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Global Boundedness and Mass Persistence in a Multi-Species Chemotaxis System

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Highlights:
- Chemotaxis
- Boundedness
- Persistence

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- Chemotaxis
- multi-species system
- regular sensitivity
- global existence
- global boundedness
- mass persistence

ABSTRACT:

This research paper concerns with the population dynamics of a multi-species and multi-chemicals chemotaxis system characterized by a parabolic-parabolic-elliptic-elliptic structure under no-flux boundary conditions in a smooth bounded domain. This research study examines the global existence, global boundedness, and persistence of mass of solutions of the system mentioned above. In all spatial dimensional settings, we first demonstrate the global L^p-boundedness of solutions under some explicit parameter conditions that notably exclude any dependence on the dimensionality. Then, it has been establihed that the global existence and boundedness of positive solutions are implied by L^p-bounds of solutions under the exact same hypotheses. In addition to these ones, we prove that any globally bounded classical solution eventually persist as a whole under the same conditions. The results obtained in this study contribute to a more profound theoretical understanding of chemotaxis models in multi-species and multi-chemical environments. In order to establish the qualitative properties of chemotaxis model mentioned in the above, some advanced mathematical techniques and strategies has been developed.

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INTRODUCTION

The term chemotaxis models explain the movement of motile species in response for some certain chemical gradient in their regions. Keller and Segel promoted a notable differential equations sytem to characterize this phenomenon both mathematically and biologically (see references in (Keller & Segel, 1970; Keller & Segel, 1971). This natural event is observed in many biological procedure such as tumor growth, immune cell migration, and population dynamics. Numerous authors investigated chemotaxis models from various perspectives, including local existence, uniqueness, finite time blow-up, global existence and boundedness, persistence, and stability. The readers are referred to the research papers (Horstmann, 2004; Hillen & Painter, 2009; Bellomo et al., 2015) for additional information.

Throughout this research article, the subsequent parabolic-parabolic-elliptic-elliptic chemotaxis model that includes two-mobile species and two-chemicals as well as two-logistics kinetics has been considered

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla z) + au - bu^2, & x \in \Omega, \\ v_t = \Delta v - \lambda \nabla \cdot (u \nabla z) + cv - dv^2, & x \in \Omega, \\ 0 = \Delta z - z + w, & x \in \Omega, \\ 0 = \Delta w - w + u + v, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

along with the initial $u(x,0) := u_0(x)$ and $v(x,0) := v_0(x)$ fulfilling

$$u_0, v_0 \in C^0(\bar{\Omega}) \quad and \quad u_0, v_0 \geq 0, \tag{2}$$

and together with a smooth domain $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ and $a, c > 0$ and $b, d, \chi, \lambda > 0$ are positive constants such that

$$b > 4\chi + \lambda, \tag{3}$$

and

$$d > \chi + 4\lambda. \tag{4}$$

From the biological aspects, model Eq. (1) represents the growing of two species that are in competition and subject to two chemical substance in their neighborhood. We point out that this is the first research work associated with the system Eq. (1). Throughout this study, we will investigate the L^p-bounds, global existence, global bounds and long-range behaviors for any global positive solution such as mass persistence of the system Eq. (1).

We point out that up to now, many variants of the system Eq. (1) have been investigated in many research works. In particular, let us mention about the related literature for one species or multi-species and one-multi type chemical substance models. First, assume that $v(x,t) = 0$ and $w(x,t) = 0$, which corresponds to

$$\begin{cases} u_t = ku - \chi \nabla \cdot (u \nabla z) + au - bu^2, & x \in \Omega, \ t > 0, \\ 0 = kz - z + u, & x \in \Omega, \ t > 0, \end{cases} \tag{5}$$

Case I: Assume $n \geq 2$ and $a = b = 0$. Then system Eq. (5) has a finite-time blows-up in solutions of Eq. (1) under some restriction on the initial data. The reader are directed to the papers (Herrero et al., 1996; Herrero & Velzquez, 1997; Nagai & Senba, 1998;Nagai, 2001) for more details.

