha|^2 (1-|w|^2)^{r+s}\,d\nu(w) \\ &= \frac{(1+N+s)_{n-m}}{(\alpha-\beta)!}\, z^{\alpha-\beta}\, \frac{N!\,\alpha!}{(1+r+s)_{N+n}}. \end{aligned}$$

For $s \leq -(1+N)$, similarly,

$$\begin{aligned} J &= \int_{\mathbb{B}} w^\alpha \overline{w}^\beta (1-|w|^2)^{r+s} \sum_{k=0}^\infty \frac{k!}{(1-(N+s))_k} \sum_{|\gamma|=k} \frac{k!}{\gamma!}\, z^\gamma \overline{w}^\gamma\,d\nu(w) \\ &= \frac{((n-m)!)^2}{(1-(N+s))_{n-m}}\, \frac{1}{(\alpha-\beta)!}\, z^{\alpha-\beta} \int_{\mathbb{B}} |w^\alpha|^2 (1-|w|^2)^{r+s}\,d\nu(w) \\ &= \frac{((n-m)!)^2}{(1-(N+s))_{n-m}}\, \frac{1}{(\alpha-\beta)!}\, z^{\alpha-\beta}\, \frac{N!\,\alpha!}{(1+r+s)_{N+n}}. \end{aligned}$$

We use Lemma 3.6 only for $m = 1$.

4. BACKWARD SHIFT OPERATORS ON SPACES ON UNIT DISC

The spaces we work on have infinite families of equivalent norms. The pairings under which the dual spaces are realized depend strongly on the particular norms used. Likewise, the adjoint operators take different forms depending on the pairings. For this reason, for each type of space, we define the adjoints anew.

Throughout this section, $N = 1$. When (7) and (11) both hold, always $s + t > -1$.

Definition 4.1. For $q > -1$, let $S_q^p : A_q^p \to A_q^p$ be the shift operator acting on a Bergman space. If $1 < p < \infty$, we define its adjoint $(S_q^p)^* : A_q^{p'} \to A_q^{p'}$ by $\big[S_q^p f, g\big]_q = \big[f, (S_q^p)^* g\big]_q$, where $f \in A_q^p$ and $g \in A_q^{p'}$. If $p = 1$, we define its adjoint $(S_q^1)^* : \mathcal{B}_\alpha^\infty \to \mathcal{B}_\alpha^\infty$ by $\big[S_q^1 f, g\big]_{q+\alpha} = \big[f, (S_q^1)^* g\big]_{q+\alpha,s,0}$, where $f \in A_q^1$ and $g \in \mathcal{B}_\alpha^\infty$.

Theorem 4.2. *The adjoint of a Bergman shift operator is*

$$(S_q^p)^* g(z) = (1+s) P_{q+\alpha}\big(\overline{z} g(z)\big) = (1+s)\int_{\mathbb{D}} \frac{\overline{w}\, g(w)}{(1-\overline{w}z)^{2+q+\alpha}}\, d\nu_{q+\alpha}(w),$$

where $g \in A_q^{p'}$, $\alpha = 0$, *and* $s = q$ *for* $1 < p < \infty$, *and* $g \in \mathcal{B}_\alpha^\infty$, $\alpha \in \mathbb{R}$, *and* s *satisfies* (11) *for* $p = 1$.

Proof. Let $1 < p < \infty$ and $g \in A_q^{p'}$ first. Definition 4.1 and (17) give

$$\begin{aligned}\big[S_q^p f,\, g\big]_q &= \int_{\mathbb{D}} z f(z)\, \overline{g(z)}\, d\nu_q(z) = \int_{\mathbb{D}} f(z)\, \overline{\overline{z} g(z)}\, d\nu_q(z)\\ &= (1+q)\int_{\mathbb{D}} f(z)\, \overline{P_q\big(\overline{z} g(z)\big)}\, d\nu_q(z) = \big[f,\, (1+q)P_q\big(\overline{z} g(z)\big)\big]_q.\end{aligned}$$

Then

$$(S_q^p)^* g(z) = (1+q)P_q\big(\overline{z} g(z)\big) \qquad (1 < p < \infty).$$

