

Monad Discrete Spaces and Limit Monad Points on Them

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Abstract

In this study, Monad discrete spaces were defined on amply soft topologies which is a more flexible and comprehensive structure from other known traditional soft topologies. The conditions under which a given amply soft set is amply soft open or amply soft closed set were given over them. Then, the amply soft set of limit monad points and closure points of any amply soft set in monad discrete spaces was investigated and the relevant properties were given. And then, as an extra, limit monad points in amply soft topological spaces produced by classical topologies were investigated and the relevant property was given. While constructing amply soft sets, specifically focused on amply soft topologies formed by selecting real numbers from both universal and parametric spaces unlike the traditional soft sets. Finally, a question was left that we hope will catch the reader's attention.

Keywords: a new extended soft set, limit monad point, monad discrete space, PAS topology

1. Introduction

When tackling complex problems in disciplines such as physics, mathematics, chemistry, psychology, astronomy, geology, and environmental science, numerous uncertainties inevitably arise. To address these uncertainties, various theoretical frameworks have been proposed, including fuzzy sets [1], interval-valued fuzzy sets [2], intuitionistic fuzzy sets [3], neutrosophic sets [4], vague sets [5], and rough sets [6]. Each of these approaches introduces its own set of complexities and methodological nuances.

However, since each of these theories addresses distinct aspects of uncertainty, the question arises: is there a unified structure capable of encompassing all these needs? This question was first posed by Molodtsov [7], who introduced the concept of soft sets as a potential solution. Building upon this foundation, Maji [8] advanced the theory by defining notions such as subset, superset, and equality within the context of soft sets, along with operations including union, intersection, and complement. He also proposed several related properties based on these definitions.

Subsequently, some of these properties were found to be incorrect. In response, Yang [9] provided counterexamples to refute certain claims, while additional counterexamples were presented by Ali et al. [10] and Neog et al. [11]. These findings necessitated a reevaluation and redefinition of soft set operations. Furthermore, other researchers [12–17] have identified similar inconsistencies and proposed revised formulations for soft set operators to address these shortcomings.

Çağman et al. [18] proposed a framework for soft topology by incorporating different sets of parameters within soft sets. In their study, they introduced fundamental topological concepts in the context of soft sets, including soft open and closed sets, soft interior, soft closure, and soft limit points. These developments collectively contributed to the formal definition of a soft topological structure.

As noted by Çağman et al. [19] in a subsequent study, the union of a soft set with its complement does not necessarily yield the universal soft set, as highlighted in Remark 3.17. This limitation introduces significant challenges, particularly in the treatment of soft set complements. In light of this issue, Jyonthis and Sunil [20] observed that "any closed set in a soft topological

space with non-constant parameters may not contain all the accumulation points".

The core of the problem lies in the definition of closed sets within soft topological spaces that involve non-constant parameters. Specifically, closed sets were defined as the complements of open sets; however, the union of a set with its complement does not always produce the universal soft set in such spaces. To address this inconsistency, two alternative approaches have been proposed in the literature.

One approach to addressing the limitations of classical soft set theory involves incorporating a fixed set of parameters, as demonstrated by Shabir and Naz [21]. In their work, they introduced several concepts related to soft sets and their topological properties, using a consistent parameter set across all soft sets. However, this fixed-parameter requirement has posed a significant obstacle in the development of soft metric spaces. Despite this, Göçür [22] introduced the concept of soft metrizable to advance the theory. To overcome existing challenges, Das and Samantha [23] proposed a soft metric on soft topological spaces with constant parameters. In their study, a new concept of a soft point was introduced, and the distance between two soft points was defined based on both the fixed parameter's distance condition and a selected metric between parameters. While this approach reduced randomness in point selection, it required the introduction of a separate metric within the parameter space.

The second approach centers on redefining soft sets and their associated operations. Zhu and Wen [24] were among the first to offer such a redefinition, particularly emphasizing the handling of parameter sets among soft sets. However, the complexity of parameter management in this redefinition continued to present difficulties.

In response, researchers began to explore hybrid soft theories, which combined soft sets with other uncertainty modeling techniques. As a result, various hybrid structures were introduced, including fuzzy soft sets [19], N-soft sets [25], hypersoft sets [26], bipolar soft sets [27], bipolar hypersoft sets [28], N-hypersoft sets [29], and N-bipolar hypersoft sets [30].

To address the ongoing need for a more flexible and comprehensive structure, Göçür [31] introduced a novel soft set framework known as the amply soft set. This model is capable of functioning with both fixed and non-fixed parameters, making it more general and inclusive than classical soft set theories. An amply soft set is constructed over a parameter space and an initial universe; however, unlike prior models, it imposes no restrictions on the nature of these spaces. They may be countable or uncountable, finite or infinite, qualitative or quantitative, or any combination thereof. Furthermore,

amply soft sets can be defined using distinct parameter sets.

