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THE MATRIX FORM OF THE k -PELL HYPERBOLIC FUNCTIONS

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Abstract

In the present paper, we introduce the matrix form of the k -Pell hyperbolic sine and cosine functions, along with their symmetrical forms. We examine their recurrence and hyperbolic properties, including Pythagorean, de Moivre, Catalan, Cassini, and d'Ocagne identities, as well as various sum and difference identities. Additionally, we define the quasi-sine k -Pell matrix functions and the matrix form of the three-dimensional k -Pell spiral, and investigate some of their fundamental properties. Matrix functions play a significant role in various scientific fields, particularly in mathematics and engineering, as they frequently arise in the solutions of differential equations. The findings presented contribute to the development of special matrix functions and their potential applications.

Keywords: k -Pell number sequence, hyperbolic functions, matrix exponential

1. Introduction

Integer sequences are fundamental in both number theory and applied mathematics because they provide the fundamental structures for studying prime distributions, divisibility, and modular arithmetic, as well as playing an important role in applications such as cryptography, signal processing, and mathematical modelling. Besides well-known integer sequences such as the Fibonacci and Lucas, Pell number sequences and some of their generalizations have been widely studied due to their recurrence relations and algebraic properties.

Let us review the definitions of several well-known integer sequences. The Horadam number sequences are defined by the second-order recurrence relation

$$H_n = pH_{n-1} + qH_{n-2},$$

where $n \geq 1$, $H_0 = a$, $H_1 = b$, p and q are non-zero real numbers [1]. The characteristic equation of H_n is $t^2 - pt - q = 0$ with the roots $\phi = \frac{p+\sqrt{p^2+4q}}{2}$ and $\bar{\phi} = \frac{p-\sqrt{p^2+4q}}{2}$ [1]. The Binet formula for H_n is

$$H_n = \frac{A\phi^n - (-1)^n Bq^n \phi^{-n}}{\sqrt{p^2+4q}} = \begin{cases} \frac{A\phi^n + Bq^n \phi^{-n}}{\sqrt{p^2+4q}}, & n \text{ is odd} \\ \frac{A\phi^n - Bq^n \phi^{-n}}{\sqrt{p^2+4q}}, & n \text{ is even,} \end{cases}$$

where $A = b - a\bar{\phi}$ and $B = b - a\phi$, also p and q are non-zero real numbers such that

$$p^2 + 4q \neq 0 \text{ [1].}$$

This general form includes many classical sequences depending on the choice of initial conditions and parameters. In particular, the Fibonacci, Lucas, Pell, and Jacobsthal sequences arise as special cases. Below, we give the definitions of some of these sequences.

The Fibonacci and Lucas number sequences are defined by the similar recurrence relations as follows

$$F_n = F_{n-1} + F_{n-2}, \quad L_n = L_{n-1} + L_{n-2}$$

with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$ [2]. The Binet formulas for the Fibonacci and Lucas number sequences are

$$F_n = \frac{\theta^n - \bar{\theta}^n}{\theta - \bar{\theta}}, \quad L_n = \theta^n + \bar{\theta}^n,$$

where $\theta = \frac{1+\sqrt{5}}{2}$ and $\bar{\theta} = \frac{1-\sqrt{5}}{2}$ denote the roots of the characteristic equation $t^2 - kt - 1 = 0$ [2]. The k -Fibonacci and k -Lucas number sequences represent significant generalizations of the Fibonacci and Lucas sequences through the introduction of a real parameter k . Note that for $k = 1$, these sequences reduce to the classical Fibonacci and Lucas sequences. These sequences extend the fundamental recurrence relations of their classical counterparts, thereby enhancing their algebraic structure and expanding their applicability across diverse mathematical and computational domains. The k -Fibonacci and k -Lucas number sequences are defined for any integers $k \geq 1$ and $n \geq 2$ as follows

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ with } F_{k,0} = 0, F_{k,1} = 1,$$

$$L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ with } L_{k,0} = 0, L_{k,1} = k,$$

respectively [3, 4]. The formulas

$$F_{k,n} = \frac{\theta_k^n - \bar{\theta}_k^n}{\theta_k - \bar{\theta}_k}, \quad L_{k,n} = \theta_k^n + \bar{\theta}_k^n$$

represent the Binet formulas of the k -Fibonacci and k -Lucas number sequences, respectively,

where $\theta_k = \frac{k+\sqrt{k^2+4}}{2}$ and $\bar{\theta}_k = \frac{k-\sqrt{k^2+4}}{2}$ denote the roots of the characteristic equation

$$t^2 - kt - 1 = 0.$$

The Pell number sequence is determined by the rule

$$P_{n+2} = 2P_{n+1} + P_n$$

with $P_0 = 0$ and $P_1 = 1$. The Binet formula for the Pell number sequence is

$$P_n = \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}},$$

where $\alpha = 1 + \sqrt{2}$ and $\bar{\alpha} = 1 - \sqrt{2}$, which are the roots of the characteristic equation $t^2 - 2t - 1 = 0$ [5]. There are various generalizations of the Pell number sequence in the literature. In the following, we examine some of these generalizations. The modified Pell sequence was defined by Horadam [5] as follows

$$q_{n+2} = 2q_{n+1} + q_n$$

with initial conditions $q_0 = 1$ and $q_1 = 1$. The Binet formula for the modified Pell number sequence is

$$q_n = \frac{\sigma^n - \bar{\sigma}^n}{2},$$

where $\sigma = 1 + \sqrt{2}$ and $\bar{\sigma} = 1 - \sqrt{2}$ [5], which can be rewritten as

$$q_n = \begin{cases} \frac{\sigma^n - \sigma^{-n}}{2}, & n \text{ is odd,} \\ \frac{\sigma^n + \sigma^{-n}}{2}, & n \text{ is even.} \end{cases}$$

Another generalization of the Pell number sequence is the k -Pell number sequence, which is defined by the recurrence relation

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}$$

with $P_{k,0} = 0$, $P_{k,1} = 1$ and $k \geq 1$ [6]. The characteristic equation of the k -Pell number sequence is $t^2 - 2t - k = 0$ with the roots $\alpha_k = 1 + \sqrt{k+1}$ and $\bar{\alpha}_k = 1 - \sqrt{k+1}$ [6]. The Binet formula for the k -Pell number sequence is [6]

$$P_{k,n} = \frac{\alpha_k^n - \bar{\alpha}_k^n}{\alpha_k - \bar{\alpha}_k}.$$

Note that for $k = 1$, the k -Pell number sequence reduces to the classical Pell number sequence. The generalization introduces a parameter that allows for an extended exploration of its structural and theoretical properties.