Let $p = 1$ and $g \in \mathcal{B}_\alpha^\infty$ next. Definition 4.1 and (18) give

$$\begin{aligned}\big[S_q^1 f,\, g\big]_{q+\alpha} &= \int_{\mathbb{D}} z f(z)\, \overline{g(z)}\, d\nu_{q+\alpha}(z) = \int_{\mathbb{D}} f(z)\, \overline{\overline{z} g(z)}\, d\nu_{q+\alpha}(z)\\ &= (1+s)\int_{\mathbb{D}} f(z)\, \overline{I_{q+\alpha}^{-q-\alpha+s} P_{q+\alpha}\big(\overline{z} g(z)\big)}\, d\nu_{q+\alpha}(z)\\ &= \big[f,\, (1+s)P_{q+\alpha}\big(\overline{z} g(z)\big)\big]_{q+\alpha,s,0}.\end{aligned}$$

Thus $(S_q^1)^* g(z) = (1+s)P_{q+\alpha}\big(\overline{z} g(z)\big)$.

We keep α when $p = 1$ for flexibility. But if we choose $\alpha = 0$, then the only difference between the two cases in Theorem 4.2 is whether the coefficient is $1+q$ or $1+s$ with $s > q$.

Definition 4.3. Let $q \le -1$ and t, s satisfy (7) and (11). Also let $s > -2$ for convenience. Let $S_q^p : B_q^p \to B_q^p$ be the shift operator acting on a proper Besov space. If $1 < p < \infty$, we define its adjoint $(S_q^p)^* : B_q^{p'} \to B_q^{p'}$ by the identity $\big[S_q^p f,\, g\big]_{q,s,t} = \big[f,\, (S_q^p)^* g\big]_{q,s+1,t}$, where $f \in B_q^p$ and $g \in B_q^{p'}$. If $p = 1$, we define its adjoint $(S_q^1)^* : \mathcal{B}_\alpha^\infty \to \mathcal{B}_\alpha^\infty$ by $\big[S_q^1 f,\, g\big]_{q+\alpha,s,t} = \big[f,\, (S_q^1)^* g\big]_{q+\alpha,s+1,t}$, where $f \in B_q^1$ and $g \in \mathcal{B}_\alpha^\infty$.

Theorem 4.4. *The adjoint of a proper Besov shift operator is*

$$(S_q^p)^* g(z) = \frac{(2+s+t)^2}{2+s} P_{q+\alpha+t}\big(\overline{z} I_{q+\alpha+t}^{-q-\alpha+s} g(z)\big),$$

where $g \in B_q^{p'}$ *and* $\alpha = 0$ *for* $1 < p < \infty$, $g \in \mathcal{B}_\alpha^\infty$ *and* $\alpha \in \mathbb{R}$ *for* $p = 1$, t, s *satisfy* (7) *and* (11), *and* $s > -2$ *for convenience.*

The explicit integral expression for $P_{q+\alpha+t}$ depends on whether $q+\alpha+t > -2$ or $q+\alpha+t \le -2$.

Proof. Let $f \in B_q^p$ be given by its Taylor series $f(z) = \sum_{k=0}^\infty f_k z^k$. Since $s > -2$ and $s+t > -1 > -2$, by (5) we have

$$D_s^t(zf(z)) = \sum_{k=0}^\infty \frac{(2+s+t)_{k+1}}{(2+s)_{k+1}} f_k z^{k+1} = \frac{2+s+t}{2+s}\, z \sum_{k=0}^\infty \frac{(2+s+1+t)_k}{(2+s+1)_k} f_k z^k = \frac{2+s+t}{2+s}\, z D_{s+1}^t f(z),$$

and hence $I_s^t(zf(z)) = \dfrac{2+s+t}{2+s}\, z I_{s+1}^t f(z)$.

Let $1 < p < \infty$ and $g \in B_q^{p'}$ first. Definition 4.3, the previous calculation, and Theorem 3.4 give

$$\begin{aligned}\big[S_q^p f,\, g\big]_{q,s,t} &= \int_{\mathbb{D}} I_s^t(zf(z))\, \overline{I_{q+t}^{-q+s} g(z)}\, d\nu_q(z)\\ &= \int_{\mathbb{D}} I_{s+1}^t f(z)\, \overline{\frac{2+s+t}{2+s}\, \overline{z} I_{q+t}^{-q+s} g(z)}\, d\nu_q(z)\\ &= (2+s+t)\int_{\mathbb{D}} I_{s+1}^t f(z)\, \overline{\frac{2+s+t}{2+s} I_{q+t}^{-q+s+1} P_{q+t}\big(\overline{z} I_{q+t}^{-q+s} g(z)\big)}\, d\nu_q(z)\\ &= \left[f(z),\, \frac{(2+s+t)^2}{2+s} P_{q+t}\big(\overline{z} I_{q+t}^{-q+s} g(z)\big)\right]_{q,s+1,t}.\end{aligned}$$

The desired formula is obtained.