In the same study, Göçür introduced several topological structures derived from this framework, including the amply soft (AS) topology, which supports the formulation of multiple topologies across different parameters; and the AS topology produced by classical topologies (PAS topology), which allows for the definition of separate classical topologies for each parameter. He also introduced the concept of monad points and a new class of parametric separation axioms P_i (where $i=0,1,2,3,4$), in contrast to traditional topological separation axioms T_i ($i=0,1,2,3,4$). While T_i axioms focus on relationships among elements in the universal set, P_i axioms evaluate the strength of connections between the associated parameters, thereby adding more flexibility and depth to the analysis of soft topological structures.

Building upon this, Göçür [32] demonstrated that it is possible to measure distances even when an object possesses multiple, potentially undefined metrics across various dimensions. He introduced the concept of monad metrics and passing points, showing that the invisibility of certain dimensions does not preclude the holistic measurement or perception of an object. This suggests that unobservable dimensions of an entity can still have a definable presence, and their existence cannot be discounted simply because they cannot be directly perceived.

In a follow-up study, Göçür [33] introduced the notions of interior, exterior, boundary, and closure points within amply soft topological spaces. Subsequently, Göçür [34] presented the concepts of neighborhood and open neighborhood in these spaces, along with their associated properties.

Later, Karakaş et al. [35] extended this line of research by introducing the notion of isolated monad points and successfully constructed amply soft sets consisting of such points within amply soft topological spaces.

Also, Göçür [36] introduced limit monad points on amply soft topological spaces. He defined the amply soft set of limit points for the monad points of any amply soft set and provide properties related to the intersections and unions of these sets. And then explored closure points through limit monad points.

In this study, Monad discrete spaces were defined on amply soft topologies which is a more flexible and comprehensive structure from other known traditional soft topologies. The conditions under which a given amply soft set is amply soft open or amply soft closed set were given over them. Then, the amply soft set of limit

monad points and closure points of any amply soft set in monad discrete spaces was investigated and the relevant properties were given. And then, as an extra, limit monad points in amply soft topological spaces produced by classical topologies were investigated and the relevant property was given. While constructing amply soft sets, specifically focused on amply soft topologies formed by selecting real numbers from both universal and parametric spaces unlike the traditional soft sets. Finally, a question was left that we hope will catch the reader's attention.

In the Materials and Methods section, we first introduce the amply soft set, followed by the operations performed on it. Note that the operations performed here are evaluated on parameters, so we don't need to see all the elements of the set. This is one of the most important features that distinguishes an amply soft set from other defined soft sets. Next, we introduce the amply soft topology, considering a family of amply soft sets. The concepts of amply soft open, amply soft closed, the concepts of a monad point and a whole point as well as neighborhood, open neighborhood and limit monad points of a monad point are introduced.

In the Results and Discussion section, monad discrete spaces are first introduced, followed by a discussion of the existence of limit monad points on these spaces. Closure points are then given. In addition, limit monad points are investigated on amply soft topological spaces generated from classical topologies, and sets of closure points are constructed. A question we believe will be of interest to the reader is then posed.

2. Materials and Methods

Definition 2.1. [31] Let G, Λ and $P(G)$ be a universe, a set of universal parameters and a power set of G respectively. We define subset $\Gamma \subseteq \Lambda$. An amply soft set on G , symbolized as an AS set, is characterized by a mapping $T: \Lambda \rightarrow P(G)$ constructed as follows:

$$T(\lambda) = \begin{cases} \text{if } \lambda \in \Gamma, & T(\lambda), \\ \text{if } \lambda \in \Lambda - \Gamma, & \phi. \end{cases}$$

This notation indicates that $T * \Gamma$ represents the collection of values assigned by T to the parameters in Γ . Thus, we can refer to T as the function defining the AS set, capturing its essential properties and relationships with the parameter set Λ .

Definition 2.2. [31] Let G be a universe, Λ denote a universal parameter set. If $T(\lambda) = G$, for in all $\lambda \in \Lambda$, then the AS set $T * \Lambda$ on \tilde{G} is referred to as an absolute AS set. This is symbolized by \tilde{G} .

In this article, unless otherwise specified, the following symbols will be used to represent the corresponding expressions: G : a universe, Λ : a set of universal

parameters; $\Gamma, \Omega, \Upsilon, \Psi \subseteq \Lambda$; $T * \Gamma, S * \Omega, V * \Upsilon, Y * \Psi$: AS sets on \tilde{G} .

Definition 2.3. [31] If $T(\lambda) \subseteq S(\lambda)$ for all $\lambda \in \Gamma$ and $\Gamma \subseteq \Omega$, then it follows that $T * \Gamma \subseteq S * \Omega$. This correlation is symbolized by $T * \Gamma \subseteq S * \Omega$, indicating that the AS set defined by T on the subset Γ is contained within the AS set defined by S on the subset Ω . This inclusion highlights a hierarchical relationship between the two AS sets based on their mappings.

