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Research Article

Transforming Skew Cyclic Codes into Generalized Quasi-Cyclic Codes via a New Gray Map over $\mathbb{Z}_4 + u\mathbb{Z}_4$

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Abstract

We introduce a new Gray map $Q(a+ub)=(a,a+3b)$, along with an automorphism $\theta(a+ub)=a+u(3b)$. Using these, we construct quasi-cyclic codes as left submodules of the skew polynomial ring $R[x;\theta]<x^n-1>$ over $R=\mathbb{Z}_4+u\mathbb{Z}_4$. Although the original codes are skew cyclic, we show that their Gray images under Q are invariant under a modified cyclic shift operator, and hence are generalized quasi-cyclic codes over $\mathbb{Z}_4$. Our analysis of the Lee weight transformation, supported by examples, demonstrates that this approach yields codes with predictable structure and favorable properties.

Keywords: codes, quasi-cyclic codes, Gray map.

$\mathbb{Z}_4 + u\mathbb{Z}_4$ Üzerinde Yeni Bir Gray Dönüşümü Kullanarak Eğik Devirli Kodların Genelleştirilmiş Yarı Devirli Kodlara Dönüştürülmesi

Öz

Bu çalışmada yeni bir Gray dönüşümü olan $Q(a+ub)=(a,a+3b)$ ve $\theta(a+ub)=a+u(3b)$ otomorfizması tanıtılmaktadır. Bu dönüşümler kullanılarak, $R=\mathbb{Z}_4+u\mathbb{Z}_4$ halkası üzerinde tanımlı, eğik devirli (skew cyclic) kodların, $R[x;\theta]<x^n-1>$ çokterimli halkasında sol altmodül olarak oluşturulması sağlanarak, elde edilen kodların Gray dönüşümleri ile yarı devirli (quasi-cyclic) kodlar elde edilmiştir. Aslında orijinal kodlar eğik devirli iken, Gray görüntüleri altında, değiştirilmiş bir devirli kaydırma operatörüne karşı değişmezlik sağlanmakta; böylece $\mathbb{Z}_4$ üzerinde genelleştirilmiş yarı devirli kodlar oluşmaktadır. Lee ağırlık dönüşümünün kuramsal analizi ve örnekler, önerilen yöntemin öngörülebilir yapıya ve istenen özelliklere sahip kodlar ürettiğini göstermektedir.

Anahtar Kelimeler: kodlar, yarı devirli kodlar, Gray dönüşümü.

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1. Introduction

Cyclic and skew cyclic codes over finite rings have attracted considerable attention due to their rich algebraic structure and applications [2], [6], [7]. In particular, the ring $Z_4 + uZ_4$, where $u^2 = 0$, has been extensively studied for its ability to produce codes with remarkable properties (see, [3], [4] for further information). Gray maps are essential tools that allow one to transform codes over such rings into codes over smaller alphabets.

In this paper, we introduce a new Gray map $Q(a + ub) = (a, a + 3b)$, which, together with the automorphism $\theta(a + ub) = a + u(3b)$, provides a novel method for constructing new codes and analyzing skew cyclic codes over $Z_4 + uZ_4$. We show that although the Gray image under Q is, in general, quasi-cyclic, it is invariant under a modified cyclic shift operator. This invariance enables us to transfer structural properties from the original code to its Gray image, facilitating theoretical analysis.

The paper is organized as follows. In Section 2, we introduce the necessary preliminaries and definitions. Section 3 presents the skew polynomial ring and skew cyclic codes over $Z_4 + uZ_4$. In Section 3, we also examine the cyclic structure preservation under the new Gray map. Section 4 is devoted to the analysis of the weight enumerator and minimum distance. Duality of these codes are discussed in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

Definition 2.1: Let Z_4 denote the ring of integers modulo 4. The ring $Z_4 + uZ_4 = \{a + ub: a, b \in Z_4, u^2 = 0\}$ is a commutative ring with unity, where addition and multiplication are defined by $(a + ub) + (c + ud) = (a + c) + u(b + d)$ and $(a + ub)(c + ud) = ac + u(ad + bc)$ respectively, with all operations over Z_4.

