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On The Some Special Biquadratic Bezier Surfaces in E^3

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Highlights:
- Bezier Surfaces
- Control Points
- Matrix Represantation

Keywords:
- Biquadratic bezier Surfaces
- Elliptic paraboloid
- Hyperbolic Paraboloid
- Parabolic cylinder

ABSTRACT:

In this study, biquadratic Bezier surfaces were investigated. These surfaces have a very important place in the scientific world. The importance of biquadratic Bézier surfaces lies in their ability to provide a smooth, flexible, and computationally efficient method for modeling surfaces. They fill the gap between simpler bilinear surfaces and more complex bicubic surfaces, offering a great solution in areas where both simplicity and precision are needed. While not as commonly used in high-end 3D modeling as their higher-order counterparts, they remain a useful tool for surface design, particularly when the trade-off between quality and computational efficiency is critical. In this study, firstly the matrix representations of biquadratic Bezier surfaces are given. Then, it is stated how to find the control points of some special surface types given as biquadratic Bezier surfaces, and the control points of parabolic cylinder, elliptic paraboloid and hyperbolic paraboloid surfaces are calculated.

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INTRODUCTION

Bézier curves and surfaces are one of the fundamental building blocks of computer graphics and animation, providing a high degree of flexibility, accuracy, control and efficiency. They are used in many areas such as design, animation, modeling and industrial applications, both in 2D and 3D areas. Their mathematical simplicity and the fast editing and easy control features they provide in visual applications increase the importance of these tools in the literature. In the literature, numerous mathematical studies and research have been conducted to improve the properties and applications of Bézier curves and surfaces. For example; In (Han et al., 2008), it has been focused on generalized versions of Bézier curves and surfaces. Furthermore, a new generalized approach is presented to make mathematical modeling of traditional Bézier curves and surfaces usable in a wider range of applications. In (Hu et al., 2018), a method is proposed that aims to make the design of evolving surfaces more efficient and flexible by providing local control of H-Bézier surfaces. This study provides an important contribution especially for free-form surfaces and complex geometric structures. In (Zhang, 1999), a survey of the mathematical properties and applications of C-Bézier curves and surfaces is presented. C-Bézier curves and surfaces are a more extended version of traditional Bézier curves and surfaces. This paper provides important contributions to the development of C-Bézier curves and surfaces, their use in design, and especially the effect of control points. In (Michael, 2003) and (Hagen, 1986) the mathematical theory and applications of Bézier curves and surfaces in computer graphics, computational geometry, and related fields. Such studies have allowed the development of more efficient algorithms, higher accuracy surface modeling, and more efficient animation processes. Finding control points of Bézier curves and surfaces is a very important process, because these points determine the shape and properties of the curves or surfaces.

Control points directly affect the shape of the Bézier curve or surface and play a major role in animation, modeling or design processes. Accurately determining these points is a critical factor that improves both visual quality and processing efficiency. Control points determine the shape and location of a Bézier curve or surface. The curve or surface is manipulated by control points. For example, the distance and direction between control points of a Bézier curve determine how the curve will curve, how curved or straight it will be. A similar situation applies to Bézier surfaces, where control points determine the curvature, smoothness, and natural transitions of the surface. Moving control points directly changes the shape of the surface.

Finally in this study, the control points of surfaces such as elliptic paraboloid, hyperbolic paraboloid and parabolic cylinder as biquadratic Bezier surfaces, which have an important place in geometry and applied sciences, were calculated. In addition, matrix representations for biquadratic Bezier surfaces were included for the first time. In this study, some papers that inspired us when finding control points and matrix represantation of biquadratic Bezier surfaces are (Kılıçoğlu & Şenyurt, 2022; Kılıçoğlu, 2023; Kılıçoğlu & Yurttançıkmaz, 2023).