Definition 2.4. [31] If $T(\lambda) \supseteq S(\lambda)$ for all $\lambda \in \Omega$ and $\Gamma \supseteq \Omega$, then it follows that $T * \Gamma \supseteq S * \Omega$. This correlation is symbolized by $T * \Gamma \supseteq S * \Omega$, indicating that the AS set defined by S on the subset Γ is a superset of the AS set defined by S on the subset Ω . This inclusion underscores the broader scope of the AS set $T * \Gamma$ in relation to $S * \Omega$.

Definition 2.5. [31] $T * \Gamma$ and $S * \Omega$ are considered an equal is denoted as $T * \Gamma \cong S * \Omega$ if and only if $\Gamma \cong \Omega$ and $T(\lambda) = S(\lambda)$. If this condition is not met, $T * \Gamma$ and $S * \Omega$ regarded as not equal, symbolized by $T * \Gamma \not\cong S * \Omega$. This definition provides a clear criterion for determining the equality or inequality of the two AS sets.

Definition 2.6. [31] If $T(\lambda) = \phi$ for all $\lambda \in \Gamma$ or $\Gamma \subseteq \phi$, then the AS set $T * \Gamma$ on G is referred to as an empty AS set. This is symbolized by $\tilde{\phi}$. This notation indicates that the AS set contains no elements, reflecting the absence of any values assigned by the mapping T for the parameters in Γ .

Definition 2.7. [31] The union of $T * \Gamma$ with $S * \Omega$ is described as follows:

$$T \cup S(\lambda) = \begin{cases} \text{if } \lambda \in \Gamma - \Omega, & T(\lambda), \\ \text{if } \lambda \in \Omega - \Gamma, & S(\lambda), \\ \text{if } \lambda \in \Gamma \cap \Omega, & T(\lambda) \cup S(\lambda), \\ \text{if } \lambda \in \Lambda - (\Gamma \cup \Omega), & \phi. \end{cases}$$

This combination is symbolized by

$$T * \Gamma \tilde{\cup} S * \Omega \cong T \cup S * \Gamma \cup \Omega.$$

This notation captures the combined elements of them, taking into account their relationships within the sets Γ and Ω .

Definition 2.8. [31] The intersection of $T * \Gamma$ with $S * \Omega$ is defined for all as follows:

$$T \cap S(\lambda) = \begin{cases} \text{if } \lambda \in \Gamma \cap \Omega, & T(\lambda) \cap S(\lambda), \\ \text{if } \lambda \in \Lambda - (\Gamma \cap \Omega), & \phi. \end{cases}$$

This intersection is denoted as

$$T * \Gamma \tilde{\cap} S * \Omega \cong T \cap S * \Gamma \cap \Omega.$$

This notation captures the common elements of them, reflecting their relationships within the sets Γ and Ω .

Definition 2.9. [31] The difference of $T * \Gamma$ from $S * \Omega$ is defined as follows:

$$T \setminus S(\lambda) = \begin{cases} \text{if } \lambda \in \Gamma - \Omega, T(\lambda), \\ \text{if } \lambda \in \Gamma \cap \Omega, T(\lambda) \setminus S(\lambda), \\ \text{if } \lambda \in \Lambda - \Gamma, \phi. \end{cases}$$

This difference is denoted as

$$T * \Gamma \setminus S * \Omega \cong T \setminus S * \Gamma.$$

This notation captures the elements of the AS set $T * \Gamma$ that are not included in $S * \Omega$, emphasizing the distinction between the two AS sets.

Definition 2.10. [31] The complement of $T * \Gamma$ on \tilde{G} is described as follows:

$$T'(\lambda) = \begin{cases} \text{if } \lambda \in \Gamma, G - T(\lambda), \\ \text{if } \lambda \in \Lambda - \Gamma, G. \end{cases}$$

This definition captures the elements of the initial universe G that are not included in the AS set $T * \Gamma$ for parameters in Γ while assigning the entire universe G for parameters outside of Γ .

Definition 2.11. [31] Let $\tilde{\wp}$ be a collection of all AS sets on \tilde{G} . If the three conditions are ensured for this collection, then $\tilde{\wp}$ is called an amply soft topological space on \tilde{G} :

1. $\tilde{\phi}, \tilde{G} \in \tilde{\wp}$,
2. Any union of AS sets is also inside $\tilde{\wp}$,
3. Any intersection of two AS sets is inside $\tilde{\wp}$.

If these conditions hold, we can say that $\tilde{G} * \tilde{\wp} * \Lambda$ constitutes an AS topological space on \tilde{G} .

Definition 2.12. [31] Let $T * \{\alpha\}$ be an AS set over \tilde{G} with $\alpha \in \Lambda$.

If $T(\alpha) = \{g\}, g \in G$, then $T * \{\alpha\}$ is referred to as a monad point of \tilde{G} . This is denoted by g_α . The monad point represents a specific element in the universe \tilde{G} corresponding to the parameter α .

Definition 2.13. [31] Let $T * \Lambda$ be an AS set on \tilde{G} . If $T(\lambda) = \{g\}$ for all $\lambda \in \Lambda$, $T * \Lambda$ is referred to as a whole point. This is symbolized by \tilde{g} .