Definition 2.2: The Lee weight on Z_4 is defined by: $w_L(0) = 0, w_L(1) = w_L(3) = 1, w_L(2) = 2$. For an element $z = a + ub \in Z_4 + uZ_4$, the Lee weight is extended as: $w_L(z) = w_L(a) + w_L(b)$.

Gray maps provide a method to transform codes over rings to codes over smaller alphabets while preserving or modifying distance properties. In the literature on codes over $Z_4 + uZ_4$, some Gray maps have been studied. For example, the traditional Gray map in [3] consider $Q_1(a + ub) = (b, a + b)$, and have been used to investigate cyclic and linear codes over $Z_4 + uZ_4$. Furthermore, Yildiz and Karadeniz [5] proposed some alternate Gray maps, based on binary decompositions of the elements of Z_4. These maps provide alternative ways to construct codes over larger alphabets from codes over smaller alphabets. Now, we introduce a new Gray map and analyze its properties.

Definition 2.3: Let the new gray map $Q: \mathbb{Z}_4 + u\mathbb{Z}_4 \to \mathbb{Z}_4^2$ be defined with $Q(a + ub) = (a, a + 3b)$. Since the matrix representation of Q is $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$, and its determinant is 3 (which is a unit in $\mathbb{Z}_4$), the map Q is a linear bijection. The introduction of the factor 3 in the second coordinate differentiates Q from previously studied Gray maps and leads to a distinct transformation of the Lee weight. In particular, for an element $c = a + ub$, we have $w_L(Q(c)) = w_L(a) + w_L(a + 3b)$. A case analysis shows that while in some cases this equals $2w_L(c)$ (for instance, when $b = 0$), in general it is a piecewise linear function of a and b. This transformation, when coupled with an appropriate automorphism described in the next section, yields a Gray image of a skew cyclic code that is invariant under a modified cyclic shift operator. The resulting code is shown to be a generalized quasi-cyclic code.

3. Skew Polynomial Rings and Skew Cyclic Codes

To construct skew cyclic codes, we first need an automorphism on the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$. Our choice of automorphism is motivated by the need to preserve the structure of the ring while interacting well with new Gray map Q.

Definition 3.1: Let $\theta: \mathbb{Z}_4 + u\mathbb{Z}_4 \to \mathbb{Z}_4 + u\mathbb{Z}_4$, we define the necessary automorphism with $\theta(a + ub) = a + u(3b)$ for all $a, b \in \mathbb{Z}_4$.

This automorphism aids in the later construction of the skew polynomial ring $R[x, \theta]$ and ensures that the cyclic structure of codes is maintained in a controlled manner under the skew shift. We now establish the following theorem to prove that θ is indeed a ring automorphism.

Theorem 3.2: The mapping $\theta: \mathbb{Z}_4 + u\mathbb{Z}_4 \to \mathbb{Z}_4 + u\mathbb{Z}_4$ with $\theta(a + ub) = a + u(3b)$ is a ring automorphism.

Proof: We must show that θ is a ring homomorphism and bijective.

i. Homomorphism: Let $x = a + ub$ and $y = c + ud$ be arbitrary elements in $\mathbb{Z}_4 + u\mathbb{Z}_4$.

Additivity: $x + y = (a + ub) + (c + ud) = (a + d) + u(b + d)$. Then

$$\theta(x + y) = (a + c) + u\big(3(b + d)\big).$$

On the other hand, $\theta(x) + \theta(y) = (a + u(3b)) + (c + u(3d)) = (a + c) + u\big(3(b + d)\big)$. Thus, $\theta(x + y) = \theta(x) + \theta(y)$.