Case II: Assume $a, b > 0$. Then model Eq. (5) has a globally bounded solution under the restriction $n < 2$ or $n \geq 3$ whenever $\chi < \frac{bn}{n-2}$, see (Tello & Winkler, 2007). This result was extended in in (Hu & Tau, 2017) and they proved that the global existence and boundedness of this model was obtained at the critical point, which is $\chi = \frac{bn}{n-2}$ with $n \geq 3$. In addition, the mass persistence of solutions of Eq. (5) was first studied in (Tao & Winkler, 2015a) and it was shown that in any space dimensional setting, when Ω is a convex domain, then all positive solutions to model Eq. (1) always persists as a whole. Then the convexity condition for the persistence of mass of solutions has just been eliminated in (Kurt, 2025b) under the the following explicit conditions

$$n \leq 2 \quad \text{or} \quad \chi \leq \frac{k}{b} \cdot \frac{n}{n-2} \quad \text{with} \quad n \geq 3.$$

For the other dynamical behaviors of solutions for similar chemotaxis models including weak solutions, stability, persistence, the readers are referred to the papers (Chaplain & Tello, 2016; Hu & Tau, 2017; Issa & Shen, 2017; Kurt, 2025a; Kurt, 2025c; Kurt & Shen, 2021; Lankeit, 2015; Lankeit, 2015b; Tello, 2004; Viglialoro, 2016; Winkler, 2010).

Now we give some known results for the similar models of Eq. (1). Consider the subsequent chemotaxis model which includes a two-species and a one-chemoattractant as well as a Lotka-Volterra kinetics:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla z) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \\ v_t = \Delta v - \lambda \nabla \cdot (u \nabla z) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \\ 0 = \Delta z - z + u + v, & x \in \Omega, \end{cases} \tag{6}$$

Tello and Winkler in (Tello & Winkler, 2012) established the global existence, boundedness and asymptotic stability for the sytem Eq. (6) under the explicit restrictions $2(\chi + \lambda) + a_2\mu_1 < \mu_2$ and $2(\chi + \lambda) + a_1\mu_2 < \mu_1$. Some developed results related to system Eq. (6) can be found in (Tello, 2004; Black et al., 2016; Lin et al., 2017; Mizukami, 2018). We refer the readers to the articles (Bai & Winkler, 2016; Issa & Salako, 2017; Kurt & Ekici, 2025; Lin & Mu, 2017; Xiang, 2018; Issa & Shen, 2019; Xie, 2019) for the existence, boundedness, long-term behavior of solutions such as asymptotic stability, persistence, competitive exclusion, coexistence, etc for the similar models of Eq. (6).

The fundamental consequences of this paper read as below.

Theorem 1 (Global existence and boundedness) Assume that Eq. (3) and Eq. (4) are valid. Then for any given initial functions u_0, v_0 fulfilling Eq. (2), the solution triple (u, v, w, z) is global, which means,

$$T_{\max}(u_0, v_0) = \infty.$$

Moreover, there exists a positive number $K_\infty \in (0, \infty)$ such that

$$\sup_{t>0} \| u + v \|_{L^\infty(\Omega)} \leq K_\infty.$$

Theorem 2 (Mass persistence) Assume that Eq. (3) and Eq. (4) hold. Then for any given initial functions u_0, v_0 fulfilling Eq. (2), there is $\sigma > 0$ such that

$$\int_\Omega u + \int_\Omega v \geq \sigma \text{ for every } t > 0.$$

MATERIALS AND METHODS

The next lemma is related to the local existence of solution of Eq. (1), which is obtained from [(Tello & Winkler, 2007), Theorem 7].

Lemma 1. Assume u_0, v_0 satisfy (2). There exists $T_{\max}(u_0, v_0) \in (0, \infty]$ fulfilling Eq. (1) derives a classical solution that is unique on $(0, T_{\max}(u_0, v_0))$. Moreover, $u, v \in C((0, T_{\max}) \times \bar{\Omega}) \cap C^{2,1}((0, T_{\max}) \times \bar{\Omega})$ and $z, w \in C^{2,0}((0, T_{\max}) \times \bar{\Omega})$. If $T_{\max} < \infty$, then $\limsup_{t \to T_{\max}} \|u + v\|_{C^0(\bar{\Omega})} = \infty$.

We now provide some basic estimates in the following.

Lemma 2. It holds that

$$\int_\Omega u \leq m_1 := \max\{\frac{a}{b}|\Omega|, \int_\Omega u_0\},$$

and

$$\int_\Omega v \leq m_2 := \max\{\tfrac{c}{d}|\Omega|, \int_\Omega v_0\},$$
for all $t \in (0, T_{\max})$.

Proof. Using integration by parts on the equations Eq.(1) $_1$ and Eq.(1) $_2$, respectively, as well as using Hölder inequality, we get

$$\frac{d}{dt}\int_\Omega u = a\int_\Omega u - b\int_\Omega u^2 \leq a\int_\Omega u - \frac{b}{|\Omega|}(\int_\Omega u)^2,$$
and
$$\frac{d}{dt}\int_\Omega v = c\int_\Omega v - d\int_\Omega v^2 \leq c\int_\Omega v - \frac{d}{|\Omega|}(\int_\Omega v)^2,$$
for all $t \in (0, T_{\max})$. The ODE's comparison principle completes the proof.