In this article, unless otherwise stated, the following symbols will be used to represent the corresponding expressions: $\tilde{G} * \tilde{\wp} * \Lambda$: an AS topological space g_α : a monad point on $\tilde{G}, g \in G, \lambda \in \Lambda$; \tilde{g} : a whole point on \tilde{G} .

Definition 2.14. [31] Let us consider the $\tilde{G} * \tilde{\wp} * \Lambda$. In this context, an AS open set is defined as any member of the collection $\tilde{\wp}$.

Definition 2.15. [31] Let us consider the $\tilde{G} * \tilde{\wp} * \Lambda$. If the complement of $T * \Gamma$, denoted $(T * \Gamma)^\gamma$, is a member of $\tilde{\wp}$, then, we can refer to $T * \Gamma$ as an AS closed set. This designation indicates that the AS closed set corresponds to those sets whose complements are open within the AS topology.

Definition 2.16. [31] Let $\tilde{G} * \tilde{\wp} * \Lambda$ be given. The intersection of entire AS closed supersets of $S * \Omega$ is said to be a closure of $S * \Omega$. This closure is symbolized by $\overline{S * \Omega}$. In this context, it is evident that, $S * \Omega$ is contained within the narrowest AS closed set that includes it, illustrating the relationship between the set and its closure in the AS topology.

Proposition 2.17. [31] Let $\tilde{G} * \tilde{\wp} * \Lambda$ be given. Then the collection $\wp_\lambda = \{T(\lambda) | T * \Lambda \in \tilde{\wp}\}$ for each $\lambda \in \Lambda$, defines topologies on \tilde{G} .

Definition 2.18. [31] Let $\tilde{G} * \tilde{\wp} * \Lambda$ be given and let \tilde{q} such that $\tilde{q} \subseteq \tilde{\wp}$. If every element of $\tilde{\wp}$ can be written as any union of elements of \tilde{q} , then \tilde{q} is called as an AS basis for the AS topology $\tilde{\wp}$. Then we can say that each element of \tilde{q} is an AS basis element.

Definition 2.19. [31] Let $\lambda_n \in \Lambda$ for $n \in \mathbb{N}$, and let (G, τ_n) represent topological spaces over identical G . If f is a function from τ_n to τ_{λ_n} and $\tilde{q} = \{\tau_{\lambda_n}\}$ serves as an AS basis for an AS topology $\tilde{\wp}$, then this topology is said to be the AS topology produced by classical topologies. For brevity, we can refer to it as a PAS topology. This concept emphasizes the relationship between classical topologies and the resulting AS topology, highlighting the foundational role of classical structures in defining new topological frameworks.

Definition 2.20. [31] Let us consider the $\tilde{G} * \tilde{\wp} * \Lambda$. The interior of $T * \Gamma$ is described as the union of all amply soft open subsets of $T * \Gamma$. This interior is denoted by $(T * \Gamma)^\delta$. This notation captures the essence of the interior, representing all points within $T * \Gamma$ that can be surrounded by open sets in the AS topology $\tilde{\wp}$.

Definition 2.21. [33] Let us consider the $\tilde{G} * \tilde{\wp} * \Lambda$. The exterior of $T * \Gamma$ is defined as the interior of $(T * \Gamma)^\gamma$ and it is symbolized by $(T * \Gamma)_\delta$. In summary, this relationship is expressed as:

$$(T * \Gamma)_\delta = (T * \Gamma)^\gamma.$$

This notation indicates that the exterior consists of all points in the universe that are not in $T * \Gamma$ and can be surrounded by open sets in the AS topology.

Definition 2.22. [34] Let $\tilde{G} * \tilde{\wp} * \Lambda$ and g_α be given. If $T * \Gamma$ is an AS open set such that $g_\alpha \in T * \Gamma$, then the open neighbourhood of g_α is defined as $T * \Gamma$. This is denoted by $\tilde{\wp}(g_\alpha)$. This definition highlights the specific open set containing the monad point g_α within the AS topology, emphasizing its local context.

Definition 2.23. [36] Let $\tilde{G} * \tilde{\wp} * \Lambda$ and g_α be given. If every amply soft open set including g_α also includes at least one monad point different from g_α from $T * \Gamma$, then g_α is said to be a limit monad point of $T * \Gamma$. In other words, let $V * Y$ be an AS open set such that $V * Y \tilde{\in} \tilde{\wp}(g_\alpha)$. g_α is considered a limit monad point of $T * \Gamma$ if

$$V * Y \tilde{\cap} (T * \Gamma \setminus \{g_\alpha\}) \tilde{\neq} \tilde{\wp}.$$

Then, the AS set of entire limit monad points of $T * \Gamma$ is denoted by $(T * \Gamma)^{lm}$.

Proposition 2.24. [36] Let $\tilde{G} * \tilde{\wp} * \Lambda$ and $T * \Gamma$ be given. Then, the equation

$$T * \Gamma \tilde{\cup} (T * \Gamma)^{lm} \cong \overline{T * \Gamma}$$

is hold.