Multiplicativity: Recall that $(a + ub)(c + ud) = ac + u(ad + bc)$, since $u^2 = 0$. Then, $\theta(xy) = ac + u\big(3(ad + bc)\big)$. Also,

$$\theta(x)\theta(y) = (a + u(3b))(c + u(3d)) = (ac) + u\big(3(ad + bc)\big).$$

Thus, $\theta(xy) = \theta(x)\theta(y)$.

ii. Bijectivity: Note that $\theta(\theta(a + ub)) = \theta(a + u(3b)) = a + u(9b)$. Since $9 \equiv 1\ mod(4)$, it follows that $\theta(\theta(a + ub)) = \theta(\theta(a + ub))$. Hence, θ is its own inverse and is bijective.

Thus, θ is a ring automorphism. ■

Definition 3.3: The skew polynomial ring $R[x;\theta]$ is defined as:

$$R[x;\theta] = \{f(x) = a_0 + a_1 x + \cdots + a_k x^k \mid a_i \in R, k \geq 0\},$$

with addition as usual and multiplication determined by the rule:

$$x \cdot a = \theta(a) \cdot x, \forall a \in R.$$

Definition 3.4: A linear code $C \subset R^n$ is called skew cyclic if for every codeword $c = (c_0, c_1, \dots, c_{n-1}) \in C$ the skew cyclic shift $\sigma(c) = (\theta(c_{n-1}), c_0, c_1, \dots, c_{n-2})$ is also in C. Equivalently, skew cyclic codes can be viewed as left submodules of the quotient ring $R[x;\theta]/< x^n - 1 >$.

Theorem 3.5: [Cyclic Structure Preservation]: Let C be a skew cyclic code of length n over R with respect to the automorphism θ defined above. Then the Gray image $Q(C) \subset \mathbb{Z}_4^{2n}$ is invariant under the operator S defined as follows. For a Gray image codeword $Q(c) = (v_0, v_1, \dots, v_{n-1})$ with $v_i = Q(c_i) = (a_i, a_i + 3b_i)$ define the operator T on $\mathbb{Z}_4^2$ by

$$T(a_i, a_i + 3b_i) = (a_i, a_i + b_i)$$

and then define the shift operator S by

$$S = (v_0, v_1, \dots, v_{n-1}) = (T(v_{n-1}), v_0, v_1, \dots, v_{n-2}).$$

Also, let the skew cyclic shift on C be defined by $\sigma(c) = (\theta(c_{n-1}), c_0, c_1, \dots, c_{n-2})$. Then for every $c \in C$,

$$S(Q(c)) = Q(\sigma(c)) \in Q(C).$$

Thus, $Q(C)$ is a generalized quasi-cyclic code over $\mathbb{Z}_4$.

Proof. Let $c = (c_0, c_1, \dots, c_{n-1}) \in C$ with $c_i = a_i + ub_i$. By definition,

$$Q(c) = (Q(c_0), Q(c_1), \dots, Q(c_{n-1})),$$

where $Q(c_i) = (a_i, a_i + 3b_i)$. Since C is skew cyclic, the shifted codeword $\sigma(c) = (\theta(c_{n-1}), c_0, c_1, \dots, c_{n-2})$ is in C. Noting that $\theta(c_{n-1}) = a_{n-1} + u(3b_{n-1})$, we have

$$Q(\theta(c_{n-1})) = (a_{n-1}, a_{n-1} + 9b_{n-1}).$$

Since $9 \equiv 1 \ mod\ 4$, it follows that $Q(\theta(c_{n-1})) = (a_{n-1}, a_{n-1} + b_{n-1})$. By the definition of T,

$$T(Q(c_{n-1})) = T(a_{n-1}, a_{n-1} + 3b_{n-1}) = (a_{n-1}, a_{n-1} + b_{n-1}).$$

Hence,

$$S(Q(c)) = \left(T(Q(c_{n-1})), Q(c_0), Q(c_1), \dots, Q(c_{n-2})\right) = Q(\sigma(c)).$$