Trigonometric and hyperbolic functions are indispensable in various scientific fields as they provide powerful tools for solving differential equation systems. The classical trigonometric and hyperbolic functions are

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2},$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The Fibonacci and Lucas hyperbolic functions were introduced by Stakhov and Tkachenko [7] as follows

$$sF(x) = \frac{\theta^{2x} - \theta^{-2x}}{\sqrt{5}}, \quad cF(x) = \frac{\theta^{2x+1} + \theta^{-2x-1}}{\sqrt{5}},$$

$$sL(x) = \theta^{2x+1} - \theta^{-2x-1}, \quad cL(x) = \theta^{2x} + \theta^{-2x},$$

respectively, where $\theta = \frac{1+\sqrt{5}}{2}$. Stakhov and Rozin [8] defined the symmetrical forms of these functions with respect to the origin, which are a similar structure to the classical hyperbolic functions, as

$$sFs(x) = \frac{\theta^x - \theta^{-x}}{\sqrt{5}}, \quad cFs(x) = \frac{\theta^x + \theta^{-x}}{\sqrt{5}},$$

$$sLs(x) = \theta^x - \theta^{-x}, \quad cLs(x) = \theta^x + \theta^{-x}.$$

The extended forms of the Fibonacci and Lucas hyperbolic functions, known as the k -Fibonacci and k -Lucas hyperbolic functions, were introduced by Falcon and Plaza [9] and Taşcı and Azman [10] as follows

$$sF_k(x) = \frac{\theta_k^{2x} - \theta_k^{-2x}}{\sqrt{k^2 + 4}}, \quad cF_k(x) = \frac{\theta_k^{2x+1} + \theta_k^{-2x-1}}{\sqrt{k^2 + 4}},$$

$$sL_k(x) = \theta_k^{2x+1} - \theta_k^{-2x-1}, \quad cL_k(x) = \theta_k^{2x} + \theta_k^{-2x},$$

where $\theta_k = \frac{k+\sqrt{k^2+4}}{2}$. The symmetrical forms of these functions were derived by the authors of [9] and [10], based on the approach of Stakhov and Rozin [8] as follows

$$sF_k s(x) = \frac{\theta_k^x - \theta_k^{-x}}{\sqrt{k^2 + 4}}, \quad cF_k s(x) = \frac{\theta_k^x + \theta_k^{-x}}{\sqrt{k^2 + 4}},$$

$$sL_k s(x) = \theta_k^x - \theta_k^{-x}, \quad cL_k s(x) = \theta_k^x + \theta_k^{-x}.$$

The fundamental properties, including recurrence relations, the Pythagorean Theorem, Catalan, Cassini, and d'Ocagne identities, as well as various formulas for sum and difference, were presented for the symmetrical k -Fibonacci and k -Lucas hyperbolic functions in [9, 10]. Also, the definitions of the quasi-sine k -Fibonacci and k -Lucas functions were presented and some of their properties were examined in [9, 10]. Furthermore, Falcon and Plaza [9] introduced the three-dimensional k -Fibonacci spiral, called the Metallic Shofar, which extends the Golden Shofar in [11], along with some of their properties.

Koçer [12] defined hyperbolic modified Pell sine and cosine functions

$$sq(x) = \frac{\alpha^{2x+1} - \alpha^{-2x-1}}{2}, \quad cq(x) = \frac{\alpha^{2x} + \alpha^{-2x}}{2},$$

and their symmetrical forms

$$sqs(x) = \frac{\alpha^x - \alpha^{-x}}{2}, \quad cqs(x) = \frac{\alpha^x + \alpha^{-x}}{2},$$

respectively. The authors examined some of their hyperbolic and recurrence properties.

Koçer et al. [13] established explicit representations of second-order recurrence sequences in terms of hyperbolic functions, covering both real and complex root cases.

Bahşi and Solak [14] introduced hyperbolic Horadam sine and cosine functions by the rules

$$sW(x) = \frac{A\phi^{2x} - Bq^{2x}\phi^{-2x}}{\sqrt{p^2 + 4q}}, \quad cW(x) = \frac{A\phi^{2x+1} + Bq^{2x+1}\phi^{-2x-1}}{\sqrt{p^2 + 4q}},$$

where $\phi = \frac{p + \sqrt{p^2 + 4q}}{2}$, and p and q are non-zero real numbers such that $p^2 + 4q \neq 0$. The symmetrical hyperbolic Horadam sine and cosine functions were given in [14] as follows

$$sWs(x) = \frac{A\phi^x - Bq^x\phi^{-x}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cWs(x) = \frac{A\phi^x + Bq^x\phi^{-x}}{\sqrt{p^2 + 4q}}.$$

The functions

$$sP(x) = \frac{\alpha^{2x} - \alpha^{-2x}}{2\sqrt{2}} \quad \text{and} \quad cP(x) = \frac{\alpha^{2x+1} + \alpha^{-2x-1}}{2\sqrt{2}}$$

were recently defined by Kuloğlu and Özkan [15] as the Pell hyperbolic sine and cosine functions, respectively. The authors also introduced the k -Pell hyperbolic functions and their symmetrical forms as

$$sP_k(x) = \frac{\alpha_k^{2x} - k^{2x}\alpha_k^{-2x}}{2\sqrt{k+1}}, \quad cP_k(x) = \frac{\alpha_k^{2x+1} + k^{2x+1}\alpha_k^{-2x-1}}{2\sqrt{k+1}},$$

$$sP_k s(x) = \frac{\alpha_k^x - k^x\alpha_k^{-x}}{2\sqrt{k+1}}, \quad cP_k s(x) = \frac{\alpha_k^x + k^x\alpha_k^{-x}}{2\sqrt{k+1}},$$

respectively, where $\alpha_k = 1 + \sqrt{k+1}$. Some recurrence properties and identities, such as the Catalan, Cassini and d'Ocagne identities for symmetrical k -Pell hyperbolic functions were investigated in [15]. Kuloğlu and Özkan [15] also examined the quasi-sine k -Pell functions and three-dimensional k -Pell spiral, along with several of their properties.