3. Results and Discussion

This section consists of two subsections and includes the main topics of our article. In the first section, we introduced monad discrete spaces and investigated limit monad points on these spaces. In the second section, we investigated limit monad points on PAS topological spaces.

3.1. Monad Discrete Spaces and Investigating Limit Monad Points on Them

Definition 3.1.1 Let us consider g_α as a monad point of $\tilde{G}, g \in G, \alpha \in \Lambda$. The AS topological space $\tilde{G} * \tilde{\wp} * \Lambda$ obtained by adding the $\tilde{\wp}$ to the structure where all monad points g_α on \tilde{G} are considered as the AS basis is called the monad discrete space.

Example 3.1.2 Let us consider the monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, defined by $G = \{1,2,3\} \subset \mathbb{R}$ and $\Lambda = \{1,2,3,4\} \subset \mathbb{R}$. We will explore the limit monad points of the AS set

$$T * \Gamma = \{\{1\}_{\{1,2\}}, \{1,3\}_{\{4\}}\}$$

Analyzing limit monad points:

We note that $1_1 \in \{1\}_{\{1\}}$, which is classified as an AS open set according to the referenced Definition 3.1.1. To determine if 1_1 is a limit monad point, we compute:

$$\begin{aligned} & \{1\}_{\{1\}} \tilde{\cap} (T * \Gamma \setminus \{1\}_{\{1\}}) \\ & \cong \{1\}_{\{1\}} \tilde{\cap} \{\{1\}_{\{2\}}, \{1,3\}_{\{4\}}\} \cong \tilde{\wp} \end{aligned}$$

Since this intersection is empty AS set, we conclude that 1_1 is not a limit monad point of $T * \Gamma$.

Similarly, it can be verified that both $1_2, 1_4$ and 3_4 also satisfy the condition for not being limit monad points. So

let us see if 1_3 is a limit monad point. $1_3 \tilde{\in} \{1\}_{\{3\}}$. Because of

$$\begin{aligned} & \{1\}_{\{3\}} \cap (T * \Gamma \setminus \{1\}_{\{3\}}) \\ & \cong \{1\}_{\{3\}} \tilde{\cap} \{\{1\}_{\{1,2\}}, \{1,3\}_{\{4\}}\} \cong \tilde{\wp}, \end{aligned}$$

1_3 is not a limit monad point of $T * \Gamma$.

Consequently, we find that there are no limit monad points of $T * \Gamma$ then

$$(T * \Gamma)^{lm} \cong \tilde{\wp}.$$

Proposition 3.1.3 In a monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, it can be established that there are no limit monad points for any AS set defined within \tilde{G} .

Proof Let us assume the monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, and $g_\alpha, \alpha \in \Lambda, T * \Gamma, \Gamma \subseteq \Lambda$ be given. Then $\{g\}_{\{\alpha\}}$ is an AS open set that included g_α from the Definition 3.1.1. There are two different situations: $g_\alpha \tilde{\in} T * \Gamma$ and $g_\alpha \tilde{\notin} T * \Gamma$. If $g_\alpha \tilde{\in} T * \Gamma$, then $g_\alpha \tilde{\notin} (T * \Gamma \setminus \{g\}_{\{\alpha\}})$. That is,

$$\{g\}_{\{\alpha\}} \tilde{\cap} (T * \Gamma \setminus \{g\}_{\{\alpha\}}) \cong \tilde{\wp}.$$

If $g_\alpha \tilde{\notin} T * \Gamma$, then $g_\alpha \tilde{\notin} (T * \Gamma \setminus \{g\}_{\{\alpha\}})$. That is,

$$\{g\}_{\{\alpha\}} \tilde{\cap} (T * \Gamma \setminus \{g\}_{\{\alpha\}}) \cong \tilde{\wp}.$$

Therefore, g_α can not be a limit monad point of $T * \Gamma$. Thus,

$$(T * \Gamma)^{lm} \cong \tilde{\wp}. \quad \blacksquare$$

Proposition 3.1.4 Let us consider the monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, g_α be given. Then, $\{g\}_{\{\alpha\}}$ is both an AS open and AS closed set.

Proof Let us assume the monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, and $g_\alpha, \alpha \in \Lambda$ be given. Then $\{g\}_{\{\alpha\}}$ is an AS open set that included g_α from the Definition 3.1.1. Then,

$$(\{g\}_{\{\alpha\}})^{\tilde{\vee}} \cong \{(G - \{g\})_{\{\alpha\}}, G_{\Lambda - \{\alpha\}}\}$$

is an element of $\tilde{\wp}$ from Definition 3.1.1 and 2.18. That is, $(\{g\}_{\{\alpha\}})^{\tilde{\vee}}$ is an AS open set so $\{g\}_{\{\alpha\}}$ is an AS closed set. \blacksquare

Corollary 3.1.5 Let us consider the monad discrete space $\tilde{G} * \tilde{\wp} * \Lambda$, g_α be given. Then,

$$\overline{\{g\}_{\{\alpha\}}} \cong \{g\}_{\{\alpha\}}$$

Proof It is clear to seen from Proposition 3.1.4 and the Definition 2.16 respectively. \blacksquare

Proposition 3.1.6 Let us consider the monad discrete space $\tilde{G} * \tilde{\varphi} * \Lambda, T * \Gamma$ be given. Then, $T * \Gamma$ is both an AS open and AS closed set.