MATERIALS AND METHODS

A Bézier curve is a parametric curve defined by a set of control points**.** These curves are widely used in computer graphics, animation, and design because they are simple to compute and offer intuitive control over the shape of the curve. A Bézier curve of degree n (also called an n-degree Bézier curve) is defined by $n+1$ control points $P_0, P_1, P_2, \dots, P_n$ and is given by the following parametric equation:

$$B(t) = \sum_{i=0}^{n} P_i B_{i,n}(t)$$

Where:
• $B(t)$ is the Bézier curve at parameter value t (where t ranges from 0 to 1).
• P_i are the control points.
• $B_{i,n}(t)$ are the Bernstein basis polynomials, which are defined as:

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

• $\binom{n}{i}$ is the binomial coefficient (Hagen, 1986; Farin, 2002; Michael, 2003).

Definition 1: A quadratic Bézier curve with given the control points P_0, P_1 and P_2 is the path traced by the function

$$\alpha(t) = (1-t)^2 P_0 + 2t(1-t)P_1 + t^2 P_2, \quad t \in [0,1]$$

and has the following matrix representation

$$\alpha(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T [B^2] \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

$$[B^2] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Bézier surfaces are a generalization of Bézier curves into two dimensions. They are used to define smooth, curved surfaces in 3D space, and are often employed in 3D modeling and computer-aided design (CAD). A Bézier surface is defined by a grid of control points**.** For a surface defined by $m+1$ control points along the u-direction and $n+1$ control points along the v-direction, the surface $B(u,v)$ is given by the following parametric equation:

$$B(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i,m}(u) B_{j,n}(v)$$

Where:
• $B(u,v)$ is the Bézier surface at parameters u and v, where $u, v \in [0,1]$.
• $P_{i,j}$ are the control points arranged in a grid.
• $B_{i,m}(u)$ and $B_{j,n}(v)$ are the Bernstein basis polynomials for u and v, respectively, and are defined as:

$$B_{i,m}(u) = \binom{m}{i} u^i (1-u)^{m-i}, \qquad B_{j,n}(v) = \binom{n}{j} v^j (1-v)^{n-j}.$$

Definition 2: Consider a Bezier surface of degree (n,m) is given by set of $P_{i,j}$ of the control points $(n+1)(m+1)$, where $i = 0, \dots, n$ and $j = 0, \dots, m$. A two-dimensional Bézier surface can be defined as a parametric surface at any point P as a function of the parameters u, v as follows

$$\mathbf{B}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} [P_{i,j}] \binom{n}{i} u^i (1-u)^{n-i} \binom{m}{j} v^j (1-v)^{n-j}$$

$$\mathbf{B}(u,v) = [v] M_v P M_u^T [u]^T.$$

For more detail see (Zheng, 2003; Anonymous, 2013).

All of the $u = constant$ and $v = constant$ lines in space (u,v) and especially the edges of the deformed (u,v) unit square are Bezier curves. Since a Bézier surface will lie completely within the convex hull of its control points, it will also completely within the bounding box of its control points in any given Cartesian coordinate system. Although the corners of the deformed unit square coincide with four of the control points, a Bézier surface generally does not pass through its other control points. A

Bézier surface transform in the same way as its control points under all linear transformations. In (Sederberg et al., 1984), it has been proposed implicitization of parametric polynomial surfaces.

RESULTS AND DISCUSSION

Biquadratic Bézier Surface

One of the most common uses of Bézier surfaces is in the form of biquadratic patches where $m = n = 2$, which means both the degree of the surface in the u-direction and the v-direction is 2. In this case, the Bézier surface is defined by a grid of 9 control points arranged in a 3×3 matrix. For more detail see (Zhang&Sederberg, 2003).