Lemma 3. For any p ≥ 1,
$$\int_\Omega w^{p+1} \leq 2^p\{\int_\Omega u^{p+1} + \int_\Omega v^{p+1}\}$$

Proof. First, by multiplying the equation Eq.(1) $_4$ by u^{p-1} and employing integration by parts, one entail
$$\int_\Omega u^{p-1} \cdot (kw - w + u + v) = 0,$$
which entails
$$p\int_\Omega w^{p-1}|\nabla w|^2 + \int_\Omega w^{p+1} = \int_\Omega uw^p + \int_\Omega vw^p.$$
Here, Hölder's inequality directs to
$$\int_\Omega uw^p \leq \left(\int_\Omega u^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega w^{p+1}\right)^{\frac{p}{p+1}},$$
and
$$\int_\Omega vw^p \leq \left(\int_\Omega v^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega w^{p+1}\right)^{\frac{p}{p+1}},$$
It then follows with the nonnegativity of $\int_\Omega w^{p-1}|\nabla w|^2$ that
$$\int_\Omega w^{p+1} \leq \left(\int_\Omega u^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega w^{p+1}\right)^{\frac{p}{p+1}} + \left(\int_\Omega v^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega w^{p+1}\right)^{\frac{p}{p+1}},$$
which gives
$$\begin{aligned}\int_\Omega w^{p+1} &\leq \left\{\left(\int_\Omega u^{p+1}\right)^{\frac{1}{p+1}} + \left(\int_\Omega v^{p+1}\right)^{\frac{1}{p+1}}\right\}^{p+1} \\ &\leq 2^p\{\int_\Omega u^{p+1} + \int_\Omega v^{p+1}\}.\end{aligned}$$
The proof is over.

Lemma 4. For all p ≥ 1, we have
$$\int_\Omega u^{p-1}\nabla u \cdot \nabla z \leq \frac{2^{\frac{p}{p+1}}}{p(p+1)}\{(2p+1)\int_\Omega u^{p+1} + \int_\Omega v^{p+1}\},$$
and
$$\int_\Omega v^{p-1}\nabla v \cdot \nabla z \leq \frac{2^{\frac{p}{p+1}}}{p(p+1)}\{\int_\Omega u^{p+1} + (2p+1)\int_\Omega v^{p+1}\},$$
for every $t \in (0, T_{\max})$.

Proof. By integration by parts over Ω together with Eq. (1) $_3$, Hölder's inequality and Lemma 3 yields
$$\begin{aligned}p\int_\Omega u^{p-1}\nabla u \cdot \nabla z &= -\int_\Omega u^p \Delta z \\ &= -\int_\Omega zu^p + \int_\Omega wu^p \\ &\leq \left(\int_\Omega w^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega u^{p+1}\right)^{\frac{p}{p+1}}\end{aligned}$$

$$\leq \left(2^p\{\int_\Omega u^{p+1} + \int_\Omega v^{p+1}\}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega u^{p+1}\right)^{\frac{p}{p+1}}$$
$$\leq 2^{\frac{p}{p+1}} \int_\Omega u^{p+1} + 2^{\frac{p}{p+1}}\left(\int_\Omega v^{p+1}\right)^{\frac{1}{p+1}} \cdot \left(\int_\Omega u^{p+1}\right)^{\frac{p}{p+1}}$$
$$\leq 2^{\frac{p}{p+1}} \cdot \frac{2p+1}{p+1} \int_\Omega u^{p+1} + 2^{\frac{p}{p+1}} \cdot \frac{1}{p+1} \int_\Omega v^{p+1}$$

for every $t \in (0, T_{\max})$. Similarly, we can obtain

$$p \int_\Omega v^{p-1} \nabla v \cdot \nabla z \leq 2^{\frac{p}{p+1}} \cdot \frac{1}{p+1} \int_\Omega u^{p+1} + 2^{\frac{p}{p+1}} \cdot \frac{2p+1}{p+1} \int_\Omega v^{p+1},$$

for every $t \in (0, T_{\max})$. The proof thus finishes.

We now establish L^p-bounds.