Proof Let us consider the monad discrete space $\tilde{G} * \tilde{\varphi} * \Lambda, T * \Gamma$ and g_α be given. Then $T * \Gamma$ is an AS open set that included for all $g_\alpha, \alpha \in \Gamma$ from the Definition 3.1.1. Then,

$$(T * \Gamma)^{\tilde{\varphi}} \cong \{(G - T(\lambda))_\Gamma, G_{\Lambda - \Gamma}\}, \lambda \in \Gamma$$

is an element of $\tilde{\varphi}$ from Definition 3.1.1 and 2.18. That is, $(T * \Gamma)^{\tilde{\varphi}}$ is an AS open set so $T * \Gamma$ is an AS closed set. ■

Corollary 3.1.7 Let us consider the monad discrete space $\tilde{G} * \tilde{\varphi} * \Lambda, T * \Gamma$ be given. Then,

$$\overline{T * \Gamma} \cong T * \Gamma.$$

Proof It is clear to seen from Proposition 3.1.6 and the Definition 2.16 respectively. ■

Example 3.1.6 Let us assume a monad discrete space $\tilde{\mathbb{R}} * \tilde{\varphi} * \mathbb{R}$. We will explore the limit monad points of the AS sets defined as follows:

$$\begin{aligned} & [2,3]_{(1,4)}, \\ & \{1,2,3,4\}_{\{1,3,5\}}, \\ & \{\mathbb{Q}_{\{1\}}, I_{\{2\}}\}, \\ & (-5,10]_{1,2}. \end{aligned}$$

For these AS sets, we can calculate their respective AS sets of limit monad points as follows based on Proposition 3.1.3:

$$\begin{aligned} ([2,3]_{(1,4)})^{lm} & \cong \tilde{\varphi}, \\ (\{1,2,3,4\}_{\{1,3,5\}})^{lm} & \cong \tilde{\varphi}, \\ (\{\mathbb{Q}_{\{1\}}, I_{\{2\}}\})^{lm} & \cong \tilde{\varphi}, \\ ((-5,10]_{1,2})^{lm} & \cong \tilde{\varphi}. \end{aligned}$$

Now let us find the closures of the above mentioned AS sets, respectively, based on Proposition 2.22. Also we can find easily from Corollary 3.1.7.

$$\begin{aligned} \overline{[2,3]_{(1,4)}} & \cong [2,3]_{(1,4)} \cup ([2,3]_{(1,4)})^{lm} \\ & \cong [2,3]_{(1,4)} \cup \tilde{\varphi} \cong [2,3]_{(1,4)}, \\ \overline{\{1,2,3,4\}_{\{1,3,5\}}} & \cong \{1,2,3,4\}_{\{1,3,5\}} \cup (\{1,2,3,4\}_{\{1,3,5\}})^{lm} \\ & \cong \{1,2,3,4\}_{\{1,3,5\}} \cup \tilde{\varphi} \cong \{1,2,3,4\}_{\{1,3,5\}}, \\ \overline{\{\mathbb{Q}_{\{1\}}, I_{\{2\}}\}} & \cong \{\mathbb{Q}_{\{1\}}, I_{\{2\}}\} \cup (\{\mathbb{Q}_{\{1\}}, I_{\{2\}}\})^{lm} \\ & \cong \{\mathbb{Q}_{\{1\}}, I_{\{2\}}\} \cup \tilde{\varphi} \cong \{\mathbb{Q}_{\{1\}}, I_{\{2\}}\}, \\ \overline{(-5,10]_{1,2}} & \cong (-5,10]_{1,2} \cup ((-5,10]_{1,2})^{lm} \\ & \cong (-5,10]_{1,2} \cup \tilde{\varphi} \cong (-5,10]_{1,2}. \end{aligned}$$

3.2. Investigating Limit Monad Points on PAS Topological Spaces

In the following example, we will construct an AS topology based on various classical topologies known on the real numbers, utilizing the framework of the PAS topology. Subsequently, we will explore the limit monad points within the specified AS sets. This approach will allow us to examine the interplay between different topological structures and identify key features of the AS sets in question.

Example 3.2.1 Let \mathbb{R} represent the set of real numbers, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ denote a universal set of parameters. In this context, we consider three distinct topological structures: (\mathbb{R}, τ_1) as a standard topological space, (\mathbb{R}, τ_2) as a finite complement topological space, and (\mathbb{R}, τ_3) as a discrete topological space. According to Definition 2.17 the PAS topology established over \mathbb{R} is denoted as $\tilde{\mathbb{R}} * \tilde{\varphi} * \Lambda$.