This completes the proof. ■

Example 3.6. Let $C \subset R^2$ be the linear code generated by $G_1 = (1 \quad u)$. Every codeword in C is of the form $c = x(1, u)$, with $x \in R$. For example, choose $x = 1$ so that $c = (1, u)$. Writing $1 + u \cdot 0$ for the first coordinate and $0 + u \cdot 1$ for the second, we have:

$$Q(1) = Q(1 + u0) = (1,1), \qquad Q(u) = Q(0 + u \cdot 1) = (0,3).$$

Thus, $Q(c) = \big((1,1), (0,3)\big)$. The skew cyclic shift of $c = (1, u)$ is $\sigma(c) = (\theta(u), 1)$, with $\theta(u) = \theta(0 + 1 \cdot u) = 3u$. Since $Q(3u) = Q(0 + 3u) = (0,9) \equiv (0,1)\ mod\ 4$, we obtain

$$Q(\sigma(c)) = \big((0,1), (1,1)\big).$$

On the other hand, applying the operator S to $Q(c)$ gives $S\big((1,1), (0,3)\big) = \Big(T\big((0,3)\big), (1,1)\Big)$, where $T\big((0,3)\big) = (0,1)$. Thus,

$$S\big(Q(c)\big) = \big((0,1), (1,1)\big) = Q(\sigma(c)).$$

4. Lee Weight and Weight Enumerator Analysis

Recall that, for an element $c = a + ub \in R$ with $a, b \in \mathbb{Z}_4$, $Q(c) = (a, a + 3b)$ and $w_L\big(Q(c)\big) = w_L(a) + w_L(a + 3b)$. A careful case analysis is given below:

Case 1: $b = 0$. Then since $a + 3 \cdot 0 = a$, $w_L\big(Q(a + u0)\big) = w_L(a) + w_L(a) = 2w_L(a)$.

Case 2: $b = 1$. Then:

If $a = 0$: $w_L(0) + w_L(3) = 0 + 1 = 1$,

If $a = 1$: $w_L(1) + w_L(1 + 3) = 1 + 0 = 1$,

If $a = 2$: $w_L(2) + w_L(2 + 3) = 2 + 1 = 3$,

If $a = 3$: $w_L(3) + w_L(3 + 3) = 1 + 2 = 3$.

Case 3: $b = 2$. Then $a + 3 \cdot 2 \equiv a + 2\ mod\ 4$ and:

If $a = 0$: $w_L(0) + w_L(2) = 0 + 2 = 2$,

If $a = 1$: $w_L(1) + w_L(3) = 1 + 1 = 2$,

If $a = 2$: $w_L(2) + w_L(0) = 2 + 0 = 2$,

If $a = 3$: $w_L(3) + w_L(1) = 1 + 1 = 2$.

Case 4: $b = 3$. Then $a + 3 \cdot 3 \equiv a + 1\ mod\ 4$ and:

If $a = 0$: $w_L(0) + w_L(1) = 0 + 1 = 1$,

If $a = 1$: $w_L(1) + w_L(2) = 1 + 2 = 3$,

If $a = 2$: $w_L(2) + w_L(3) = 2 + 1 = 3$,

If $a = 3$: $w_L(3) + w_L(0) = 1 + 0 = 1$.

Thus, while in the cases $b = 0$ and $b = 2$, the Lee weight of the Gray image is given by a uniform value (in the sense that when $b = 0$ it equals $2w_L(a)$, in the cases $b = 1$ and $b = 3$ the outcome depends on a in a piecewise linear manner. Note that the original Lee weight of $a + ub$ is $w_L(a) + w_L(b)$. In many structured instances (for example, when the b-component is 0) one may observe $w_L(Q(a + ub)) = 2w_L(a + ub)$, which implies that the minimum Lee distance of $Q(C)$ is twice that of C in these cases. In general, however, the transformation must be analyzed on a case-by-case basis.