For more papers on hyperbolic functions, we refer the reader to [16-23].

Studying matrix analogues of hyperbolic functions derived from well-known number sequences provides insights into both matrix theory and differential equations. For example, matrix exponential and trigonometric matrix functions are in the solutions of some differential

equations. Furthermore, investigating matrix functions derived from well-known number sequences enables researchers to obtain meaningful and applicable results.

The matrix forms of some trigonometric and hyperbolic functions were defined in [24] as follows

$$\sin(M) = \frac{e^{iM} - e^{-iM}}{2i}, \quad \cos(M) = \frac{e^{iM} + e^{-iM}}{2},$$

$$\sinh(M) = \frac{e^M - e^{-M}}{2}, \quad \cosh(M) = \frac{e^M + e^{-M}}{2},$$

where M is a matrix in $\mathbb{C}^{n \times n}$, the space of $n \times n$ complex matrices. Symmetrical hyperbolic Fibonacci sine and cosine matrix functions

$$sFs(M) = \frac{\theta^M - \theta^{-M}}{\sqrt{5}}, \quad cFs(M) = \frac{\theta^M + \theta^{-M}}{\sqrt{5}},$$

and the corresponding Lucas type functions

$$sLs(M) = \theta^M - \theta^{-M}, \quad cLs(M) = \theta^M + \theta^{-M}$$

were introduced by Bahşı and Solak [25], for $M \in \mathbb{C}^{n \times n}$ and $\theta = \frac{1+\sqrt{5}}{2}$. The authors examined some properties of these functions, including recurrence relations, de Moivre and Cassini type identities, and the Pythagorean Theorem. Bahşı and Mersin [26] defined hyperbolic Horadam sine and cosine matrix functions

$$sW(M) = \frac{A\phi^{2M} - Bq^{2M}\phi^{-2M}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cW(M) = \frac{A\phi^{2M+1} + Bq^{2M+1}\phi^{-2M-1}}{\sqrt{p^2 + 4q}}$$

and their symmetrical forms

$$sWs(M) = \frac{A\phi^M - Bq^M\phi^{-M}}{\sqrt{p^2 + 4q}} \quad \text{and} \quad cWs(M) = \frac{A\phi^M + Bq^M\phi^{-M}}{\sqrt{p^2 + 4q}}$$

for $\phi = \frac{p+\sqrt{p^2+4q}}{2}$ and non-zero real numbers p and q such that $p^2 + 4q > 0$. Some hyperbolic and recurrence properties of these functions were examined in [26]. The authors also defined quasi-sine Horadam matrix function and the matrix form of the Metallic shofars that are related to the hyperbolic Horadam sine and cosine matrix functions.

By the motivation of these papers, we aim to introduce the matrix form of the k -Pell hyperbolic functions and their symmetrical forms. We investigate some properties including some recurrence relations, sum and difference identities, Pythagorean Theorem, Catalan, Cassini, De-Moivre type identities for the newly defined functions. In addition, we define the matrix form of the three-dimensional k -Pell spiral.

First, we present some of the basic properties of the matrix exponential:

i. For $0 \in \mathbb{C}^{n \times n}$ and $p \in \mathbb{R}^+$, $p^0 = I$,

- ii. For $p \in \mathbb{R}^+$ and $r \in \mathbb{Z}$, $p^I = pI$ and $p^{rI} = (p^r)I$,
 - iii. For $M \in \mathbb{C}^{n \times n}$ and $p \in \mathbb{R}^+$, $(p^M)^{-1} = p^{-M}$,
 - iv. For commutable matrices M and N of order n , and $p \in \mathbb{R}^+$, $p^M p^N = p^{M+N}$,
- where I denotes the identity matrix of order n [25].

Lemma 1.1 The following equalities hold true

- i. $\alpha_k^I = 2I + \frac{k}{\alpha_k^I}$,
- ii. $\alpha_k^{2I} + 2kI + k^{2I} \alpha_k^{-2I} = 4(k+1)I$,

where I denotes the identity matrix of order n .

Proof.

- i. By means of [15], we have the scalar identity $\alpha_k = 2 + \frac{k}{\alpha_k}$. For the matrix expressions, this gives

$$2I + \frac{k}{\alpha_k^I} = 2I + \frac{k}{\alpha_k} I = \left(2 + \frac{k}{\alpha_k}\right) I = \alpha_k I = \alpha_k^I,$$

where the last equality follows from the definition of matrix exponential with scalar exponents.

- ii. Considering $\alpha_k = 1 + \sqrt{k+1}$, we have $\alpha_k^2 + 2k + k^2 \alpha_k^{-2} = 4(k+1)$. Then,

$$\begin{aligned} \alpha_k^{2I} + 2kI + k^{2I} \alpha_k^{-2I} &= \alpha_k^2 I + 2kI + k^2 I \alpha_k^{-2} I \\ &= (\alpha_k^2 + 2k + k^2 \alpha_k^{-2}) I \\ &= 4(k+1)I. \end{aligned}$$

2. The k -Pell hyperbolic matrix function

In this section, we extend the concept of k -Pell hyperbolic functions to the matrix domain, following on approach analogous to that used for hyperbolic Horadam functions in [26]. We first define these matrix functions and then systematically investigate their fundamental properties, including recurrence relations and identities that generalize the scalar case.