Let $p = 1$ and $g \in \mathcal{B}^\infty_\alpha$ next. Similar to the previous case, Definition 4.3, the above calculation, and Theorem 3.4 give

$$\begin{aligned}
[S^p_q f, g]_{q+\alpha,s,t} &= \int_{\mathbb{D}} I^t_s(zf(z))\, \overline{I^{-q-\alpha+s}_{q+\alpha+t} g(z)}\, d\nu_{q+\alpha}(z) \\
&= \int_{\mathbb{D}} I^t_{s+1} f(z)\, \overline{\frac{2+s+t}{2+s}\, \overline{z} I^{-q-\alpha+s}_{q+\alpha+t} g(z)}\, d\nu_{q+\alpha}(z) \\
&= \int_{\mathbb{D}} I^t_{s+1} f(z)\, \overline{\frac{(2+s+t)^2}{2+s} I^{-q-\alpha+s+1}_{q+\alpha+t} P_{q+\alpha+t}\big(\overline{z} I^{-q-\alpha+s}_{q+\alpha+t} g(z)\big)}\, d\nu_{q+\alpha}(z) \\
&= \left[f(z),\, \frac{(2+s+t)^2}{2+s} P_{q+\alpha+t}\big(\overline{z} I^{-q-\alpha+s}_{q+\alpha+t} g(z)\big) \right]_{q+\alpha,s+1,t}.
\end{aligned}$$

The desired formula follows.

We check some well-known Hilbert spaces to see the differences between our formulas and the more commonly known ones. Such differences are bound to happen since we base our formulas on integral inner products while many formulas in the literature are based on the norms derived from reproducing kernels. Since all the spaces involved consist of holomorphic functions on $\mathbb{D}$, it is enough to check the results on $g(z) = z^n$ for $n = 0, 1, 2, \ldots$.

Remark 4.5. Let $q > -1$ and consider first the Bergman Hilbert spaces A^2_q. Here there must be no difference in the literature among the backward shift operators since the reproducing kernels of Bergman spaces are derived from standard integral norms. Theorem 4.2 and Lemma 3.6 give

$$(S^2_q)^*(z^n) = (1+q) P_q(z^n \overline{z}) = (1+q) \frac{(2+q)_{n-1}}{(1+q)_{1+n}}\, n z^{n-1} = \frac{n}{1+q+n} z^{n-1},$$

which agrees with (Kaptanoğlu 2014, (26)), as expected.

Remark 4.6. For the proper Besov spaces B^2_q with $q \le -1$, there are two possibilities, $q + t > -2$ or $q + t \le -2$. In the first possibility ($q + t > -2$), we have $D^{-q+s}_{q+t}(z^n) = \dfrac{(2+s+t)_n}{(2+q+t)_n} z^n$ by (5). Then Theorem 4.4 and Lemma 3.6 give

$$\begin{aligned}
(S^2_q)^*(z^n) &= \frac{(2+s+t)^2}{2+s} \frac{(2+s+t)_n}{(2+q+t)_n} P_{q+t}\big(\overline{z}(1-|z|^2)^{-q+s} z^n\big) \\
&= \frac{(2+s+t)^2}{2+s} \frac{(2+s+t)_n}{(2+q+t)_n} \frac{(2+q+t)_{n-1}}{(1+s+t)_{1+n}}\, n z^{n-1} \\
&= \frac{(2+s+t)^2}{2+s} \frac{1}{1+s+t} \frac{n}{1+q+t+n} z^{n-1}.
\end{aligned}$$

Let $s \to \infty$ since it can be as large as we wish; then essentially

$$(S^2_q)^*(z^n) = \frac{n}{1+q+t+n} z^{n-1}. \tag{20}$$

For $q = -1$, we have $B^2_{-1} = H^2$ for which essentially $(S^2_{-1})^*(z^n) = \dfrac{n}{t+n} z^{n-1}$. For H^2, any small $t > 0$ works. If we further let $t \to 0^+$, we obtain $(S^2_{-1})^*(z^n) \to z^{n-1}$, which is what (1) says.