Let $C \subset R^n$ be a skew cyclic code and denote its Lee weight enumerator by

$$W_C(x, y) = \sum_{c \in C} x^{n - w_L(c)} y^{w_L(c)}.$$

To relate the weight enumerators of C and its Gray image $Q(C)$, define for each coordinate $c_i = a_i + ub_i$ the transformed weight

$$\phi(a_i, b_i) = w_L(a_i) + w_L(a_i + 3b_i).$$

For a codeword $c = (c_1, c_2, \dots, c_n)$, set $\Phi(c) = \sum_{i=1}^{n} \phi(a_i, b_i)$. Then the Lee weight of $Q(c)$ is exactly $\Phi(c)$, and the weight enumerator of $Q(C)$ is given by

$$W_{Q(C)}(x, y) = \sum_{c \in C} x^{2n - \Phi(c)} y^{\Phi(c)}.$$

Theorem 4.1. [Weight Enumerator Transformation]: Let $C \subset R^n$ be a skew cyclic code and let the Gray map Q be defined as above. Then the Lee weight enumerator of the Gray image $Q(C)$ is

$$W_{Q(C)}(x, y) = \sum_{c \in C} x^{2n - \Phi(c)} y^{\Phi(c)},$$

Where $\Phi(c) = \sum_{i=1}^{n} [w_L(a_i) + w_L(a_i + 3b_i)]$ for $c = (a_1 + ub_1, \dots, a_n + ub_n)$. In particular, if every codeword $c \in C$ satisfies $b_i = 0$ for all i (i.e., each coordinate is of the form $a + u.0$), then $\phi(a_i, 0) = 2w_L(a_i)$, and consequently,

$$W_{Q(C)}(x, y) = \sum_{c \in C} x^{2n - 2w_L(c)} y^{2w_L(c)} = W_C(x^2, y^2).$$

Proof. For each coordinate $c_i = a_i + ub_i$, the Gray map gives $Q(c_i) = (a_i, a_i + 3b_i)$, and by definition, the Lee weight is $w_L(Q(c_i)) = w_L(a_i) + w_L(a_i + 3b_i) = \phi(a_i, b_i)$. Since the Gray map is applied coordinatewise, the total Lee weight of $Q(c)$ is $\Phi(c) = \sum_{i=1}^{n} \phi(a_i, b_i)$. The weight enumerator of $Q(C)$ is then

$$W_{Q(C)}(x, y) = \sum_{c \in C} x^{2n-\Phi(c)} y^{\Phi(c)}.$$

In the special case when $b_i = 0$ for all coordinates, we have $a_i + 3 \cdot 0 = a_i$, so that $\phi(a_i, 0) = 2w_L(a_i)$. Since the original Lee weight of $c_i = a_i + u0$ is $w_L(c_i) = w_L(a_i)$, it follows that

$$W_{Q(C)}(x, y) = \sum_{c \in C} x^{2n-2w_L(c)} y^{2w_L(c)} = W_C(x^2, y^2).$$

This completes the proof.■

Example 4.2: Consider the ring $Z_4 + uZ_4, u^2 = 0$, with the automorphism θ. Let $g(x) = x + (1 + u)$ be a generator polynomial in the quotient $R[x; \theta]/< x^2 - 1 >$. Since $\deg(g(x)) = 1$, every codeword in the skew cyclic code $C = \{q(x)g(x) \bmod (x^2 - 1) | \, q(x) \in R[x; \theta]$, $deq(q(x)) = 1\}$, is obtained by multiplying $g(x)$ by a constant $q \in R$. Writing an arbitrary element $q = a + ub$ with $a, b \in Z_4$ and using the skew relation $x \cdot q = \theta(q)x$, one may verify that $q \cdot g(x) = (a + u(a + b)) + x(a + u(3b))$. Thus,

$$C = \{c_{a,b} = (a + u(a + b), a + u(3b)) | \, a, b \in Z_4\}.$$

Applying the Gray map Q coordinatewise, the Gray image of a codeword becomes

$$Q(c_{a,b}) = \big(Q(a + u(a + b)), Q(a + u(3b))\big) = ((a, 3b), (a, a + b)).$$