Lemma 5. (L^p-bounds) Assume that u_0, v_0 satisfy Eq. (2), and the parameters χ, λ, b, d satisfy Eq. (3) and Eq. (4). Then for any given $p \geq 1$, one have $K(p) > 0$ such that

$$\int_\Omega u^p + \int_\Omega v^p \leq K(p) \quad \forall t \in (0, T_{\max}).$$

Proof. First, testing the Eq. $(1)_1$ by u^{p-1} with $p > 1$ and the Eq.$(1)_2$ by v^{p-1} with $p > 1$ and adding both equations arrives at

$$\frac{1}{p} \cdot \frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega v^p\right) = -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 - (p-1)\int_\Omega v^{p-2}|\nabla v|^2$$
$$+(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla z + (p-1)\lambda \int_\Omega v^{p-1}\nabla v \cdot \nabla z$$
$$+a\int_\Omega u^p + c\int_\Omega v^p - b\int_\Omega u^{p+1} - d\int_\Omega v^{p+1}, \tag{7}$$

for $t \in (0, T_{\max})$. In view of Lemma 4, we have

$$(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla z \leq \frac{(p-1)(2p+1)}{p(p+1)} 2^{\frac{p}{p+1}} \chi \int_\Omega u^{p+1} + \frac{p-1}{p(p+1)} 2^{\frac{p}{p+1}} \chi \int_\Omega v^{p+1},$$

and

$$(p-1)\lambda \int_\Omega v^{p-1}\nabla v \cdot \nabla z \leq \frac{p-1}{p(p+1)} 2^{\frac{p}{p+1}} \lambda \int_\Omega u^{p+1} + \frac{(p-1)(2p+1)}{p(p+1)} 2^{\frac{p}{p+1}} \lambda \int_\Omega v^{p+1},$$

which yields

$$(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla z + (p-1)\lambda \int_\Omega v^{p-1}\nabla v \cdot \nabla z$$
$$\leq \frac{(p-1)((2p+1)\chi+\lambda)2^{\frac{p}{p+1}}}{p(p+1)} \int_\Omega u^{p+1} + \frac{(p-1)(\chi+(2p+1)\lambda)2^{\frac{p}{p+1}}}{p(p+1)} \int_\Omega v^{p+1} \tag{8}$$

for $t \in (0, T_{\max})$. In addition, in lights of Young's inequality,

$$a\int_\Omega u^p \leq \varepsilon \int_\Omega u^{p+1} + C(a, p, \varepsilon, |\Omega|), \tag{9}$$

and

$$c\int_\Omega v^p \leq \varepsilon \int_\Omega v^{p+1} + C(c, p, \varepsilon, |\Omega|), \tag{10}$$

for $t \in (0, T_{\max})$. Collecting Eq. (7)-(10) entails

$$\frac{1}{p} \cdot \frac{d}{dt}\left(\int_\Omega u^p + \int_\Omega v^p\right) \leq \left\{\frac{(p-1)(2p+1)2^{\frac{p}{p+1}}}{p(p+1)}\chi + \frac{(p-1)2^{\frac{p}{p+1}}}{p(p+1)}\lambda + \varepsilon - b\right\} \int_\Omega u^{p+1}$$
$$\leq \left\{\frac{(p-1)2^{\frac{p}{p+1}}}{p(p+1)}\chi + \frac{(p-1)(2p+1)2^{\frac{p}{p+1}}}{p(p+1)}\lambda + \varepsilon - d\right\} \int_\Omega v^{p+1}$$
$$\leq -\tilde{\varepsilon}\left(\int_\Omega u^p + \int_\Omega u^p\right) + \tilde{C} \quad \text{for all } t \in (0, T_{\max}),$$

due to the fact

$$\max\left\{\frac{(p-1)(2p+1)2^{\frac{p}{p+1}}}{p(p+1)}\right\} < 4 \quad \text{and} \quad \frac{(p-1)2^{\frac{p}{p+1}}}{p(p+1)} < 1, \quad \forall p > 1,$$

and the main assumptions Eq. (3) and Eq. (4) that are $b > 4\chi + \lambda$ and $d > \chi + 4\lambda$, which implies that

$$b > \frac{(p-1)(2p+1)2^{\frac{p}{p+1}}}{p(p+1)}\chi + \frac{(p-1)2^{\frac{p}{p+1}}}{p(p+1)}\lambda + \varepsilon,$$

and

$$d > \frac{(p-1)2^{\frac{p}{p+1}}}{p(p+1)}\chi + \frac{(p-1)(2p+1)2^{\frac{p}{p+1}}}{p(p+1)}\lambda + \varepsilon.$$

Therefore, the Gronwall's inequality concludes that

$$\int_\Omega u^p + \int_\Omega v^p \leq \mathrm{K}(p) := \max\left\{\int_\Omega u_0^p + \int_\Omega v_0^p, \frac{2\tilde{C}}{\tilde{\varepsilon}}\right\} \quad \text{for all } t \in (0, T_{\max}).$$

The proof is over.

