ISTANBUL UNIVERSITY PRESS

Istanbul Journal of Mathematics
ijmath 2 (2), 82–86, 2024
DOI: 10.26650/ijmath.2024.00018

RESEARCH ARTICLE

The nX-complementary generations of the group $PSL(3,5)$

A. B. M. Basheer[1*] and T. T. Seretlo[2]

[1]School of Mathematical and Computer Sciences, University of Limpopo (Turfloop), P Bag X1106, Sovenga 0727, South Africa
[2]School of Mathematical and Statistical Sciences, PAA Focus Area, North-West University (Mahikeng), P. Bag X2046, Mmabatho 2790, South Africa

ABSTRACT
Let G be a finite non-abelian simple group and nX be a non-trivial conjugacy class of elements of order n in G. We say that G is *nX-complementary generated*, if for every $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. In this paper we study the nX-complementary generations for all the non-trivial conjugacy classes of the projective special linear group $PSL(3,5)$. We approach this kind of generation using the structure constant method. GAP [The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.9.3;* 2018. (http://www.gap-system.org)] is used frequently in our computations.

Mathematics Subject Classification (2020): 20C15, 20C40, 20D08

Keywords: Conjugacy classes, generation, nX-complementary generation, structure constant

1. INTRODUCTION

The problem of generation of finite groups has great interest and has many applications to groups and their representations. The classification of finite simple groups is involved heavily and plays a pivotal role in most general results on the generation of finite groups. The study of generating sets in finite groups has a rich history, with numerous applications. We are interested in three kinds of generations of a finite simple group G, namely the (p,q,r)-generation, the nX-complementary generation and the ranks of the non-trivial conjugacy classes of G.

Definition 1.1. A finite group G is called *(l,m,n)-generated*, if it is a quotient group of the triangle group $T(l,m,n) = \left\langle x, y, z | x^l = y^m = z^n = xyz = 1 \right\rangle$.

In our work we generally restrict ourselves to the cases when l, m and n are primes and we use the notation (p,q,r)-generation rather than (l,m,n)-generation.

Definition 1.2. Let G be a finite simple group and nX be a non-trivial conjugacy class of elements of G. We define the *rank* of nX in G, denoted by $rank(G:nX)$, to be the minimum number of elements in nX that generate G.

Definition 1.3. A finite group G is said to be *nX-complementary generated*, if for every $x \in G$, there exists an element $y \in nX$ such that $G = \langle x, y \rangle$. The element y is called *complementary*.

In Woldar (1994), Woldar proved that every sporadic simple group G is pX-complementary generated where p is the largest prime dividing the order of G. In Ganief (1997), Ganief conjectured that every finite simple group is nX-complementary generated for some conjugacy class nX. In an attempt to further the theory on nX-complementary generation, he posed the problem:
Problem 1: Given a finite non-abelian simple group G, find all conjugacy classes nX of G such that G is nX-complementary generated.

In Ganief (1997) Ganief gave a complete answer to Problem 1 for the following sporadic simple group: the Janko groups J_1, J_2, J_3, J_4; the Higman-Sims group HS; the McLaughlin group McL; the Conway groups Co_3, Co_2 and the Fischer group Fi_{22}. The previous results have been published by Ganief and Moori in a series of papers Ganief and Moori (1997, 1998), Ganief and Moori (1998, 1999). Ashrafi in Ashrafi (2003, 2004) did the same for the sporadic groups He, Th, HN and $O'N$. Also Darafsheh, Ashrafi and Moghani established in Ashrafi et. al (2006); Darafsheh et. al. (2003, 2004), the nX-complementary generations for the sporadic groups Fi_{23}, Ru, Ly and Co_1.

With nX being a non-trivial conjugacy class of the group $G = PSL(3,5)$ as in the Atlas ATLAS (2024), the main result on the nX-complementary generation of the projective special linear group $PSL(3,5)$ can be summarized in the following theorem.