Thus,

$$Q(C) = \{((a, 3b), (a, a + b)) | \, a, b \in Z_4\} \subset Z_4^4.$$

A complete enumeration shows that the 16 codewords in Q (C) have the following Lee weight frequencies: Weight 0: 1 occurrence, Weight 2: 2 occurrences, Weight 3: 4 occurrences, Weight 4: 1 occurrence, Weight 5: 4 occurrences, Weight 6: 4 occurrences. Hence, the Lee weight enumerator of $Q(C)$ is

$$W_{Q(C)}(y) = 1 + 2y^2 + 4y^3 + y^4 + 4y^5 + 4y^6.$$

Remark 4.3: While the transformation $w_L(Q(a + ub))$ is completely predictable, it is not uniformly equal to $2w_L(a + ub)$. This piecewise linear behavior enables precise analysis of how the weight distribution of the original code C is mapped to that of the Gray image $Q(C)$. By carefully choosing the structure of C, one may design Gray images with favorable minimum Lee distances.

5. Duality and Quasi-Cyclicity of Gray Images

In this section we prove that the dual of a skew cyclic code (with respect to the standard inner product on R^n is also skew cyclic, and that the Gray map Q induces a bijection between the dual of a code and the dual of its Gray image. As a consequence, the dual of the Gray image of a skew cyclic code is quasi-cyclic. For further theoretical information, please see [1].

Theorem 5.1: Let $C \subset R^n$ be a skew cyclic code with respect to θ and that R be Frobenius. Define the dual of C by $C^{\perp} = \{x \in R^n | < x, c >_R = 0\ \forall c \in C\}$. Further, define on $\mathbb{Z}_4^{2n}$ the induced inner product $< v, w >_Q := < Q^{-1}(v), Q^{-1}(w) >_R$. Then the dual of the Gray image of C is given by $\big(Q(C)\big)^{\perp Q} = Q(C^{\perp})$. In particular, when $C^{\perp}$ is skew cyclic, its Gray image $Q(C^{\perp})$ is quasi-cyclic; hence, $\big(Q(C)\big)^{\perp Q}$ is a quasi-cyclic code over $\mathbb{Z}_4$.

Proof: Since Q is a linear bijection from R^n to $\mathbb{Z}_4^{2n}$, for any $c \in C$ and any $x \in R^n$ we have

$$< x, c >_R = 0 \iff < Q(x), Q(c) >_Q = < Q^{-1}(Q(x)), Q^{-1}(Q(c)) >_R = 0.$$

Therefore,

$$x \in C^{\perp} \iff Q(x) \in \{v \in \mathbb{Z}_4^{2n} | < v, Q(c) >_Q = 0\ \forall c \in C\} = \big(Q(C)\big)^{\perp Q}.$$

This shows that $\big(Q(C)\big)^{\perp Q} = Q(C^{\perp})$.

When $C^{\perp}$ is skew cyclic (see [1]), Theorem 3.5 implies that the Gray image $Q(C^{\perp})$ is invariant under the modified cyclic shift operator, that is, $Q(C^{\perp})$ is quasi-cyclic. Hence, the dual $\big(Q(C)\big)^{\perp Q}$ is quasi-cyclic.