Definition 2.1 The k -Pell hyperbolic sine and cosine matrix functions are defined as

$$sP_k(M) = \frac{\alpha_k^{2M} - k^{2M} \alpha_k^{-2M}}{2\sqrt{k+1}} \quad \text{and} \quad cP_k(M) = \frac{\alpha_k^{2M+I} + k^{2M+I} \alpha_k^{-2M-I}}{2\sqrt{k+1}},$$

where $M \in \mathbb{C}^{n \times n}$, $\alpha_k = 1 + \sqrt{k+1}$ and $k \geq 1$.

Throughout the paper, we use $\alpha_k = 1 + \sqrt{k+1}$ with $k \geq 1$, and this is not stated repeatedly.

Since the hyperbolic functions sP_k and cP_k are not symmetrical with respect to the origin, we construct their symmetrical representations using the approach of Stakhov and Rozin [8].

Definition 2.2 Symmetrical k -Pell hyperbolic sine and cosine matrix functions are defined as

$$sP_k s(M) = \frac{\alpha_k^M - k^M \alpha_k^{-M}}{2\sqrt{k+1}} \quad \text{and} \quad cP_k s(M) = \frac{\alpha_k^M + k^M \alpha_k^{-M}}{2\sqrt{k+1}}.$$

To illustrate the newly defined functions in Definition 2.2, we present the following example.

Example 2.1 Consider the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

We evaluate the symmetrical k -Pell hyperbolic sine and cosine matrix functions defined in Definition 2.2 for the matrix M with integer values $k = 1, 2, \dots, 5$. The computed results are presented in Table 1.

Table 1. The values of the matrix functions $sP_k s(M)$ and $cP_k s(M)$ for M with $k = 1, 2, \dots, 5$.

k	α_k	$sP_k s(M)$	$cP_k s(M)$
1	2.414	$\begin{bmatrix} 2 & 1.293 \\ 0 & 0.707 \end{bmatrix}$	$\begin{bmatrix} 2.121 & 1.121 \\ 0 & 1 \end{bmatrix}$
2	2.732	$\begin{bmatrix} 2 & 1.423 \\ 0 & 0.577 \end{bmatrix}$	$\begin{bmatrix} 2.309 & 1.309 \\ 0 & 1 \end{bmatrix}$
3	3	$\begin{bmatrix} 2 & 1.5 \\ 0 & 0.5 \end{bmatrix}$	$\begin{bmatrix} 2.5 & 1.5 \\ 0 & 1 \end{bmatrix}$
4	3.236	$\begin{bmatrix} 2 & 1.553 \\ 0 & 0.447 \end{bmatrix}$	$\begin{bmatrix} 2.683 & 1.683 \\ 0 & 1 \end{bmatrix}$
5	3.449	$\begin{bmatrix} 2 & 1.592 \\ 0 & 0.408 \end{bmatrix}$	$\begin{bmatrix} 2.858 & 1.858 \\ 0 & 1 \end{bmatrix}$

According to Table 1, the results show that the matrix structure is preserved, with both $sP_k s(M)$ and $cP_k s(M)$ maintaining the upper triangular form of M . As k increases, clear patterns emerge in the entries: the diagonal entries of $cP_k s(M)$ increase, while those of $sP_k s(M)$ decrease, and the off-diagonal entries change steadily. These results help in understanding the fundamental behavior of the matrix functions and provide a basis for exploring their theoretical properties and applications in matrix analysis.

Now, we investigate some properties of symmetrical k -Pell hyperbolic sine and cosine matrix functions.

Proposition 2.1 The Pythagorean Theorem for symmetrical k -Pell hyperbolic function is as follows

$$[cP_k s(M)]^2 - [sP_k s(M)]^2 = \frac{k^M}{k+1},$$

where $M \in \mathbb{C}^{n \times n}$.

Proof.

$$\begin{aligned} & [cP_k s(M)]^2 - [sP_k s(M)]^2 \\ &= \frac{(\alpha_k^M + k^M \alpha_k^{-M})^2 - (\alpha_k^M - k^M \alpha_k^{-M})^2}{4(k+1)} \\ &= \frac{\alpha_k^{2M} + 2k^M \alpha_k^M \alpha_k^{-M} + k^{2M} \alpha_k^{-2M} - \alpha_k^{2M} + 2k^M \alpha_k^M \alpha_k^{-2M} - k^{2M} \alpha_k^{-2M}}{4(k+1)} \\ &= \frac{4k^M \alpha_k^M \alpha_k^{-M}}{4(k+1)} \\ &= \frac{k^M}{k+1}. \end{aligned}$$

This Pythagorean-type relation parallels the classical identity $\cosh^2(x) - \sinh^2(x) = 1$. However, due to the k -Pell structure, the right-hand side becomes $\frac{k^M}{k+1}$. In the scalar case where $M = x$, this simplifies to $\frac{k^x}{k+1}$ which is not identically 1, thus generalizing the classical identity.

Proposition 2.2 The following sum and difference equalities hold true for symmetrical k -Pell hyperbolic functions

- i. $cP_k s(M+N) = \sqrt{k+1}[cP_k s(M)cP_k s(N) + sP_k s(M)sP_k s(N)],$
- ii. $cP_k s(M-N) = k^{-N}\sqrt{k+1}[cP_k s(M)cP_k s(N) - sP_k s(M)sP_k s(N)],$
- iii. $sP_k s(M+N) = \sqrt{k+1}[sP_k s(M)cP_k s(N) + cP_k s(M)sP_k s(N)],$
- iv. $sP_k s(M-N) = k^{-N}\sqrt{k+1}[sP_k s(M)cP_k s(N) - cP_k s(M)sP_k s(N)],$

where M and N are commutable matrices of order n .

Proof. To avoid repetition, we only give the proof of the first equality.