RESULTS AND DISCUSSION

Now, one will first examine global existence and global boundedness, and then mass persistence in Eq. (1).

Global existence and boundedness

We point out that if $p > \frac{n}{2}$, then global L^p-boundedness of solutions in time implies the L^∞-boundedness in time of solutions. Since we showed the L^p-bounds of solutions without any restrictions in Lemma 5, one simply prove the existence and boundedness results using the well-known approach.

Proof (Theorem 1). We shall show that $T_{\max} = \infty$ through the contradiction method. First, thanks to the constant formula, one can write

$$u(t,\cdot) = e^{-(I-\Delta)t}u_0 - \chi \int_0^t e^{-(I-\Delta)t}\nabla \cdot (u(\cdot,s)\nabla z(\cdot,s))ds$$
$$+ \int_0^t e^{-(I-\Delta)t}u(\cdot,s)(a+1-bu(\cdot,s))ds$$
$$=: I_1 + I_2 + I_3 \quad \text{for all} \quad t \in (0, T_{\max}), \tag{11}$$

and

$$v(t,\cdot) = e^{-(I-\Delta)t}v_0 - \chi \int_0^t e^{-(I-\Delta)t}\nabla \cdot (v(\cdot,s)\nabla z(\cdot,s))ds$$
$$+ \int_0^t e^{-(I-\Delta)t}v(\cdot,s)(c+1-dv(\cdot,s))ds$$
$$=: J_1 + J_2 + J_3 \quad \text{forall} \quad t \in (0, T_{\max}), \tag{12}$$

where $I - \Delta : \mathcal{D}(I-\Delta) \subset L^p(\Omega) \to L^p(\Omega)$ with $\mathcal{D}(I-\Delta) = \{u \in W^{2,p}(\Omega) \mid \partial u \cdot \nu = 0 \text{ on } \partial\Omega\}$, and $e^{-t(I-\Delta)}$ represents an analytic semigroup which is generated by $-(I-\Delta)$ on $L^p(\Omega)$. We now give the following estimates for I_1, I_2, I_3 and J_1, J_2, J_3. First,

$$\| I_1 \|_{L^\infty(\Omega)} = \| e^{-(I-\Delta)t}u_0 \|_{L^\infty(\Omega)} \leq \| u_0 \|_{L^\infty} \quad \forall t > 0. \tag{13}$$

and

$$\| J_1 \|_{L^\infty(\Omega)} = \| e^{-(I-\Delta)t}v_0 \|_{L^\infty(\Omega)} \leq \| v_0 \|_{L^\infty} \quad \forall t > 0. \tag{14}$$

Next,

$$\| I_3 \|_{L^\infty(\Omega)} = \int_0^t e^{-(I-\Delta)t}u(\cdot,s)(a+1-bu(\cdot,s))ds \leq \frac{(a+1)^2}{4b} \quad \forall t \in [0, T_{\max}). \tag{15}$$

and

$$\| J_3 \|_{L^\infty(\Omega)} = \int_0^t e^{-(I-\Delta)t}v(\cdot,s)(c+1-dv(\cdot,s))ds \leq \frac{(c+1)^2}{4d} \quad \forall t \in [0, T_{\max}). \tag{16}$$

Third, let $p \geq 1$ and assume that $\frac{n}{2} < p < n < q$ and $r \in (1, \infty)$ such that $\frac{1}{p} - \frac{1}{n} < \frac{1}{q}$ and $\frac{1}{r} < 1 - q(\frac{1}{p} - \frac{1}{n})$. Now the Gagliardo-Nirenberg inequality and Lemma 5 direct to

$$\| \nabla z \|_{L^{\frac{qr}{r-1}}(\Omega)} \leq C \| \nabla z \|_{L^{\frac{np}{n-p}}(\Omega)} \leq C \| u + v \|_{L^p(\Omega)} \leq C(\| u \|_{L^p(\Omega)} + \| v \|_{L^p(\Omega)}) \leq C_p, \tag{17}$$

for all $t \in (0, T_{\max})$, due to $\frac{qr}{r-1} = \frac{q}{1-\frac{1}{r}} < \frac{q}{(\frac{1}{p}-\frac{1}{n})q} = \frac{1}{\frac{1}{p}-\frac{1}{n}} = \frac{np}{n-p}$. Now, Lemma 2 and Hölder inequality, and Eq. (17), we get

$$\| u\nabla z \|_{L^q(\Omega)} \leq \| u \|_{L^{qr}(\Omega)} \cdot \| \nabla z \|_{L^{\frac{qr}{r-1}}(\Omega)}$$

$$\leq \| u \|_{L^1(\Omega)}^{\frac{1}{qr}} \cdot \| u \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \cdot \| \nabla z \|_{L^{\frac{np}{n-p}}(\Omega)}$$

$$\leq (m_1)^{\frac{1}{qr}} C_p \cdot \| u \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{18}$$