Remark 4.7. In the second possibility ($q+t \le -2$) when $q \le -1$, by (5) again, we have $D^{-q+s}_{q+t}(z^n) = \dfrac{(2+s+t)_n}{n!} \dfrac{(-(q+t))_n}{n!} z^n$. Then Theorem 4.4 and Lemma 3.6 give

$$\begin{aligned}
(S^2_q)^*(z^n) &= \frac{(2+s+t)^2}{2+s} \frac{(2+s+t)_n}{n!} \frac{(-(q+t))_n}{n!} P_{q+t}\big(\overline{z}(1-|z|^2)^{-q+s} z^n\big) \\
&= \frac{(2+s+t)^2}{2+s} \frac{(2+s+t)_n}{n!} \frac{(-(q+t))_n}{n!} \frac{(n-1)!}{(-(q+t))_{n-1}} \frac{n!}{(1+s+t)_{1+n}} z^{n-1} \\
&= \frac{(2+s+t)^2}{2+s} \frac{1}{1+s+t} \frac{-(1+q+t)+n}{n} z^{n-1}.
\end{aligned}$$

As $s \to \infty$ again, essentially

$$(S^2_q)^*(z^n) = \frac{-(1+q+t)+n}{n} z^{n-1}. \tag{21}$$

For $q = -2$, B^2_{-2} is the Dirichlet space for which $(S^2_{-2})^*(z^n) = \dfrac{1-t+n}{n} z^{n-1}$ essentially. For this space, we must have $t > 1/2$. In spite of this, if we further let $t \to 0^+$, we obtain $(S^2_{-2})^*(z^n) \to \dfrac{1+n}{n} z^{n-1}$, which is what (Kaptanoğlu 2014, (26)) says, contrary to intuition.

5. WANDERING SUBSPACE PROPERTY

In this section, we identify some shift operators acting on Bergman-Besov Hilbert spaces B^2_q whose invariant subspaces have the wandering subspace property.

Let T be a left-invertible operator on a Hilbert space H and let $E \subset H$ be a closed T-invariant subspace of H. We say E has the *wandering subspace property* if E is the smallest closed T-invariant subspace including $E \ominus TE$, where $\ominus$ indicates orthogonal complement, that is, if $E = \bigvee_{n=0}^{\infty} T^n(E \ominus TE)$, where $\bigvee$ indicates closed linear span.

In (Richter 1988, Theorem 1 and Corollary), it is shown that the invariant subspaces of the shift operator on the Besov Hilbert spaces B^2_q with $-2 \le q \le -1$, that is, on those spaces between the Hardy Hilbert and Dirichlet spaces, have the wandering subspace property. In (Aleman et al. 1996, Theorem 3.5), it is shown that the invariant subspaces of the shift operator on the unweighted Bergman Hilbert space has the wandering subspace property. It should be noted that all results of this form are norm (or inner product) dependent since the adjoint depends on it, except perhaps those on Bergman spaces in whose norms there is universal agreement. In fact, in Gallardo-Gutiérrez et al. (2020) it is shown that by renorming, one can force the wandering subspace property.

In (Shimorin 2001, Theorem 4.1), a very practical sufficient condition is given for the wandering subspace property in which $A \le B$ means $B - A$ is a positive operator.

Theorem 5.1. (Shimorin (2001)) *If S is the shift operator on a space of holomorphic functions on $\mathbb{D}$ and $SS^* + (S^*S)^{-1} \le 2I$, then the invariant subspaces of S have the wandering subspace property.*

Checking the hypothesis of this theorem is especially easy since both $S^2_q(S^2_q)^*$ and $(S^2_q)^*S^2_q$ are diagonal operators on the orthogonal basis $\{1, z, z^2 \dots\}$ for all B^2_q. We also see that renorming does have an effect.