Before starting the algebraic expansion, we note that the combination of exponential terms such as $\alpha_k^M \alpha_k^N = \alpha_k^{M+N}$ and $k^N k^M = k^{M+N}$ are valid only when M and N commute. This is the step in the proof where the commutativity assumption is needed.

$$\begin{aligned}
& cP_k s(M) cP_k s(N) + sP_k s(M) sP_k s(N) \\
&= \frac{(\alpha_k^M + k^M \alpha_k^{-M})(\alpha_k^N + k^N \alpha_k^{-N}) + (\alpha_k^M - k^M \alpha_k^{-M})(\alpha_k^N - k^N \alpha_k^{-N})}{4(k+1)} \\
&= \frac{\alpha_k^{M+N} + k^N \alpha_k^{M-N} + k^M \alpha_k^{-M+N} + k^{M+N} \alpha_k^{-M-N} + \alpha_k^{M+N} - k^N \alpha_k^{M-N} - k^M \alpha_k^{-M+N}}{4(k+1)} \\
&= \frac{2\alpha_k^{M+N} + 2k^{M+N} \alpha_k^{-M-N}}{4(k+1)} \\
&= \frac{\alpha_k^{M+N} + k^{M+N} \alpha_k^{-M-N}}{2(k+1)} \\
&= \frac{1}{\sqrt{k+1}} cP_k s(M+N).
\end{aligned}$$

Corollary 2.1 If we take $M = N$ in Proposition 2.2 (i) and (iii), we have

- i. $cP_k s(2M) = \sqrt{k+1}([cP_k s(M)]^2 + [sP_k s(M)]^2),$
- ii. $sP_k s(2M) = 2\sqrt{k+1}sP_k s(M)cP_k s(M).$

Proposition 2.3 The following recurrence relation holds between the symmetrical k -Pell hyperbolic sine and cosine matrix functions

$$sP_k s(M+I) = 2cP_k s(M) + ksP_k s(M-I),$$

where $M \in \mathbb{C}^{n \times n}$ and I denotes the identity matrix of order n .

Proof. Since the identity matrix I commutes with any matrix M , the usual rules of exponents apply when combining terms such as α_k^M , α_k^{M-I} , k^M , and k^{M-I} . This justifies each algebraic manipulation in the derivation. Considering Lemma 1.1, we obtain

$$\begin{aligned}
2cP_k s(M) + ksP_k s(M-I) &= \frac{2(\alpha_k^M + k^M \alpha_k^{-M}) + k(\alpha_k^{M-I} - k^{M-I} \alpha_k^{-M+I})}{2\sqrt{k+1}} \\
&= \frac{1}{2\sqrt{k+1}} \left(\alpha_k^M \left(2 + \frac{k}{\alpha_k} \right) I + k^M \alpha_k^{-M} (2 - \alpha_k) I \right) \\
&= \frac{\alpha_k^{M+I} - k^{M+I} \alpha_k^{-M-I}}{2\sqrt{k+1}} \\
&= sP_k s(M+I).
\end{aligned}$$

Proposition 2.4 The Catalan type identities for k -Pell hyperbolic functions are as follows

- i. $cP_k s(M-N)cP_k s(M+N) - (cP_k s(M))^2 = k^{M-N}(sP_k s(N))^2,$

- ii. $cP_kS(M - N)cP_kS(M + N) - (sP_kS(M))^2 = k^{M-N}(cP_kS(N))^2$,
- iii. $sP_kS(M - N)sP_kS(M + N) - (sP_kS(M))^2 = -k^{M-N}(sP_kS(N))^2$,
- iv. $sP_kS(M - N)sP_kS(M + N) - (cP_kS(M))^2 = -k^{M-N}(cP_kS(N))^2$,

where M and N are commutable matrices of order n .

Proof. To avoid repetition, we show the proof for the first equality only.

Before carrying out the expansions, we recall that the algebraic manipulations involving expressions such as $\alpha_k^{M \mp N}$ and $k^{M \mp N}$ are valid only when M and N commute. This ensures that the usual exponent rules extend to the matrix setting and justifies each step in the simplification.

$$\begin{aligned}
 & cP_kS(M - N)cP_kS(M + N) - (cP_kS(M))^2 \\
 &= \frac{(\alpha_k^{M-N} + k^{M-N}\alpha_k^{-M+N})(\alpha_k^{M+N} + k^{M+N}\alpha_k^{-M-N}) - (\alpha_k^M + k^M\alpha_k^{-M})^2}{4(k+1)} \\
 &= \frac{\alpha_k^{2M} + k^{M+N}\alpha_k^{-2N} + k^{M-N}\alpha_k^{2N} + k^{2M}\alpha_k^{-2M} - \alpha_k^{2M} - 2k^M I - k^{2M}\alpha_k^{-2M}}{4(k+1)} \\
 &= \frac{k^{M+N}\alpha_k^{-2N} + k^{M-N}\alpha_k^{2N} - 2k^M}{4(k+1)} \\
 &= \frac{k^{M-N}(k^{2N}\alpha_k^{-2N} + \alpha_k^{2N} - 2k^N)}{4(k+1)} \\
 &= k^{M-N}(sP_kS(N))^2.
 \end{aligned}$$

Proposition 2.5 The Cassini type identities for k -Pell hyperbolic functions are as follows

- i. $cP_kS(M - I)cP_kS(M + I) - (sP_kS(M))^2 = k^{M-I}$,
- ii. $sP_kS(M - I)sP_kS(M + I) - (cP_kS(M))^2 = -k^{M-I}$,

where $M \in \mathbb{C}^{n \times n}$ and I denotes the identity matrix of order n .

Proof. Since the identity matrix I commutes with any matrix M , the manipulations involving expressions such as $\alpha_k^{M \mp N}$ and $k^{M \mp N}$ are valid. The proof then follows analogously to Proposition 2.4.