Similarly, we have

$$\| v\nabla z \|_{L^q(\Omega)} \leq \| v \|_{L^{qr}(\Omega)} \cdot \| \nabla z \|_{L^{\frac{qr}{r-1}}(\Omega)}$$

$$\leq \| v \|_{L^1(\Omega)}^{\frac{1}{qr}} \cdot \| v \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \cdot \| \nabla z \|_{L^{\frac{np}{n-p}}(\Omega)}$$

$$\leq (m_2)^{\frac{1}{qr}} C_p \cdot \| v \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{19}$$

Now let us fix $\beta \in (\frac{N}{2p}, \frac{1}{2})$ and $\zeta \in (0, \frac{1}{2} - \beta)$ as well as $T \in (0, T_{\max})$. Hence, by using well-known semigroup estimates and Lemma 4, we obtain

$$\| I_2 \|_{L^\infty(\Omega)} = \| \chi \int_0^t e^{-(I-\Delta)t} \nabla \cdot (u\nabla z) \|_{L^\infty(\Omega)}$$

$$\leq C_1 \chi \int_0^t \| e^{-(I-\Delta)t} \nabla \cdot (u\nabla z) \|_{X_p^\beta(\Omega)}$$

$$= C_1 \chi \int_0^t \| (I-\Delta)^\beta e^{-(I-\Delta)t} \nabla \cdot (u\nabla z) \|_{L^p(\Omega)}$$

$$\leq C_2 \chi \int_0^t (t-s)^{-\beta} (1 + (t-s)^{-\frac{1}{2}}) e^{-\zeta(t-s)} \| u\nabla z \|_{L^p(\Omega)} \, ds$$

$$\leq C_3 \chi \times \int_0^t (t-s)^{-\beta-\frac{1}{2}} e^{-\zeta(t-s)} \| u \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \, ds$$

$$\leq C_4 \chi \times \int_0^\infty (t-s)^{-\beta-\frac{1}{2}} e^{-\zeta(t-s)} ds \times \sup_{s\in[0,T]} \| u(\cdot,s) \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}}$$

$$\leq \tilde{C}_p \sup_{s\in[0,T]} \| u(\cdot,s) \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \tag{20}$$

for every $t \in [0, T]$, where $\tilde{C}_p \in (0, \infty)$. Similarly, we have

$$\| J_2 \|_{L^\infty(\Omega)} = \| \lambda \int_0^t e^{-(I-\Delta)t} \nabla \cdot (v\nabla z) \|_{L^\infty(\Omega)} \leq \tilde{C}_p \sup_{s\in[0,T]} \| v(\cdot,s) \|_{L^\infty(\Omega)}^{1-\frac{1}{qr}} \tag{21}$$

Substituting I_1, I_2, I_3 and J_1, J_2, J_3 into Eq. (11), we obtain

$$\sup_{t\in[0,T]} \| u(t,\cdot) \|_{L^\infty} + \sup_{t\in[0,T]} \| v(t,\cdot) \|_{L^\infty} \leq \| u_0 \|_{L^\infty(\Omega)} + \| v_0 \|_{L^\infty(\Omega)} + \frac{(a+1)^2}{4b} + \frac{(c+1)^2}{4d}$$
$$+ \tilde{C}_p \cdot \left(\sup_{t\in[0,T]} \| u(\cdot,t) \|_{L^\infty(\Omega)} + \sup_{t\in[0,T]} \| v(\cdot,t) \|_{L^\infty(\Omega)} \right)^{1-\frac{1}{qr}}$$

for every $T \in (0, T_{\max})$, where $0 < 1 - \frac{1}{qr} < 1$ and $\tilde{C}_p > 0$. Thus we get

$$\limsup_{t\to T_{\max}} \|u + v\|_{L^\infty(\Omega)} < \infty,$$

which contradicts to Lemma 7. This entails that $T_{\max} = \infty$ and $\sup\|u + v\|_{L^\infty(\Omega)}$ is bounded for all $t > 0$ such that there is $K > 0$ such that

$$\sup\|u + v\|_{L^\infty(\Omega)} \leq K \quad \forall t > 0.$$

Therefore, the proof is complete.