Definition 3: For $n = m = 2$, a given biquadratic Bézier surface (order $2x2$) is defined by a set of $(2+1)(2+1) = 9$ control points $P_{i,j}$ where $0 \leq i \leq 2$, $0 \leq j \leq 2$.

$$\boldsymbol{B}(u,v) = \sum_{j=0}^{2}\sum_{i=0}^{2}\binom{2}{i}u^i(1-u)^{n-i}\binom{2}{j}v^j(1-v)^{n-j}[P_{i,j}]$$

$$\begin{aligned}\boldsymbol{B}(u,v) &= [v]M_vPM_u^T[u]^T\\ &= [v][B^2]P[B^2]^T[u]^T\\ &= [v][B^2]P[B^2][u]^T\\ &= \begin{bmatrix}v^2\\ v\\ 1\end{bmatrix}^T\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}S_{00} & S_{01} & S_{02}\\ S_{10} & S_{11} & S_{12}\\ S_{20} & S_{21} & S_{22}\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}u^2\\ u\\ 1\end{bmatrix}\end{aligned}$$

Also

$$\boldsymbol{B}(u,v) = \begin{bmatrix}[v]C_x[u]^T\\ [v]C_y^T[u]\\ [v]C_z^T[u]\end{bmatrix} \quad \text{and} \quad \begin{aligned}C_x &= [B^2]S_x[B^2]\\ C_y &= [B^2]S_y[B^2]\\ C_z &= [B^2]S_z[B^2]\end{aligned}$$

$$\boldsymbol{B}(u,v) = \begin{bmatrix}[v][B^2]S_x[B^2][u]^T\\ [v][B^2]S_y[B^2][u]^T\\ [v][B^2]S_y[B^2][u]^T\end{bmatrix}$$

where stores the coefficients S_x, S_y, and S_z of the biquadratic equation for $x, y,$ and z respectively

$$S_x = \begin{bmatrix}S_{0x} & S_{3x} & S_{6x}\\ S_{1x} & S_{4x} & S_{7x}\\ S_{2x} & S_{5x} & S_{8x}\end{bmatrix}, S_y = \begin{bmatrix}S_{0y} & S_{3y} & S_{6y}\\ S_{1y} & S_{4y} & S_{7y}\\ S_{2y} & S_{5y} & S_{8y}\end{bmatrix}, S_z = \begin{bmatrix}S_{0z} & S_{3z} & S_{6z}\\ S_{1z} & S_{4z} & S_{7z}\\ S_{2z} & S_{5z} & S_{8z}\end{bmatrix}.$$

Paraboloids As Biquadratic Bézier Surface

A paraboloid is a special type of quadric surface with a distinct shape and properties. Whether elliptic or hyperbolic, paraboloids have important applications in fields like optics, architecture, and signal processing due to their unique reflective properties and smooth curves. They also serve as useful models in computer graphics and surface design (Weir et al., 2005).

Theorem 1: The components of the control points for the any parametric surfaces as biquadratic Bézier surface are

$$\begin{aligned}S_x &= [B^2]^{-1}[CMPS]_x[B^2]^{-1}\\ S_y &= [B^2]^{-1}[CMPS]_y[B^2]^{-1}\\ S_z &= [B^2]^{-1}[CMPS]_z[B^2]^{-1}\end{aligned}$$

where $[CMPS]_x =$ coefficients matrix of parametric surface for x components,
$[CMPS]_y =$ coefficients matrix of parametric surface for y components,
$[CMPS]_z =$ coefficients matrix of parametric surface for z components.

Proof: Since we can write any parametric surface and also in biquadratic Bezier surface as in the following matrix forms

$$\varphi(u,v) = \left([v][CMPS]_x^T[u], [v][CMPS]_y^T[u], [v][CMPS]_z^T[u]\right)$$

and biquadratic Bézier surface has the following matrix form

$$\varphi(u,v) = \begin{bmatrix} [v][B^2]S_x[B^2][u]^T \\ [v][B^2]S_y[B^2][u]^T \\ [v][B^2]S_z[B^2][u]^T \end{bmatrix}.$$

By using the equality of both left side ve get the proof easily. Where

$$[v] = [v^2 \quad v \quad 1], \quad [u] = [u^2 \quad u \quad 1]$$

and

$$S_x = \begin{bmatrix} S_{0x} & S_{3x} & S_{6x} \\ S_{1x} & S_{4x} & S_{7x} \\ S_{2x} & S_{5x} & S_{8x} \end{bmatrix}, S_y = \begin{bmatrix} S_{0y} & S_{3y} & S_{6y} \\ S_{1y} & S_{4y} & S_{7y} \\ S_{2y} & S_{5y} & S_{8y} \end{bmatrix}, S_z = \begin{bmatrix} S_{0z} & S_{3z} & S_{6z} \\ S_{1z} & S_{4z} & S_{7z} \\ S_{2z} & S_{5z} & S_{8z} \end{bmatrix}.$$