Proposition 2.6 The d'Ocagne identity for k -Pell hyperbolic functions are as follows

- i. $cP_kS(M)cP_kS(N + L) - sP_kS(M + L)sP_kS(N) = k^N cP_kS(L)cP_kS(M - N)$,
- ii. $cP_kS(M)sP_kS(N + L) - cP_kS(M + L)sP_kS(N) = k^N sP_kS(L)cP_kS(M - N)$,

where M, N and L are pairwise commutable matrices of order n .

Proof. The proof relies on the fact that M, N and L commute pairwise, which guarantees that the exponential terms appearing in expressions such as α_k^{M+L} , k^{N+L} , and α_k^{M-N} can be combined using the usual exponent rules. This ensures that each algebraic manipulation in the expansion is valid within the matrix setting.

$$\begin{aligned}
 & cP_k s(M) cP_k s(N+L) - sP_k s(M+L) sP_k s(N) \\
 &= \frac{(\alpha_k^M + k^M \alpha_k^{-M})(\alpha_k^{N+L} + k^{N+L} \alpha_k^{-N-L}) - (\alpha_k^{M+L} - k^{M+L} \alpha_k^{-M-L})(\alpha_k^N - k^N \alpha_k^{-N})}{4(k+1)} \\
 &= \frac{k^{N+L} \alpha_k^{M-N-L} + k^M \alpha_k^{-M+N+L} + k^N \alpha_k^{M-N+L} + k^{M+L} \alpha_k^{-M+N-L}}{4(k+1)} \\
 &= \left[\frac{\alpha_k^L + k^L \alpha_k^{-L}}{2\sqrt{k+1}} \right] \left[\frac{k^N \alpha_k^{M-N} - k^M \alpha_k^{-M+N}}{2\sqrt{k+1}} \right] \\
 &= \left[\frac{\alpha_k^L + k^L \alpha_k^{-L}}{2\sqrt{k+1}} \right] k^N \left[\frac{\alpha_k^{M-N} - k^{M-N} \alpha_k^{-M+N}}{2\sqrt{k+1}} \right] \\
 &= k^N cP_k s(L) cP_k s(M-N).
 \end{aligned}$$

This completes the proof of (i), and the proof of (ii) follows by a similar argument.

Proposition 2.7 The de Moivre type identities for k -Pell hyperbolic functions are as follows

$$\begin{aligned}
 \text{i. } [cP_k s(M) + sP_k s(M)]^n &= \left[\frac{2\alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} [cP_k s(M) + sP_k s(M)], \\
 \text{ii. } [cP_k s(M) - sP_k s(M)]^n &= \left[\frac{2k^M \alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} [cP_k s(M) - sP_k s(M)],
 \end{aligned}$$

for $M \in \mathbb{C}^{n \times n}$.

Proof. In the following proof, we utilize the properties of matrix functions to verify the de Moivre type identities. When we write $[cP_k s(M) + sP_k s(M)]^n$, we are referring to the n -th power of the matrix $cP_k s(M) + sP_k s(M)$, which is well-defined since the involved matrices commute.

$$\begin{aligned}
 \text{i. } \left[\frac{2\alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} [cP_k s(M) + sP_k s(M)] &= \left[\frac{2\alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} \frac{2\alpha_k^M}{2\sqrt{k+1}} \\
 &= \left[\frac{2\alpha_k^M}{2\sqrt{k+1}} \right]^n \\
 &= \left[\frac{\alpha_k^M + k^M \alpha_k^{-M}}{2\sqrt{k+1}} + \frac{\alpha_k^M - k^M \alpha_k^{-M}}{2\sqrt{k+1}} \right]^n \\
 &= [cP_k s(M) + sP_k s(M)]^n.
 \end{aligned}$$

ii.

$$\begin{aligned}
 \left[\frac{2k^M \alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} [cP_k s(M) - sP_k s(M)] &= \left[\frac{2k^M \alpha_k^M}{2\sqrt{k+1}} \right]^{n-1} \frac{2k^M \alpha_k^M}{2\sqrt{k+1}} \\
 &= \left[\frac{2k^M \alpha_k^M}{2\sqrt{k+1}} \right]^n \\
 &= \left[\frac{\alpha_k^M + k^M \alpha_k^{-M}}{2\sqrt{k+1}} - \frac{\alpha_k^M - k^M \alpha_k^{-M}}{2\sqrt{k+1}} \right]^n \\
 &= [cP_k s(M) - sP_k s(M)]^n.
 \end{aligned}$$

We note that the n -th powers in the identities are well-defined, since the matrices involved commute and the usual rules for multiplying commuting matrices apply.

3. The quasi-sine k -Pell matrix functions

Considering the equation $(-1)^n = \cos(n\pi)$ for $n \in \mathbb{Z}$, the Binet formula of the k -Pell numbers can be rewritten as

$$P_{k,n} = \frac{\alpha_k^n - k^n \cos(n\pi) \alpha_k^{-n}}{2\sqrt{k+1}}.$$

Now, we introduce the quasi-sine k -Pell matrix function and examine some of its properties.

Definition 3.1 The quasi-sine k -Pell matrix function is defined as

$$PP_k(M) = \frac{\alpha_k^M - k^M \cos(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}},$$

where $M \in \mathbb{C}^{n \times n}$.

Proposition 3.1 The following recurrence relation holds for the quasi-sine k -Pell matrix function

$$PP_k(M + 2I) = 2PP_k(M + I) + kPP_k(M)$$

for $M \in \mathbb{C}^{n \times n}$ and n th order identity matrix I .