Theorem 5.2. *For $-1 < q \le 0$, the shift operator S^2_q on the Bergman space A^2_q has the wandering subspace property.*

Proof. By Remark 4.5, we have $S^2_q(S^2_q)^*(z^n) = \dfrac{n}{1+q+n} z^n$ for $n \ge 1$, $S^2_q(S^2_q)^*(1) = 0$, $(S^2_q)^*S^2_q(z^n) = \dfrac{1+n}{2+q+n} z^n$, and $((S^2_q)^*S^2_q)^{-1}(z^n) = \dfrac{2+q+n}{1+n} z^n$. Applying Theorem 5.1, $S^2_q(S^2_q)^* + (S^2_q)^*S^2_q \le 2I$ if and only if

$$\frac{2+q}{1} \le 2 \qquad \text{and} \qquad \frac{n}{1+q+n} + \frac{2+q+n}{1+n} \le 2.$$

The first inequality gives $q \le 0$ and the second $q \ge -1$.

Theorem 5.3. *For $q \le -1$, the shift operator S^2_q on the Besov space B^2_q using the adjoints from (Kaptanoğlu 2014, (26)) has the wandering subspace property if $q = -1$, that is, for the Hardy space H^2 with the usual norm and adjoint.*

Proof. For $-2 < q \le -1$, $(S^2_q)^*(z^n) = \dfrac{n}{1+q+n} z^{n-1}$, which is identical to the adjoints in Theorem 5.2. So we have only $q = -1$ from the proof of that theorem.

For $q \le -2$, $(S^2_q)^*(z^n) = \dfrac{-1-q+n}{n} z^{n-1}$. Then $S^2_q(S^2_q)^*(z^n) = \dfrac{-1-q+n}{n} z^n$ and $((S^2_q)^*S^2_q)^{-1}(z^n) = \dfrac{1+n}{-q+n} z^n$. Hence $S^2_q(S^2_q)^* + (S^2_q)^*S^2_q \le 2I$ if and only if

$$\frac{1}{-q} \le 2 \qquad \text{and} \qquad \frac{-1-q+n}{n} + \frac{1+n}{-q+n} \le 2.$$

The first inequality gives $q \le -1/2$ and the second $q \ge -1$. Thus there is no $q \le -2$ with desired properties.

Theorem 5.4. *For $q \le -1$, the shift operator S^2_q on the Besov space B^2_q using the adjoints in Theorem 4.4 has the wandering subspace property if t is chosen to obtain $-1 \le q + t \le 0$.*

Note that it is compulsory to have $q + 2t > -1$ by (7).

Proof. To make the formulas amenable to computation, we use the adjoints in Remarks 4.6 and 4.7 in their limiting form as $s \to \infty$.

For $q + t > -2$, by using (20) we have $(S^2_q)^*(z^n) = \dfrac{n}{1+q+t+n} z^{n-1}$. Then also $S^2_q(S^2_q)^*(z^n) = \dfrac{n}{1+q+t+n} z^n$ and

$((S_q^2)^*S_q^2)^{-1}(z^n) = \dfrac{2+q+t+n}{1+n}\, z^n$. Hence $S_q^2(S_q^2)^* + (S_q^2)^*S_q^2 \leq 2I$ if and only if

$$\frac{2+q+t}{1} \leq 2 \qquad \text{and} \qquad \frac{n}{1+q+t+n} + \frac{2+q+t+n}{1+n} \leq 2.$$

The first inequality gives $q+t \leq 0$ and the second $q+t \geq -1$. Thus for $q \leq -1$, if we choose t with $-1 \leq q+t \leq 0$, then S_q^2 on B_q^2 has the wandering subspace property.

For $q+t \leq -2$, by using (21) we have $(S_q^2)^*(z^n) = \dfrac{-1-q-t+n}{n}\, z^{n-1}$. Then also $S_q^2(S_q^2)^*(z^n) = \dfrac{-1-q-t+n}{n}\, z^n$ and $((S_q^2)^*S_q^2)^{-1}(z^n) = \dfrac{1+n}{-q-t+n}\, z^n$. Hence $S_q^2(S_q^2)^* + (S_q^2)^*S_q^2 \leq 2I$ if and only if

$$\frac{1}{-q-t} \leq 2 \qquad \text{and} \qquad \frac{-1-q-t+n}{n} + \frac{1+n}{-q-t+n} \leq 2.$$

The first inequality gives $q+t \geq 1/2$ and the second $q+t \geq -1$, which contradict $q+t \leq -2$. So no q and t can be found with the desired properties in this case.

Some other shift operators are checked in Gu and Luo (2024).