Elliptic Paraboloid As Biquadratic Bézier Surface

The elliptical paraboloid surface has great importance in the literature, especially in the fields of optics, engineering and mathematical modeling. The elliptical paraboloid surface is a surface with important applications in both technical and aesthetic terms and is used in a wide range from engineering to optics, architecture to 3D modeling.

Theorem 2: The control points of the oriented elliptic paraboloid surface $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ as biquadratic Bézier surface are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{b^2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{b^2} \\ 1 & 0 & \frac{1}{a^2} \\ 1 & \frac{1}{2} & \frac{1}{a^2} \\ 1 & 1 & \frac{1}{a^2}+\frac{1}{b^2} \end{bmatrix}.$$

Proof. We can write the oriented elliptic paraboloid surface $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ as in the parametric form $\varphi(u,v) = \left(u, v, \frac{u^2}{a^2} + \frac{v^2}{b^2}\right)$ with the matrix representation easily

$$\varphi(u,v) = \left([v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[u]^T, [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}[u]^T, [v]\begin{bmatrix} 0 & 0 & \frac{1}{b^2} \\ 0 & 0 & 0 \\ \frac{1}{a^2} & 0 & 0 \end{bmatrix}[u]^T \right). \qquad (1)$$

Also as biquadratic Bézier surface it has the following matrix form

$$\varphi(u,v)=\begin{bmatrix}[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}S_x\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T,\\ [v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}S_y\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T,\\ [v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}S_z\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\end{bmatrix}. \quad (2)$$

Using the equality of the both sides, we get

$$\begin{bmatrix}[v]\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\\ [v]\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\\ [v]\begin{bmatrix}0&0&\frac{1}{b^2}\\0&0&0\\\frac{1}{a^2}&0&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{00}&S_{01}&S_{02}\\S_{10}&S_{11}&S_{12}\\S_{20}&S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\end{bmatrix}.$$

If we simplify these matrix equations we have the following matrix equations

$$\begin{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\\ \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\\ \begin{bmatrix}0&0&\frac{1}{b^2}\\0&0&0\\\frac{1}{a^2}&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{00}&S_{01}&S_{02}\\S_{10}&S_{11}&S_{12}\\S_{20}&S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\end{bmatrix}.$$

Hence we have the coordinates the control points as in the following way

$$\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}=\begin{bmatrix}0&\frac{1}{2}&1\\0&\frac{1}{2}&1\\0&\frac{1}{2}&1\end{bmatrix},$$

$$\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}=\begin{bmatrix}0&0&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\1&1&1\end{bmatrix},$$

$$\begin{bmatrix}S_{0z}&S_{3z}&S_{6z}\\S_{1z}&S_{4z}&S_{7z}\\S_{2z}&S_{5z}&S_{8z}\end{bmatrix}=\begin{bmatrix}0&0&\frac{1}{a^2}\\0&0&\frac{1}{a^2}\\\frac{1}{b^2}&\frac{1}{b^2}&\frac{1}{a^2}+\frac{1}{b^2}\end{bmatrix}.$$

This complete the proof.