Let us define the AS sets

$$T * \Gamma = (2,5)_\Lambda \text{ and } S * \Omega = \{(3,7)_{\{\lambda_1, \lambda_2\}}, \{7,9,12\}_{\{\lambda_3\}}\}.$$

Our goal is to identify their respective AS sets of their limit monad points. Let $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

We choose the AS set $(2 - \varepsilon, 2 + \varepsilon)_{\{\lambda_1\}} \tilde{\in} \tilde{\vartheta}(2_{\lambda_1})$. We then analyze the intersection:

$$\begin{aligned} & (2 - \varepsilon, 2 + \varepsilon)_{\{\lambda_1\}} \tilde{\cap} (T * \Gamma \setminus \{2\}_{\{\lambda_1\}}) \\ & \cong (2 - \varepsilon, 2 + \varepsilon)_{\{\lambda_1\}} \tilde{\cap} (2,5)_\Lambda \cong (2,2 + \varepsilon)_{\{\lambda_1\}} \not\cong \tilde{\varphi}. \end{aligned}$$

This indicates that 2_{λ_1} is a limit monad point of $T * \Gamma$.

Similarly, we conclude that 3_{λ_1} is a limit monad point of $S * \Omega$ eighter. Next, we choose the AS set

$$(3 - \varepsilon, 3 + \varepsilon)_{\{\lambda_1\}} \tilde{\in} \tilde{\vartheta}(3_{\lambda_1}).$$

We then analyze the intersection:

$$\begin{aligned} & (3 - \varepsilon, 3 + \varepsilon)_{\{\lambda_1\}} \tilde{\cap} (T * \Gamma \setminus \{3\}_{\{\lambda_1\}}) \\ & \cong (3 - \varepsilon, 3 + \varepsilon)_{\{\lambda_1\}} \tilde{\cap} \{(2,3) \cup (3,5)_{\{\lambda_1\}}, (2,5)_{\{\lambda_2, \lambda_3\}}\} \\ & \cong (3 - \varepsilon, 3) \cup (3,3 + \varepsilon)_{\{\lambda_1\}} \not\cong \tilde{\varphi}. \end{aligned}$$

That is, 3_{λ_1} is a limit monad point of $T * \Gamma$.

Moving on to the AS set

$$(\mathbb{R} - \{4\})_{\{\lambda_2\}} \tilde{\in} \tilde{\vartheta}(3_{\lambda_2}).$$

Now, we analyze the intersection:

$$\begin{aligned} & (\mathbb{R} - \{4\})_{\{\lambda_2\}} \tilde{\cap} (T * \Gamma \setminus \{3\}_{\{\lambda_2\}}) \\ & \cong (-\infty, 4) \cup (4, \infty)_{\{\lambda_2\}} \tilde{\cap} \{(2,5)_{\{\lambda_1, \lambda_3\}}, (2,3) \cup (3,5)_{\{\lambda_2\}}\} \\ & \cong (2,3) \cup (3,4) \cup (4,5)_{\{\lambda_2\}} \not\cong \tilde{\varphi}. \end{aligned}$$

This shows that 3_{λ_2} is a limit monad point of $T * \Gamma$. Similarly, we can conclude that 3_{λ_2} is also a limit monad point of $S * \Omega$ eighter.

Next, we evaluate the AS set

$$\{4\}_{\{\lambda_3\}} \tilde{\in} \tilde{\vartheta}(4_{\lambda_3})$$

we then analyze the intersection:

$$\begin{aligned} & \{4\}_{\{\lambda_3\}} \tilde{\cap} (T * \Gamma \setminus \{4\}_{\{\lambda_3\}}) \\ &= \{4\}_{\{\lambda_3\}} \tilde{\cap} (2,4) \cup (4,5)_{\{\lambda_3\}} \cong \tilde{\phi}. \end{aligned}$$

Thus, we can summarize our findings as follows:

$$\begin{aligned} (T * \Gamma)^{lm} &\cong \{[2,5]_{\{\lambda_1\}}, \mathbb{R}_{\{\lambda_2\}}\}, \\ (S * \Omega)^{lm} &\cong \{[3,7]_{\{\lambda_1\}}, \mathbb{R}_{\{\lambda_2\}}\}. \end{aligned}$$

This analysis allows us to further explore the topological properties and the nature of the limit monad points within these AS sets.

Proposition 3.2.2 Let $\tilde{G} * \tilde{\wp} * \Lambda$ denote a PAS topological space. When we aim to identify the limit monad points of any AS set $T * \Gamma$ within this PAS topology, we adopt a methodical approach: we calculate the limit points associated with each parameter of $T * \Gamma$ independently. For each parameter in $T * \Gamma$, the limit monad points are determined based on the topology defined by the specific parameter. These points are then aggregated and expressed as an AS set, allowing for a comprehensive representation of the limit monad points across all parameters involved. In other words, assume that $(T(\lambda))^l$ represent set of limit points of set $T(\lambda)$ in (G, τ_{λ_n}) topological spaces.

$$(T * \Gamma)^{lm} \cong \{(T(\lambda))^l \text{ in } (G, \tau_{\lambda_n}), \lambda \in \Gamma, i \in \mathbb{N}\}$$