Corresponding Author: A. B. M. Basheer **E-mail:** ayoubbasheer@gmail.com
Submitted: 09.06.2024 • **Last Revision Received:** 18.12.2024 • **Accepted:** 19.12.2024

Theorem 1.4. *The group $G = PSL(3,5)$ is nX-complementary generated for all the conjugacy classes of G except when $nX \in \{1A, 2A, 4A, 4B, 5A\}$.*

The proof of Theorem 1.4 will be done through sequence of propositions and corollaries that will be established in Section 3.

The (p,q,r)-generations and the ranks of the conjugacy classes of the group $PSL(3,5)$ have been determined by the authors in Basheer and Seretlo (2019). We mainly used the structure constant method. In this paper we study the nX-complementary generation for the non-trivial conjugacy classes of the group $PSL(3,5)$. For the notation, description of the structure constant method and known results, we follow precisely Basheer and Seretlo (2019); Basheer et. al. (2019); Basheer and Seretlo (2021, 2020); Basheer et. al. (2024).

2. THE PROJECTIVE SPECIAL LINEAR GROUP $PSL(3,5)$

The projective special linear group $PSL(3,5)$ is a simple group of order $372000 = 2^5 \times 3 \times 5^3 \times 31$. By the Atlas ATLAS (2024), the group $PSL(3,5)$ has exactly 30 conjugacy classes of its elements, of which 14 of these classes have elements of prime orders. These are the classes $2A$, $3A$, $5A$, $5B$, $31A$, $31B$, $31C$, $31D$, $31E$, $31F$, $31G$, $31H$, $31I$ and $31J$. Also $PSL(3,5)$ has 5 conjugacy classes of maximal subgroups, where representatives of these classes of maximal subgroups can be taken as follows:

Table 1. The maximal subgroups of $G = PSL(3,5)$

	$\lvert H_i \rvert$	$[G : H_i]$
$H_1 = 5^2{:}GL(5,2)$	12000	31
$H_2 = 5^2{:}GL(5,2)$	12000	31
$H_3 = S_5$	120	3100
$H_4 = 4^2{:}S_3$	96	3875
$H_5 = 31{:}3$	93	4000

Throughout this paper and unless otherwise stated, by G we always mean the projective special linear group $PSL(3,5)$.

3. THE NX-COMPLEMENTARY GENERATIONS OF THE $PSL(3,5)$

The following lemma is very crucial in the determination of which conjugacy classes nX of G are nX-complementary generated.

Lemma 3.1. *A group G is nX-complementary generated if and only if for every conjugacy class pY of G, p prime, there exists a conjugacy class $t_{pY}Z$, depending on pY, such that the group G is $(pY, nX, t_{pY}Z)$-generated. Moreover; if G is a finite simple group, then G is not a $2X$-complementary generated, for any conjugacy class of involutions.*

Proof. See Lemma 2.3.8 of Ganief Ganief (1997).

Remark 3.2. Lemma 3.1 gives us a necessary and sufficient condition to determine whether a group G is nX-complementary generated or not. To check that the group G is nX-complementary generated, we need to check that G is $(pY, nX, t_{pY}Z)$-generated group for all the classes pY, where p is a prime number divides the order of G.

Remark 3.3. Recall that two involutions generate a dihedral group. Thus if G is a finite simple group and $2X$ is an involution class of G, then G is not a $2X$-complementary generated. To see this, suppose that G is a $2X$-complementary generated group. Then by Lemma 3.1, it follows that G is $(2X, 2X, t_{2X}Z)$-generated, which means that G is a dihedral group, contradicting the fact that G is a finite simple group. Therefore in the investigation on the classes of G whether they are nX-complementary generated or not we start by those classes where elements are of order at least 3.

The following result also helps a lot in determining whether G is an nX-complementary generated with the information that G is complementary generated by some conjugacy class sY and a power map of nX gives sY.

Lemma 3.4. *If G is sY-complementary generated and $(rX)^n = sY$, then G is rX-complementary generated.*

Proof. Let rX and sY be non-trivial conjugacy classes of G such that $(rX)^n = sY$ for some positive integer n. Now assume that G is not rX-complementary generated group. It follows that there exits an element x of prime order such that $\langle x, y\rangle < G$ for all $y \in rX$. Since $x, y^n \in \langle x, y\rangle$, it follows that $\langle x, y^n\rangle \leq \langle x, y\rangle < G$, for all $y^n \in sY$. We conclude that if G is not rX-complementary generated, then it is also not sY-complementary generated. The contrapositive of this gives the result.