Example 1: Lets find the control points of the elliptic paraboloid $z=\frac{x^2}{25}+\frac{y^2}{9}$ as biquadratic Bézier surface. First we can write as

$$\varphi(u,v)=\left(u,v,\frac{u^2}{25}+\frac{v^2}{9}\right)$$

Considering that $a=5$ and $b=3$, the control points are

$$\begin{bmatrix} S_0\\ S_1\\ S_2\\ S_3\\ S_4\\ S_5\\ S_6\\ S_7\\ S_8 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{1}{3^2}\\ \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 1 & \frac{1}{3^2}\\ 1 & 0 & \frac{1}{5^2}\\ 1 & \frac{1}{2} & \frac{1}{5^2}\\ 1 & 1 & \frac{1}{5^2}+\frac{1}{3^2} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{1}{9}\\ \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 1 & \frac{1}{9}\\ 1 & 0 & \frac{1}{25}\\ 1 & \frac{1}{2} & \frac{1}{25}\\ 1 & 1 & \frac{34}{225} \end{bmatrix}.$$

Corollary 1: The control points of the oriented circular paraboloid surfaces $z=\frac{x^2}{a^2}+\frac{y^2}{a^2}$ as biquadratic Bézier surface are

$$\begin{bmatrix} S_0\\ S_1\\ S_2\\ S_3\\ S_4\\ S_5\\ S_6\\ S_7\\ S_8 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 1 & \frac{1}{a^2}\\ \frac{1}{2} & 0 & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & 1 & \frac{1}{a^2}\\ 1 & 0 & \frac{1}{a^2}\\ 1 & \frac{1}{2} & \frac{1}{a^2}\\ 1 & 1 & \frac{2}{a^2} \end{bmatrix}.$$

Corollary 2: Lets find the control points of the elliptic-circular paraboloid $z=x^2+y^2$ as biquadratic Bézier surface. First we can write as

$$\varphi(u,v)=(u,v,u^2+v^2).$$

Considering that $a=1$, the control points are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 1 \\ 1 & 0 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 2: Lets find the control points of the elliptic paraboloid $z = \frac{x^2}{25} + \frac{y^2}{25}$ as biquadratic Bézier surface. First we can write as

$$\varphi(u,v) = \left(u, v, \frac{u^2}{25} + \frac{v^2}{25}\right)$$

Considering that $a = 5$, the control points are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{5^2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{5^2} \\ 1 & 0 & \frac{1}{5^2} \\ 1 & \frac{1}{2} & \frac{1}{5^2} \\ 1 & 1 & \frac{2}{5^2} \end{bmatrix}.$$

Hyperbolic Paraboloid As Biquadratic Bézier Surface

The hyperbolic paraboloid surface is an important quadratic surface that mathematically shows different curvatures on two axes and therefore has a saddle-shaped structure. It is used in many engineering and architectural fields both theoretically and practically. This surface is a fundamental shape in geometry and plays an important role in structural design, engineering, and 3D modeling. Its unique curvature properties make it ideal for applications requiring strong, efficient, and aesthetically distinctive surfaces (Weisstein, 2008).

Theorem 3: The control points of the oriented hyperbolic paraboloid surfaces $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ as biquadratic Bézier surface are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{b^2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{b^2} \\ 1 & 0 & \frac{1}{a^2} \\ 1 & \frac{1}{2} & \frac{1}{a^2} \\ 1 & 1 & \frac{1}{a^2}-\frac{1}{b^2} \end{bmatrix}.$$

Proof. We can write the oriented elliptic paraboloid surface as in the parametric form $\varphi(u,v) = \left(u, v, \frac{u^2}{a^2} - \frac{v^2}{b^2}\right)$ with the matrix representation

$$\varphi(u,v) = \left([v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[u]^T, [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}[u]^T, [v]\begin{bmatrix} 0 & 0 & -\frac{1}{b^2} \\ 0 & 0 & 0 \\ \frac{1}{a^2} & 0 & 0 \end{bmatrix}[u]^T \right). \quad (3)$$

Also as biquadratic Bézier surface it has the following matrix form

$$\varphi(u,v) = \begin{bmatrix} [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[S_x]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[S_y]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}S_z\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T \end{bmatrix}. \quad (4)$$

Using the equality of both sides we get

$$\begin{bmatrix} [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}[u]^T = [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_{0x} & S_{3x} & S_{6x} \\ S_{1x} & S_{4x} & S_{7x} \\ S_{2x} & S_{5x} & S_{8x} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T \\ [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}[u]^T = [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_{0y} & S_{3y} & S_{6y} \\ S_{1y} & S_{4y} & S_{7y} \\ S_{2y} & S_{5y} & S_{8y} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T \\ [v]\begin{bmatrix} 0 & 0 & -\frac{1}{b^2} \\ 0 & 0 & 0 \\ \frac{1}{a^2} & 0 & 0 \end{bmatrix}[u]^T = [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T \end{bmatrix}.$$