Proof Let $\tilde{G} * \tilde{\wp} * \Lambda$ represent a PAS topological space. According to Definition 2.17, there exists $n \in \mathbb{N}$ and $\lambda_n \in \Lambda$. We consider (G, τ_n) as topological spaces defined over the same initial universe G , with a mapping f from τ_n to τ_{λ_n} for all $\lambda \in \Lambda$. The collection $\tilde{\wp} \cong \{\tau_{\lambda_n}\}$ serves as an AS basis for the PAS topology $\tilde{\wp}$. Now, let $g_i \in G$ be a limit point of a set U_i in the topological space (G, τ_i) where $i \in \mathbb{N}$. This implies there exists an open set O_i in (G, τ_i) such that

$$(U_i - \{g_i\}) \cap O_i \neq \phi.$$

Because O_i is an open set in each (G, τ_i) , it follows that $O_{\{\lambda_i\}}$ is an AS open set in the PAS topological space $\tilde{G} * \tilde{\wp} * \Lambda$ as per Definition 2.17.

Given that $g_i \in G$ and $\lambda_i \in \Lambda$, we have g_{λ_i} classified as a monad point according to Definition 2.12. Consequently, since

$$(U_{\{\lambda_i\}} - \{g_{\{\lambda_i\}}\}) \tilde{\cap} \{O_{\{\lambda_i\}}\} \not\cong \tilde{\phi},$$

it can be concluded that g_{λ_i} is a limit monad point of the AS set $U_{\{\lambda_i\}}$. This reasoning holds true for all $g \in G, \lambda \in \Lambda$ and $i \in \mathbb{N}$. ■

We will not address the following problem, as it falls outside the scope of this study. However, we will present

it as a question for those who may be interested in exploring it further.

Problem 3.2.3 In the context of a PAS topological space $\tilde{G} * \tilde{\wp} * \Lambda$, when we calculate the interior points of any AS set $T * \Gamma$, can we determine these points by collecting the interior points associated with the topology corresponding to each parameter of the given the AS set $T * \Gamma$, and then expressing them in an AS set format? Furthermore, can we apply the same reasoning to the calculation of exterior points and closure points?

4. Conclusion

Jyothis and Sunil [20] showed that in any variable parameter classical soft topological spaces, the closure of any soft set may not contain soft limit points. This situation limited the studies in variable parameter soft topological spaces. In other words, if a closed set does not contain accumulation points, its being a closed set does not have much meaning in the topological sense. In this study, we gave many properties related to accumulation points due to the fact that amply soft sets and hence monad discrete and PAS spaces can be selected more comprehensively. We tested these properties especially on these amply soft topologies created by selecting sub-intervals of real numbers and even any subsets of real numbers while creating amply soft sets and showed their accuracy.

Especially when we consider infinite parameters, it may give meaning to the fact that subatomic particles can be in different places at the same time. The physicist Michio Kaku [37] declares in his book name is Future of Humanity that, introduces the concept of quantum fuzziness, emphasizing that quantum particles inherently lack precise localization. He explains that an electron, for instance, is not confined to a single position or velocity but instead exists as a superposition of states each corresponding to a distinct location and motion. Kaku notes that Einstein strongly resisted this interpretation, favoring a deterministic universe where the properties of particles could be known exactly.

Quantum theory, however, suggests a more probabilistic framework. According to Kaku, when one looks in a mirror, the reflection is not an image of a singular, welldefined self but rather the sum of countless wave functions. Some of these waves might scatter, extending into the environment or even traveling through space. It is theoretically possible that certain wave components might reach Mars or even beyond. These phenomena are called quantum fluctuations.

In this context, we believe that the concept of limit monad points of PAS spaces discussed in this study will allow for broader applications. For example, in the fields mentioned by Kaku above, a method could be developed

by considering the existence of a point with infinite parameters.

In addition to the usability of existing structures, amply soft topologies will demonstrate their uniqueness in the unlimited range of parameters. This can be exemplified by the unlimited distribution of subatomic particles or the unlimited range of time. For example, we believe that, instead of applications where time is considered a static instant (for an example [38]), may be selecting specific variations at an unlimited instant will yield more consistent results across a wider range of applications.

In this study, monad discrete spaces are defined on amply soft topologies which is a more flexible and comprehensive structure from other known traditional soft topologies. The conditions under which a given amply soft set is amply soft open or amply soft closed set are given over them. Then, the amply soft set of limit monad points and closure points of any amply soft set in monad discrete spaces is investigated and the relevant properties are given. And then, as an extra, limit monad points in amply soft topological spaces produced by classical topologies are investigated and the relevant property is given. While constructing amply soft sets, specifically focused on amply soft topologies formed by selecting real numbers from both universal and parametric spaces unlike the traditional soft sets. Finally, a question is left that we hope will catch the reader's attention

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Author's Contributions

The author solely contributed to the conception, design, analysis, and writing of this manuscript.

Ethics

There are no ethical issues after the publication of this manuscript journal.

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