It is clear that the group $G = PSL(3,5)$ is neither $1A$- nor $2A$-complementary generated. In the next proposition we show that G is not $3A$-complementary generated.

Proposition 3.5. *The group G is $3A$-complementary generated.*

Proof. By Propositions 3.1, 3.7, 3.11 and 3.12 of Basheer and Seretlo (2019), the group G is $(2A,3A,31X)$-, $(3A,3A,5B)$-, $(3A,5Y,31X)$- and $(3A,31X,31X)$-generated, for $Y \in \{A,B\}$ and $X \in \{A,B,C,D,E,F,G,H,I,J\}$. We know that if $G = \langle x,y \rangle$, then it is also $\langle y,x \rangle$. Therefore G is $(5Y,3A,31X)$- and $(31X,3A,31X)$-generated. The result follows by Lemma 3.1.

Proposition 3.6. *The group G is neither $4A$- nor $4B$-complementary generated.*

Proof. Here we show that G is not $(2A,4X,nZ)$-generated for $X \in \{A,B\}$ and all the conjugacy classes of G. The direct computations with GAP show that $\Delta_G(2A,4X,nZ) = 0$ for $X \in \{A,B\}$ and all the classes nZ of G except for the cases $(2A,4X,4Y)$, $(2A,4X,4C)$, $(2A,4X,20Y)$, $(2A,4A,24Z)$ and $(2A,4B,24Y)$ for $X,Y \in \{A,B\}$, $X \neq Y$ and $Z \in \{C,D\}$. Here we have

$$\begin{aligned}
\Delta_G(2A,4X,4Y) &= 49 < 480 = |C_G(g)|, g \in 4Y, Y \in \{A,B\},\\
\Delta_G(2A,4X,4C) &= 9 < 16 = |C_G(g)|, g \in 4C,\\
\Delta_G(2A,4X,8Y) &= 6 < 24 = |C_G(g)|, g \in 8Y, Y \in \{A,B\},\\
\Delta_G(2A,4X,20Y) &= 4 < 20 = |C_G(g)|, g \in 20Y, Y \in \{A,B\},\\
\Delta_G(2A,4A,24Z) &= 6 < 24 = |C_G(g)|, g \in 24Z, Z \in \{C,D\},\\
\Delta_G(2A,4B,24Y) &= 6 < 24 = |C_G(g)|, g \in 24Y, Y \in \{A,B\}.
\end{aligned}$$

It follows by Lemma 2.7 of Basheer and Seretlo (2020) that G is not generated by any of the previous 3-tuples. Therefore G is not generated by $(2A,4X,nZ)$, $X \in \{A,B\}$ and all the classes nZ of G. This completes the proof that G is not $4X$-complementary generated for $X \in \{A,B\}$.

Proposition 3.7. *The group G is $4C$-complementary generated.*

Proof. Here we do some computations regarding the triples $(pY,4C,31A)$ for all the primes p dividing the order of G. We firstly prove that G is $(2A,4C,31A)$-generated group. The direct computations with GAP show that $\Delta_G(2A,4C,31A) = 31$. Now from Table 1, we can see that only $H_5 = 31{:}3$ is the maximal subgroup of G that contains elements of order 31. However it is clear that H_5 neither contains elements of order 2 nor of order 4. Thus in the computations of $\Delta^*_G(2A,4C,31A)$ there is no contribution from any maximal subgroup of G. It follows that $\Delta^*_G(2A,4C,31A) = \Delta_G(2A,4C,31A) = 31$ and thus G is $(2A,4C,31A)$-generated group. Next we show that G is $(3A,4C,31A)$-generated group. The direct computations with GAP gives $\Delta_G(3A,4C,31A) = 961$. Again since H_5 is the only maximal subgroup of G with elements of order 31, however it does not contain an element of order 4. Therefore $\Delta^*_G(3A,4C,31A) = \Delta_G(3A,4C,31A) = 961$ and consequently G is $(3A,4C,31A)$-generated group. Similar arguments show that G is $(5X,4C,31A)$-generated group for $X \in \{A,B\}$ ($\Delta^*_G(5A,4C,31A) = \Delta_G(5A,4C,31A) = 31$, while $\Delta^*_G(5B,4C,31A) = \Delta_G(5B,4C,31A) = 930$). Finally using similar arguments we obtain for $X \in \{A,B,C,D,E,F,G,H,I,J\}$ that $\Delta^*_G(31X,4C,31A) = \Delta_G(31X,4C,31A) = 806$, establishing the generation of G by the triple $(31X,4C,31A)$ for X in the previous set. Now by applying Lemma 3.1 we deduce that G is $4C$-complementary generated group.