Proof. The identities

$$\cos(\pi(M + 2I)) = \cos(\pi M) = -\cos(\pi(M + I)) \quad (3.1)$$

in [24] follows from extending basic trigonometric properties to matrix functions. This extension uses the periodicity and phase shift properties of the cosine function in the scalar case, applied to matrices based on their eigenvalue structure. The validity of this identity for all square matrices is confirmed through eigenvalue analysis for diagonalizable matrices, Jordan form for non-diagonalizable ones, and by using power series expansions via analytic continuation. Before proving the theorem, we demonstrate these matrix cosine identities with a

simple example using a diagonalizable matrix M of order 2 and the identity matrix I of the same order. The eigenvalues of the matrix

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

are $\lambda_1 = 2$, $\lambda_2 = 1$ with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, respectively. Thus,

$$M = R_1 D_1 R_1^{-1}$$

is valid, where

$$R_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } R_1^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Since

$$\cos(\pi D_1) = \begin{bmatrix} \cos(2\pi) & 0 \\ 0 & \cos(\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we have

$$\begin{aligned} \cos(\pi M) &= R_1 \cos(\pi D_1) R_1^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}. \end{aligned}$$

Now, we compute $\cos(\pi(M + 2I))$ for the matrix

$$M + 2I = \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix}.$$

The eigenvalues of $M + 2I$ are $\mu_1 = 4$, $\mu_2 = 3$, and corresponding eigenvectors are $w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Then,

$$M + 2I = R_2 D_2 R_2^{-1}$$

is valid, where

$$R_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D_2 = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \text{ and } R_2^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \cos(\pi(M + 2I)) &= R_2 \cos(\pi D_2) R_2^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(4\pi) & 0 \\ 0 & \cos(3\pi) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}.
\end{aligned}$$

Finally, we compute $\cos(\pi(M + I))$ for the matrix

$$M + I = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

with eigenvalues $\gamma_1 = 3$, $\gamma_2 = 2$ with corresponding eigenvectors $\sigma_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\sigma_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Then,

$$M + I = R_3 D_3 R_3^{-1}$$

is valid, where

$$R_3 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, D_3 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } R_3^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned}
\cos(\pi(M + I)) &= R_3 \cos(\pi D_3) R_3^{-1} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(3\pi) & 0 \\ 0 & \cos(2\pi) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

As a result, the equalities given in equation (3.1) have been demonstrated. We now proceed with the proof of the theorem.

$$\begin{aligned}
&2PP_k(M + I) + kPP_k(M) \\
&= 2 \frac{\alpha_k^{M+I} - k^{M+I} \cos(\pi(M + I)) \alpha_k^{-M-I}}{2\sqrt{k+1}} + k \frac{\alpha_k^M - k^M \cos(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}} \\
&= \frac{\alpha_k^M (2\alpha_k^I + kI) + \cos(\pi M) \alpha_k^{-M-2I} k^{M+I} (2\alpha_k^I - \alpha_k^{2I})}{2\sqrt{k+1}} \\
&= \frac{\alpha_k^{M+2I} - k^{M+2I} \cos(\pi(M + 2I)) \alpha_k^{-M-2I}}{2\sqrt{k+1}} \\
&= PP_k(M + 2I).
\end{aligned}$$

Proposition 3.2 The Cassini identity for the quasi-sine k -Pell matrix function is

$$PP_k(M + I)PP_k(M - I) - (PP_k(M))^2 = k^{M-I} \cos(\pi M)$$

for $M \in \mathbb{C}^{n \times n}$ and n th order identity matrix I .

Proof. The trigonometric identities

$$\cos(\pi(M + I)) = \cos(\pi(M - I)) = -\cos(\pi M),$$

which follow from the periodic relation in equation (3.1) and the even symmetry of the cosine function, were established in [24] and yield

$$\begin{aligned} & PP_k(M + I)PP_k(M - I) - (PP_k(M))^2 \\ &= \frac{(\alpha_k^{M+I} - k^{M+I} \cos(\pi(M + I)) \alpha_k^{-M-I})(\alpha_k^{M-I} - k^{M-I} \cos(\pi(M - I)) \alpha_k^{-M+I})}{4(k + 1)} \\ &\quad - \frac{(\alpha_k^M - k^M \cos(\pi M) \alpha_k^{-M})^2}{4(k + 1)} \\ &= \frac{(\alpha_k^{M+I} + k^{M+I} \cos(\pi M) \alpha_k^{-M-I})(\alpha_k^{M-I} + k^{M-I} \cos(\pi M) \alpha_k^{-M+I})}{4(k + 1)} \\ &\quad - \frac{(\alpha_k^M k^M \cos(\pi M) \alpha_k^{-M})^2}{4(k + 1)} \\ &= \frac{k^{M-I} \cos(\pi M) \alpha_k^{2I} + k^{M+I} \cos(\pi M) \alpha_k^{-2I} + 2k^M \cos(\pi M)}{4(k + 1)} \\ &= k^{M-I} \cos(\pi M) \frac{\alpha_k^{2I} + k^{2I} \alpha_k^{-2I} + 2k^I}{4(k + 1)} \\ &= k^{M-I} \cos(\pi M). \end{aligned}$$

4. The matrix form of the three-dimensional k -Pell spiral

Kuloğlu and Özkan [15] introduced the function

$$\begin{aligned} CPP_k(x) &= \frac{\alpha_k^x - k^x \cos(\pi x) \alpha_k^{-x}}{2\sqrt{k + 1}} + i \frac{k^x \sin(\pi x) \alpha_k^{-x}}{2\sqrt{k + 1}} \\ &= \frac{\alpha_k^x - k^x + i e^{i\pi(\frac{1}{2}-x)} \alpha_k^{-x}}{2\sqrt{k + 1}} \end{aligned}$$

which is called the three-dimensional k -Pell spiral. Finally, we give the matrix form of the function $CPP_k(x)$.

Definition 4.1 The matrix form of the three-dimensional k -Pell spiral is defined by

$$\begin{aligned} CPP_k(M) &= \frac{\alpha_k^M - k^M \cos(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}} + i \frac{k^M \sin(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}} \\ &= \frac{\alpha_k^M - k^M + i e^{i\pi(\frac{1}{2}-M)} \alpha_k^{-M}}{2\sqrt{k+1}}, \end{aligned}$$

where $M \in \mathbb{C}^{n \times n}$ and $i^2 = -1$.