Mass persistence of solutions

This subsection analyzes mass persistence of solutions to Eq. (1). Observe that, by Theorem 1, $T_{\max}(u_0, v_0) = \infty$, and (u, v, w, z) is the globally bounded classical solution of system Eq. (1) on $(0, \infty)$. Let us begin providing an estimate for $(u + \mathrm{v})$ from below.

Lemma 6. Assume that $k \in (0,1)$. Then there is $\sigma > 0$ such that

$$\int_\Omega u^k + \int_\Omega v^k \geq \sigma \quad \text{forall} \quad t > 0.$$

Proof. First, by multiplying the equation Eq. $(1)_1$ by u^{k-1} with $k \in (0,1)$ and the equation Eq. $(1)_2$ by v^{k-1} with $k \in (0,1)$, respectively, and then integrating over Ω and adding theses equations to obtain

$$\frac{1}{k} \cdot \frac{d}{dt}\left(\int_\Omega u^k + \int_\Omega v^k\right) = (1-k)\int_\Omega u^{k-2}|\nabla u|^2 + (1-k)\int_\Omega v^{k-2}|\nabla v|^2$$
$$-(1-k)\chi\int_\Omega u^{k-1}\nabla u \cdot \nabla z - (1-k)\lambda\int_\Omega v^{k-1}\nabla v \cdot \nabla z$$
$$+a\int_\Omega u^k + c\int_\Omega v^k - b\int_\Omega u^{k+1} - d\int_\Omega v^{k+1}, \tag{22}$$

for all $t > 0$. Observe, by Lemma 5, that we can find $p > \min\{2, k+1\}$ such that

$$\begin{aligned}
(1-k)\chi\int_\Omega u^{k-1}\nabla u \cdot \nabla z &= -\frac{(1-k)\chi}{k}\int_\Omega u^k \Delta z \\
&= -\frac{(1-k)\chi}{k}\int_\Omega z u^k + \frac{(1-k)\chi}{k}\int_\Omega w u^k \\
&\leq \frac{(1-k)\chi}{k(k+1)}\int_\Omega w^{k+1} + \frac{(1-k)\chi}{k+1}\int_\Omega u^{k+1} \\
&\leq \frac{(1-k)\chi}{k(k+1)} 2^k\left(\int_\Omega u^{k+1} + \int_\Omega v^{k+1}\right) + \frac{(1-k)\chi}{k+1}\int_\Omega v^{k+1} \\
&\leq C_1(k,\chi)\left(\int_\Omega u^{k+1} + \int_\Omega v^{k+1}\right)
\end{aligned}$$

and

$$(1-k)\lambda\int_\Omega v^{k-1}\nabla v \cdot \nabla z \leq C_2(k,\lambda)\left(\int_\Omega u^{k+1} + \int_\Omega v^{k+1}\right),$$

which gives

$$(1-k)\chi\int_\Omega u^{k-1}\nabla u \cdot \nabla z + (1-k)\lambda\int_\Omega v^{k-1}\nabla v \cdot \nabla z \leq C_3(k,\chi,\lambda)\left(\int_\Omega u^{k+1} + \int_\Omega v^{k+1}\right), \tag{23}$$

for all $t > 0$. Note that by Theorem 1, we have the boundedness of u and v, which means

$$\sup_{t>0}\|u\|_{L^\infty(\Omega)}, \sup_{t>0}\|v\|_{L^\infty(\Omega)} \leq K_\infty.$$

This together with Gagliardo-Nirenberg inequality and Young's inequality, we get

$$\begin{aligned}
C_3(k,\chi,\lambda)\int_\Omega u^{k+1} &\leq C_3(k,\chi,\lambda)(K_\infty)^{1-\frac{k}{n}}\int_\Omega u^{\frac{k(n+1)}{n}} \\
&\leq C_4(k,\chi,\lambda,n,K_\infty)\left(\int_\Omega u^{k-2}|\nabla u|^2\right)^{\frac{1}{2}}\left(\int_\Omega u^k\right)^{\frac{1}{2}+\frac{1}{n}} + C_4(k,\chi,\lambda,n,K_\infty)\left(\int_\Omega u^k\right)^{1+\frac{1}{n}} \\
&\leq (1-k)\int_\Omega u^{k-2}|\nabla u|^2 + C_5\left(\int_\Omega u^k\right)^{1+\frac{2}{n}} + C_4\left(\int_\Omega u^k\right)^{1+\frac{1}{n}}
\end{aligned}$$