Proposition 3.8. *The group G is not $5A$-complementary generated.*

Proof. To show that G is not $5A$-complementary generated, we prove that G is not $(2A,5A,nZ)$-generated for all the conjugacy classes nZ of G. The direct computations with GAP reveal that $\Delta_G(2A,5A,nZ) = 0$ for all the classes nZ except when $nZ \in \{2A,10A,12A,12B,20A,20B\}$. Clearly the group G cannot be $(2A,5A,2A)$-generated as this violets the condition $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$ for the group to be (l,m,n)-generated. Now we have the following:

$$\begin{aligned}
\Delta_G(2A,5A,10A) &= 9 < 20 = |C_G(g)|, g \in 10A,\\
\Delta_G(2A,5A,12X) &= 6 < 24 = |C_G(g)|, g \in 12X, X \in \{A,B\},\\
\Delta_G(2A,5A,20X) &= 5 < 20 = |C_G(g)|, g \in 20X, X \in \{A,B\}.
\end{aligned}$$

It follows by Lemma 2.7 of Basheer and Seretlo (2020) that G is not generated by any of the previous 3-tuples. Therefore G is not generated by $(2A,5A,nZ)$ and all the classes nZ of G. Hence G is not $5A$-complementary generated.

Proposition 3.9. *The group G is $5B$-complementary generated.*

Proof. By Propositions 3.4, 3.11, 3.15 and 3.17 of Basheer and Seretlo (2019), the group G is $(2A,5B,31X)$-, $(3A,5B,31X)$-, $(5Z,5B,31Y)$- and $(5B,31X,31Y)$-generated, for $Z \in \{A,B\}$ and $X,Y \in \{A,B,C,D,E,F,G,H,I,J\}$. We know that if $G = \langle x,y \rangle$, then it is also $\langle y,x \rangle$. Therefore G is $(31X,5B,31Y)$-generated for X and Y in the previous set. The result follows by Lemma 3.1.

Proposition 3.10. *The group G is $6A$-complementary generated.*

Proof. The proof is similar to the one of Proposition 3.7. We show that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$, $X \in \{A, B, C, D, E, F, G, H, I, J\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 6. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta_G^*(pY, 6A, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta_G^*(2A, 6A, 31A) &= \Delta_G(2A, 6A, 31A) = 31,\\
\Delta_G^*(3A, 6A, 31A) &= \Delta_G(3A, 6A, 31A) = 651,\\
\Delta_G^*(5A, 6A, 31A) &= \Delta_G(5A, 6A, 31A) = 31,\\
\Delta_G^*(5B, 6A, 31A) &= \Delta_G(5B, 6A, 31A) = 620,\\
\Delta_G^*(31X, 6A, 31A) &= \Delta_G(31X, 6A, 31A) = 496,
\end{aligned}$$

for all $X \in \{A, B, C, D, E, F, G, H, I, J\}$. It follows that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. By Lemma 3.1 we deduce that G is $6A$-complementary generated.

Proposition 3.11. *The group G is $12A$- and $12B$-complementary generated.*

Proof. Since G has only one class of elements of order 6, it becomes obvious that the power maps $(12A)^2 = (12B)^2 = 6A$. By Proposition 3.10 we have G is $6A$-complementary generated. It follows by Lemma 3.4 that G is also $12A$- and $12B$-complementary generated group.