6. BACKWARD SHIFT OPERATORS ON SPACES ON UNIT BALL

Shift operators S_j, $j = 1, \ldots, N$, on holomorphic function spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^N$ are investigated from many perspectives in Kaptanoğlu (2014). Here we concentrate only on their adjoints represented as Bergman-Besov projections. For readability, we refrain from attaching the parameters q, p of the spaces to the shift operators since they are clear from the context. We also take $\alpha = 0$ when $p = 1$ again for simplicity.

Definition 6.1. For $q > -1$, let $S_j : A_q^p \to A_q^p$ be a shift operators acting on a Bergman space, $j = 1, \ldots, N$. If $1 < p < \infty$, we define its adjoint $S_j^* : A_q^{p'} \to A_q^{p'}$ by $\left[S_j f, g\right]_q = \left[f, S_j^* g\right]_q$, where $f \in A_q^p$ and $g \in A_q^{p'}$. If $p = 1$, we define its adjoint $S_j^* : \mathcal{B}^\infty \to \mathcal{B}^\infty$ by $\left[S_j f, g\right]_q = \left[f, S_j^* g\right]_{q,s,0}$, where $f \in A_q^1$ and $g \in \mathcal{B}^\infty$.

Theorem 6.2. *For $j = 1, \ldots, N$, the adjoint of the Bergman shift operator S_j is*

$$S_j^* g(z) = \frac{(1+s)_N}{N!}\, P_q\big(\overline{z}_j g(z)\big) = \frac{(1+s)_N}{N!} \int_{\mathbb{B}} \frac{\overline{w}_j\, g(w)}{(1-\langle z, w\rangle)^{1+N+q}}\, d\nu_q(w),$$

where $g \in A_q^{p'}$ and $s = q$ for $1 < p < \infty$, and $g \in \mathcal{B}^\infty$ and $s > q$ for $p = 1$.

Proof. The proof is very similar to that of Theorem 4.2 and we omit it. The only thing that requires attention is that now we work in $\mathbb{C}^N$ with $N > 1$. The same are true for the proof of Theorem 6.4 below.

Definition 6.3. Let $q \leq -1$ and t, s satisfy (7) and (11). Also let $s > -(1+N)$ for convenience. Let $S_j : B_q^p \to B_q^p$ be a shift operator acting on a proper Besov space, $j = 1, \ldots, N$. If $1 < p < \infty$, we define its adjoint $S_j^* : B_q^{p'} \to B_q^{p'}$ by the identity $\left[S_j f, g\right]_{q,s,t} = \left[f, S_j^* g\right]_{q,s+1,t}$, where $f \in B_q^p$ and $g \in B_q^{p'}$. If $p = 1$, we define its adjoint $S_j^* : \mathcal{B}^\infty \to \mathcal{B}^\infty$ by the same identity, where $f \in B_q^1$ and $g \in \mathcal{B}^\infty$.

Theorem 6.4. *For $j = 1, \ldots, N$, the adjoint of the proper Besov shift operator S_j is*

$$S_j^* g(z) = \frac{1+N+s+t}{1+N+s}\, \frac{(2+s+t)_N}{N!}\, P_{q+t}\big(\overline{z}_j I_{q+t}^{-q+s} g(z)\big),$$

where $g \in B_q^{p'}$ for $1 < p < \infty$, $g \in \mathcal{B}^\infty$ for $p = 1$, t, s satisfy (7) and (11), and $s > -(1+N)$ for convenience.

The explicit integral expression for P_{q+t} depends on whether $q+t > -(1+N)$ or $q+t \leq -(1+N)$.

Let's evaluate the formulas in Theorems 6.2 and 6.4 on a monomial for all values of q and see their actual effects on certain standard reproducing kernel Hilbert spaces. In the Remarks below $g(z) = z^\alpha$, $n = |\alpha|$, $\beta = e_j$, $e_j = (0, \ldots, 0, 1, 0 \ldots, 0)$ with 1 in jth position, $j = 1, \ldots, N$, and $|\beta| = m = 1$. Note that $\dfrac{\alpha!}{(\alpha - e_j)!} = \alpha_j$.