and similarly,

$$C_3(k,\chi,\lambda)\int_\Omega v^{k+1} \leq (1-k)\int_\Omega v^{k-2}|\nabla v|^2 + C_6\left(\int_\Omega v^k\right)^{1+\frac{2}{n}} + C_7\left(\int_\Omega v^k\right)^{1+\frac{1}{n}}$$

for all $t > 0$. Hence we get

$$(1-k)\chi\int_\Omega u^{k-1}\nabla u\cdot\nabla z+(1-k)\lambda\int_\Omega v^{k-1}\nabla v\cdot\nabla z$$
$$\le C_3(k,\chi,\lambda)\left(\int_\Omega u^{k+1}+\int_\Omega v^{k+1}\right)$$
$$\le (1-k)\int_\Omega u^{k-2}|\nabla u|^2+1-k)\int_\Omega v^{k-2}|\nabla v|^2$$
$$+C_5\left(\int_\Omega u^k\right)^{1+\frac{2}{n}}+C_4\left(\int_\Omega u^k\right)^{1+\frac{1}{n}}+C_6\left(\int_\Omega v^k\right)^{1+\frac{2}{n}}+C_7\left(\int_\Omega v^k\right)^{1+\frac{1}{n}} \tag{24}$$

for all $t>0$. Thus, we get from Eq. (22) to Eq. (24) that

$$\frac{1}{k}\cdot\frac{d}{dt}\left(\int_\Omega u^k+\int_\Omega v^k\right)\ge \mathrm{a}\int_\Omega u^k+\mathrm{c}\int_\Omega v^k-C_5\left(\int_\Omega u^k\right)^{1+\frac{2}{n}}-C_4\left(\int_\Omega u^k\right)^{1+\frac{1}{n}}$$
$$-C_6\left(\int_\Omega v^k\right)^{1+\frac{2}{n}}-C_7\left(\int_\Omega v^k\right)^{1+\frac{1}{n}}$$
$$\ge \min\{a,c\}\left(\int_\Omega u^k+\int_\Omega v^k\right)-C_8\left(\int_\Omega u^k+\int_\Omega v^k\right)^{1+\frac{2}{n}}-C_9\left(\int_\Omega u^k+\int_\Omega v^k\right)^{1+\frac{1}{n}}$$

for all $t>0$. One can conclude, by the Lemma (Tao & Winkler, 2015b), that there is $\sigma>0$ fulfilling

$$\int_\Omega u^k+\int_\Omega v^k\ge\sigma \quad \text{for all} \quad t>0.$$

The proof is hence over.

Proof (Theorem 2). First of all, by Hölder inequality, for any given $k\in(0,1)$, we get

$$\int_\Omega (u+v)\ge|\Omega|^{\frac{k-1}{k}}\left\{\int_\Omega (u+v)^k\right\}^{\frac{1}{k}}$$
$$\ge|\Omega|^{\frac{k-1}{k}}\left\{\int_\Omega u^k+\int_\Omega v^k\right\}^{\frac{1}{k}} \quad \text{for all} \quad t>0.$$

Next, by Lemma 6, for any given $k\in(0,1)$, there is $\sigma>0$ such that

$$\int_\Omega u^k+\int_\Omega v^k\ge\sigma \quad \text{for all} \quad t>0.$$

Hence, we get

$$\int_\Omega (u+v)\ge|\Omega|^{\frac{k-1}{k}}\left\{\int_\Omega u^k+\int_\Omega v^k\right\}^{\frac{1}{k}}\ge|\Omega|^{\frac{k-1}{k}}\sigma^{\frac{1}{k}} \quad \text{for all} \quad t>0.$$

The theorem thus follows.

CONCLUSION

In this research artcile, the followings has been shown: First, in Lemma 5, L^p-bounds of u is established under the conditions Eq. (3) and Eq. (4), $b>4\chi+\lambda$ and $d>\chi+4\lambda$, which are independent of the parameter p, hence the dimension n. Second, thanks to this achievement, one is able to derive global existence & boundedness, and persistence of mass as independent of p, hence the dimension n. This is very exciting result and also shows that dimension n is not a critical parameter in obtaing the results presented in Theorem 1 and Theorem 2.

Conflict of Interest

The article author declare that there is no conflict of interest between them.

Author's Contributions

The author declare that he has contributed totally to the article.

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