Proposition 3.12. *The group G is $24X$-complementary generated group for $X \in \{A, B, C, D\}$.*

Proof. The power maps of the classes of elements of order 24 as follows: $(24A)^2 = 12A$, $(24B)^2 = 12A$, $(24C)^2 = 12B$ and $(24D)^2 = 12B$. By Corollary 3.11, we know that G is $12A$- and $12B$-complementary generated. It follows by Lemma 3.4 that G is $24X$-complementary generated group for $X \in \{A, B, C, D\}$.

Proposition 3.13. *The group G is $8A$- and $8B$-complementary generated.*

Proof. The proof is similar to the one of Propositions 3.7 and 3.10. We show that G is $(pY, 8X, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31Z\}$, $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 8. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta_G^*(pY, 8X, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta_G^*(2A, 8X, 31A) &= \Delta_G(2A, 8X, 31A) = 31,\\
\Delta_G^*(3A, 8X, 31A) &= \Delta_G(3A, 8X, 31A) = 651,\\
\Delta_G^*(5A, 8X, 31A) &= \Delta_G(5A, 8X, 31A) = 31,\\
\Delta_G^*(5B, 8X, 31A) &= \Delta_G(5B, 8X, 31A) = 620,\\
\Delta_G^*(31Z, 8X, 31A) &= \Delta_G(31Z, 8X, 31A) = 496,
\end{aligned}$$

for all $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. It follows that G is $(pY, 8X, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31Z\}$ and $Z \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \in \{A, B\}$. By Lemma 3.1 we deduce that G is $8X$-complementary generated group for $X \in \{A, B\}$.

Proposition 3.14. *The group G is $10A$-complementary generated.*

Proof. The proof is similar to the previous ones handling the cases $4C$, $6A$, $8A$ and $8B$. We show that G is $(pY, 10A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. From Table 1, the only maximal subgroup that has elements of order 31 is $H_5 = 31{:}3$. However it is clear that H_5 does not contain an element of order 10. Thus there is no contribution from any maximal subgroup of G in the computations of $\Delta_G^*(pY, 10A, 31A)$. Using GAP we obtain that

$$\begin{aligned}
\Delta_G^*(2A, 10A, 31A) &= \Delta_G(2A, 10A, 31A) = 31,\\
\Delta_G^*(3A, 10A, 31A) &= \Delta_G(3A, 10A, 31A) = 775,\\
\Delta_G^*(5A, 10A, 31A) &= \Delta_G(5A, 10A, 31A) = 31,\\
\Delta_G^*(5B, 10A, 31A) &= \Delta_G(5B, 10A, 31A) = 774,\\
\Delta_G^*(31X, 10A, 31A) &= \Delta_G(31X, 10A, 31A) = 620,
\end{aligned}$$

for all $X \in \{A, B, C, D, E, F, G, H, I, J\}$. It follows that G is $(pY, 6A, 31A)$-generated group for all $pY \in \{2A, 3A, 5A, 5B, 31X\}$ and $X \in \{A, B, C, D, E, F, G, H, I, J\}$. By Lemma 3.1 we deduce that G is $6A$-complementary generated.

Proposition 3.15. *The group G is $20A$-complementary generated.*

Proof. Since G has only one class of elements of order 10, it becomes obvious that the power map $(20A)^2 = 10A$. By Proposition 3.14 we have G is $10A$-complementary generated. It follows by Lemma 3.4 that G is also $20A$-complementary generated group.

Proposition 3.16. *The group G is $31X$-complementary generated for $X \in \{A, B, C, D, E, F, G, H, I, J\}$.*

Proof. By Propositions 3.5, 3.12, 3.17 and 3.18 of Basheer and Seretlo (2019), the group G is $(2A, 31X, 31Y)$-, $(3A, 31X, 31Y)$-, $(5B, 31X, 31Y)$- and $(31X, 31Y, 31Z)$-generated, for $X, Y \in \{A, B, C, D, E, F, G, H, I, J\}$. Also for $X, Y \in \{A, B, C, D, E, F, G, H, I, J\}$ and $X \neq Y$, the group G is $(5A, 31X, 31Y)$-generated. Therefore G is $(pY, 31X, nZ)$-generated for all the conjugacy classes pY of G, where p is a prime divides the order of G. The result follows by Lemma 3.1.