Remark 6.5. Let $q > -1$ and consider the Bergman Hilbert spaces A_q^2 on $\mathbb{B}$. Theorem 6.2, Lemma 3.6, and (2) give

$$S_j(z^\alpha) = \frac{(1+q)_N}{N!}\, P_q(\overline{z}_j z^\alpha) = \frac{(1+q)_N}{N!}\, \frac{N!\,(1+N+q)_{n-1}}{(1+q)_{N+n}}\, \frac{\alpha!}{(\alpha-e_j)!}\, z^{\alpha - e_j} = \frac{\alpha_j}{N+q+n}\, z^{\alpha - e_j},$$

which agrees with (Kaptanoğlu 2014, (26)).

Remark 6.6. For the proper Besov spaces B^2_q with $q \le -1$, when $q+t > -2$, we have $D^{-q+s}_{q+t}(z^\alpha) = \dfrac{(1+N+s+t)_n}{(1+N+q+t)_n} z^\alpha$ by (5). Theorem 6.4, Lemma 3.6, and (2) give

$$\begin{aligned}(S^2_q)^*(z^\alpha) &= \frac{1+N+s+t}{1+N+s}\,\frac{(2+s+t)_N}{N!}\,\frac{(1+N+s+t)_n}{(1+N+q+t)_n}\,P_{q+t}\big(\overline{z}_j(1-|z|^2)^{-q+s}z^n\big)\\ &= \frac{1+N+s+t}{1+N+s}\,\frac{(2+s+t)_N}{N!}\,\frac{(1+N+s+t)_n}{(1+N+q+t)_n}\,\frac{N!\,(1+N+q+t)_{n-1}}{(1+s+t)_{N+n}}\,\alpha_j z^{\alpha-e_j}\\ &= \frac{(1+N+s+t)^2}{(1+N+s)(1+s+t)}\,\frac{\alpha_j}{N+q+t+n}\,z^{\alpha-e_j}.\end{aligned}$$

Let $s \to \infty$ as before since it can be as large as we wish; then essentially

$$(S^2_q)^*(z^n) = \frac{\alpha_j}{N+q+t+n}\,z^{\alpha-e_j}.$$

The cases $q=-1$ and $q=-N$ pertain to the Hardy space H^2 and the Drury-Arveson space. We must have $q+2t>0$ for this formula to make sense by Definition 6.3. But again contrary to intuition, if we let $t \to 0+$, we obtain the adjoint formulas in (Kaptanoğlu 2014, (26)) that are derived from the reproducing kernel norms.

Remark 6.7. For the proper Besov spaces B^2_q with $q \le -1$, when $q+t \le -2$, we have

$$D^{-q+s}_{q+t}(z^\alpha) = \frac{(1+N+s+t)_n}{n!}\,\frac{(1-(N+q+t))_n}{n!}\,z^\alpha$$

by (5). Theorem 6.4, Lemma 3.6, and (2) give

$$\begin{aligned}(S^2_q)^*(z^\alpha) &= \frac{1+N+s+t}{1+N+s}\,\frac{(2+s+t)_N}{N!}\,\frac{(1+N+s+t)_n}{n!}\,\frac{(1-(N+q+t))_n}{n!}\,P_{q+t}\big(\overline{z}_j(1-|z|^2)^{-q+s}z^n\big)\\ &= \frac{1+N+s+t}{1+N+s}\,\frac{(2+s+t)_N}{N!}\,\frac{(1+N+s+t)_n}{n!}\,\frac{(1-(N+q+t))_n}{n!}\,\frac{N!\,(n-1)!\,(n-1)!}{(1-(N+q+t))_{n-1}\,(1+s+t)_{N+n}}\,\alpha_j z^{\alpha-e_j}\\ &= \frac{(1+N+s+t)^2}{(1+N+s)(1+s+t)}\,\frac{-(N+q+t)+n}{n^2}\,\alpha_j z^{\alpha-e_j}.\end{aligned}$$

Let $s \to \infty$ again; then essentially

$$(S^2_q)^*(z^n) = \frac{-(N+q+t)+n}{n^2}\,\alpha_j z^{\alpha-e_j}.$$

The case $q=-(1+N)$ pertains to the Dirichlet space. We must have $q+2t>0$ for this formula to make sense by Definition 6.3. But again contrary to intuition, if we let $t \to 0+$, we obtain the adjoint formulas in (Kaptanoğlu 2014, (26)) that are derived from the reproducing kernel norms.

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