Proposition 4.1 The following recurrence relation holds for the function $CPP_k(M)$ as follows

$$CPP_k(M + 2I) = 2CPP_k(M + I) + kCPP_k(M),$$

where $M \in \mathbb{C}^{n \times n}$ and n th order identity matrix I .

Proof. The trigonometric identities

$$\sin(\pi(M + 2I)) = \sin(\pi M) = -\sin(\pi(M + I)) \quad (3.2)$$

are valid for $M \in \mathbb{C}^{n \times n}$ and the n th order identity matrix I [24], follow from the periodicity and odd symmetry of the sine function. Using the trigonometric equalities given in equations (3.1) and (3.2), we obtain

$$\begin{aligned} &2CPP_k(M + I) + kCPP_k(M) \\ &= 2 \left(\frac{\alpha_k^{M+I} - k^{M+I} \cos(\pi(M + I)) \alpha_k^{-M-I}}{2\sqrt{k+1}} + i \frac{k^{M+I} \sin(\pi(M + I)) \alpha_k^{-M-I}}{2\sqrt{k+1}} \right) \\ &\quad + k \left(\frac{\alpha_k^M - k^M \cos(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}} + i \frac{k^M \sin(\pi M) \alpha_k^{-M}}{2\sqrt{k+1}} \right) \\ &= \frac{\alpha_k^M (2\alpha_k^I + kI) - \cos(\pi M) (-2k^{M+I} \alpha_k^{-M-I} + k^{M+I} \alpha_k^{-M})}{2\sqrt{k+1}} \\ &\quad + i \frac{\sin(\pi(M + 2I)) (-2k^{M+I} \alpha_k^{-M-I} + k^{M+I} \alpha_k^{-M})}{2\sqrt{k+1}} \\ &= \frac{\alpha_k^{M+2I} - \cos(\pi M) (k^{M+I} \alpha_k^{-M-I} (-2I + \alpha_k^{-I}))}{2\sqrt{k+1}} + i \frac{\sin(\pi M) (k^{M+I} \alpha_k^{-M-I} (-2I + \alpha_k^{-I}))}{2\sqrt{k+1}} \\ &= \frac{\alpha_k^{M+2I} - k^{M+2I} \cos(\pi(M + 2I)) \alpha_k^{-(M+2I)}}{2\sqrt{k+1}} + i \frac{k^{M+2I} \sin(\pi(M + 2I)) \alpha_k^{-M-2I}}{2\sqrt{k+1}} \\ &= CPP_k(M + 2I). \end{aligned}$$

This completes the proof.

Finally, we provide the matrix representation of the shrinking incomplete Metallic Shofar. The function $CPP_k(X)$ has the following real and imaginary components

$$Re(CPP_k(X)) = \frac{\alpha_k^X - k^X \cos(\pi X) \alpha_k^{-X}}{2\sqrt{k+1}} = y(X),$$

$$Im(CPP_k(X)) = \frac{k^X \sin(\pi X) \alpha_k^{-X}}{2\sqrt{k+1}} = z(X),$$

respectively, for $X \in \mathbb{C}^{n \times n}$. By choosing the y axis as the real axis and the z axis as the imaginary axis, the system of matrix equations depends on the matrix form of the three-dimensional k -Pell spiral is

$$y(X) - \frac{\alpha_k^X}{2\sqrt{k+1}} = \frac{-k^X \cos(\pi X) \alpha_k^{-X}}{2\sqrt{k+1}},$$

$$z(X) = \frac{k^X \sin(\pi X) \alpha_k^{-X}}{2\sqrt{k+1}}.$$

Hence, we have the equation of the matrix form of the shrinking incomplete Metallic Shofar as follows

$$\left(y(X) - \frac{\alpha_k^X}{2\sqrt{k+1}}\right)^2 + z^2(X) = \left(\frac{k^X \alpha_k^{-X}}{2\sqrt{k+1}}\right)^2.$$

For specific values of k , we obtain special cases of mathematical interest. When $k = 1, 2$ or 3 , we derive the matrix forms of what are known in the literature as the Shrinking Incomplete Golden, Silver, and Bronze Shofars, respectively. These are defined as follows

$$\left(y(X) - \frac{\alpha_1^X}{2\sqrt{2}}\right)^2 + z^2(X) = \left(\frac{\alpha_1^{-X}}{2\sqrt{2}}\right)^2,$$

$$\left(y(X) - \frac{\alpha_2^X}{2\sqrt{3}}\right)^2 + z^2(X) = \left(\frac{2^X \alpha_2^{-X}}{2\sqrt{3}}\right)^2,$$

$$\left(y(X) - \frac{\alpha_3^X}{4}\right)^2 + z^2(X) = \left(\frac{3^X \alpha_3^{-X}}{4}\right)^2,$$

where α_1 , α_2 and α_3 correspond to Golden, Silver, and Bronze proportions, respectively.

5. Conclusion

In this paper, we introduced the k -Pell hyperbolic functions and their symmetrical forms. We demonstrated an example for a specific matrix and some values of k . We explored recurrence relations, sum and difference equations, Pythagorean identities, and several important identities (Catalan, Cassini, d'Ocagne, and de Moivre-type) for the symmetrical k -Pell hyperbolic functions. Furthermore, we presented the matrix form of the three-dimensional k -Pell spiral and discussed some of its properties.

While the k -Pell sequence generalizes the classical Pell numbers, the Horadam sequence extends various classical sequences, such as Fibonacci, Lucas, and Pell. Therefore, similar patterns appear in the matrix forms of both the k -Pell and Horadam sequences. Specifically, we observed similar properties between the matrix forms of the k -Pell and Horadam hyperbolic functions. For $k = 1$, the findings in this paper align with those in [26] for hyperbolic Horadam matrix functions. However, for $k \neq 1$, our results are specific to hyperbolic matrix functions defined using the k -Pell number sequence. These similarities may provide a useful framework for generalizing other number sequences.

Since matrix functions are important in differential equations and many other fields, we believe that these matrix functions, based on well-known number sequences, will make a meaningful contribution to both theoretical and applied mathematics.

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