Peer Review: Externally peer-reviewed.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: The first author would like to thank the University of Limpopo for the financial support received.

LIST OF AUTHOR ORCIDS

A. B. M. Basheer https://orcid.org/0000-0003-3320-4221
T. T. Seretlo https://orcid.org/0000-0003-2834-1942

REFERENCES

A. R. Ashrafi, 2003, nX-complementary generations of the Harada-Norton group HN, Acta Mathematica et Informatica Universitatis Ostraviensis, 11, no. 1, 3-9.
A. R. Ashrafi, 2004, nX-Complementary generations of the sporadic group $O'N$, Southeast Asian Bulletin of Mathematics, 28, no. 6, 1011-1019.
A. R. Ashrafi and G. A. Moghani, 2006, nX-Complementary generations of the Fischer group Fi_{23}, J. Appl. Math. & Computing, 21, no. 1-2, 393-409.
A. B. M. Basheer and T. T. Seretlo, 2019, On two generation methods for the simple linear group $PSL(3, 5)$, Khayyam J. of Math., 5, no. 1, 125-139.
A. B. M. Basheer, M. J. Motalane and T. T. Seretlo, 2021 The (p, q, r)-generations of the alternating group A_{11}, Khayyam J. of Math., 7, no. 2, 165-186.
A. B. M. Basheer and T. T. Seretlo, 2021, The (p, q, r)-generations of the Matheiu group M_{22}, Southeast Asian Bulletin of Mathematics, 45, no. 1, 11-28.
A. B. M. Basheer and T. T. Seretlo, 2020, The (p, q, r)-generations of the alternating group A_{10}, Quaestiones Mathematicae, 43, no. 3, 395-408.
A. B. M. Basheer, M. J. Motalane, M. G. Sehoana and T. T. Seretlo, 2024 The nX-complem-entary generations of the classes of the Chevalley group $G_2(3)$, Mathematics Interdisciplinary Research, 9 no. 4, 443–460.
J. H. Conway et al., 1985, Atlas of Finite Groups, Clarendon Press, Oxford.
M. R. Darafsheh, A. R. Ashrafi and G.A. Moghani, 2001, (p, q, r)-generations of the Conway group Co_1, for odd p, Kumamoto J. Math., 14, 1 - 20.
M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, 2003, (p, q, r)-generations and nX-complementary generations of the sporadic group Ly, Kumamoto J. Math., 16, 13-25.
M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani, 2004, nX-complementary generations of the sporadic group Co_1, Acta Mathematica Vietnamica, 29, no. 1, 57-75.
M. R. Darafsheh, A. R. Ashrafi and G. A. Moghani,2001, (p, q, r)-generations of the sporadic group O'N, London Mathematical Society Lecture Note Series, Groups St. Andrews in Oxford, Cambridge University Press, 304, 101 - 109.
S. Ganief, 1997, 2-Generations of the Sporadic Simple Groups, PhD Thesis, University of Natal, South Africa.
S. Ganief and J. Moori, 1997, 2-generations of the smallest Fischer group Fi_{22}, Nova J. Math. Game Theory Algebra 6, no. 2-3, 127-145.
S. Ganeif and J. Moori, 1997, (p, q, r)-generations and nX-complementary generations of the sporadic groups HS and McL., J. Algebra, 188, no. 2, 531-6.
S. Ganief and J. Moori, 1998, Generating pairs for the Conway groups Co_2 and Co_3, J. Group Theory, 1, no. 3, 237-256.
S. Ganief and J. Moori, 1999, 2-generations of the fourth Janko group J_4, J. Algebra, 212, no. 3, 305-322.
The GAP Group, 2018, GAP - Groups, Algorithms, and Programming, Version 4.9.3; (http://www.gap-system.org)
A. J. Woldar, 1994, $\frac{3}{2}$-Generation of the sporadic simple groups, Comm. Algebra 22, no. 2, 675-685.