Example 5.2: Consider the skew cyclic code $C = \{x(1, u) | x \in R\} \subset R^2$. Any codeword in C is of the form $c = x(1, u) = (x, xu)$ with $x = a + ub, a, b \in \mathbb{Z}_4$. Thus, $c = (a + ub, ua)$. Applying the gray map Q coordinatewise, we obtain $Q(c) = \big(Q(a + ub), Q(ua)\big) = \big((a, a + 3b), (0,3a)\big)$. Thus, $Q(C) = \{\big((a, a + 3b), (0,3a)\big) \big| a, b \in \mathbb{Z}_4\}$, so that C has 16 codewords and $Q(C) \subset \mathbb{Z}_4^4$ is quasi-cyclic. Now, the dual of C with respect to the standard inner product on R^2 is $C^{\perp} = \{(y_0, y_1) \in R^2 | < (x, xu), (y_0, y_1) >_R = 0, \forall x \in R\}$. And

$$C^{\perp} = \{(uc, -c + ud) | c, d \in \mathbb{Z}_4\}.$$

Then, applying Q we have: $Q(uc) = (0,3c)$, $Q(-c + ud) = (-c, -c + 3d)$. Thus,

$$Q(C^{\perp}) = \{\big((0,3c), (-c, -c + 3d)\big) \big| c, d \in \mathbb{Z}_4\}$$

6. Conclusion

In this paper, we introduced a new Gray map $(a + ub) = (a, a + 3b)$ for the ring $\mathbb{Z}_4 + u\mathbb{Z}_4$ and studied its properties and applications to skew cyclic codes. We defined an automorphism

$\theta(a+ub) = a + u(3b)$ which allowed us to construct the skew polynomial ring $R[x,\theta]$ and to define skew cyclic codes as left submodules of $R[x,\theta]/< x^n - 1 >$. We then proved that the Gray image $Q(C)$ of a skew cyclic code C is invariant under a modified cyclic shift operator S, meaning that $Q(C)$ is generalized quasi-cyclic over $\mathbb{Z}_4$. Our theoretical analysis, supported by examples, demonstrates that the Lee weight transformation under Q is given by $w_L(Q(a+ub)) = w_L(a) + w_L(a+3b)$ which, although not uniformly equal to $2w_L(a+ub)$, is completely predictable. In structured cases (e.g., when $b = 0$ or $b = 2$ the weight is either doubled or attains a fixed value, allowing one to design codes whose Gray images exhibit good minimum distance properties. Moreover, the duality is preserved under Q, leading to quasi-cyclic dual codes. As a future research direction, it would be valuable to investigate the practical error-correction capabilities of these newly constructed codes by simulating their performance on standard communication channels. Furthermore, developing efficient decoding algorithms that exploit the quasi-cyclic structure of the Gray images would be a crucial step toward their practical implementation.

Ethics in Publishing:

There are no ethical issues regarding the publication of this study.

Conflicts of Interest:

The author declares no conflict of interest.

References

[1] Boucher, D., & Ulmer, F. (2007). Coding with skew polynomial rings. *Journal of Symbolic Computation*, 44(12), 1644–1656.

[2] Hammons, A. R., Jr., Kumar, P. V., Calderbank, A. R., Sloane, N. J. A., & Solé, P. (1994). The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes. *IEEE Transactions on Information Theory*, 40(2), 301–319.

[3] Sharma, A., & Bhaintwal, M. (2018). A class of skew-cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ with derivation. *Advances in Mathematics of Communications*, 12(4), 723–739.

[4] Yildiz, B., & Aydin, N. (2014). On cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their $\mathbb{Z}_4$-images. *International Journal of Information and Coding Theory*, 2(4), 226–237.

[5] Yildiz, B., & Karadeniz, S. (2014). Linear codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$: MacWilliams identities, projections, and formally self-dual codes. *Finite Fields and Their Applications*, 27, 24–40.

[6] Yildirim, T. (2024). Linear Skew Cyclic Codes over F_qS. *Analele Stiintifice ale Universitatii Ovidius Constanta: Seria Matematica*, 32(3).

[7] Yildirim, T. (2024). Construction of cyclic DNA codes over $\mathbb{Z}_4R$. *Indian Journal of Pure and Applied Mathematics*, 55(4), 1465-1476.