If we simplify these matrix equations we have the following matrix equations

$$\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}$$

$$\begin{bmatrix}0&0&-\frac{1}{b^2}\\0&0&0\\\frac{1}{a^2}&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{00}&S_{01}&S_{02}\\S_{10}&S_{11}&S_{12}\\S_{20}&S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}$$

and

$$\begin{bmatrix}S_0&S_{3x}&S_{6x}\\S_x&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}=\begin{bmatrix}0&\frac{1}{2}&1\\0&\frac{1}{2}&1\\0&\frac{1}{2}&1\end{bmatrix}$$

$$\begin{bmatrix}S_0&S_{3y}&S_{6y}\\S_y&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}=\begin{bmatrix}0&0&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\1&1&1\end{bmatrix}$$

$$\begin{bmatrix}S_{0z}&S_{3z}&S_{6z}\\S_{1z}&S_{4z}&S_{7z}\\S_{2z}&S_{5z}&S_{8z}\end{bmatrix}=\begin{bmatrix}0&0&\frac{1}{a^2}\\0&0&\frac{1}{a^2}\\-\frac{1}{b^2}&-\frac{1}{b^2}&\frac{1}{a^2}-\frac{1}{b^2}\end{bmatrix}.$$

Corollary 3: The control points of the oriented hyperbolic paraboloid surfaces $z=\frac{x^2}{a^2}-\frac{y^2}{a^2}$ as biquadratic Bézier surface are

$$\begin{bmatrix}S_0\\S_1\\S_2\\S_3\\S_4\\S_5\\S_6\\S_7\\S_8\end{bmatrix}=\begin{bmatrix}0&0&0\\0&\frac{1}{2}&0\\0&1&-\frac{1}{a^2}\\\frac{1}{2}&0&0\\\frac{1}{2}&\frac{1}{2}&0\\\frac{1}{2}&1&-\frac{1}{a}\\1&0&\frac{1}{a^2}\\1&\frac{1}{2}&\frac{1}{a^2}\\1&1&0\end{bmatrix}.$$

Example 3: Lets find the control points of the hyperbolic paraboloid $z=\frac{x^2}{25}-\frac{y^2}{9}$ as biquadratic Bézier surface. First we can write as

$$\varphi(u,v)=\left(u,v,\frac{u^2}{25}-\frac{v^2}{9}\right)$$

Considering that $a=5$ and $b=3$, the control points are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3^2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{3^2} \\ 1 & 0 & \frac{1}{5^2} \\ 1 & \frac{1}{2} & \frac{1}{5^2} \\ 1 & 1 & \frac{1}{5^2}-\frac{1}{3^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{9} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & -\frac{1}{9} \\ 1 & 0 & \frac{1}{25} \\ 1 & \frac{1}{2} & \frac{1}{25} \\ 1 & 1 & -\frac{16}{225} \end{bmatrix}.$$

Parabolic Cylinder As Biquadratic Bézier Surface

A parabolic cylinder is a type of quadratic surface that can be thought of as a parabola extended along one direction (usually the zzz-axis), forming a cylindrical shape**.** Unlike a typical cylinder, which is defined by a circular or elliptical cross-section, a parabolic cylinder has a parabolic cross-section**.** Parabolic cylinder can be represented by the equation $z = ax^2$.

Theorem 4: The control points of the oriented parabolic cylinder surfaces $z = ax^2$ as biquadratic Bézier surface are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & a^2 \\ 1 & \frac{1}{2} & a^2 \\ 1 & 1 & a^2 \end{bmatrix}.$$

Proof. We can write the oriented parabolic cylinder surfaces as in the parametric form $\varphi(u,v) = (u, v, au^2)$ with the matrix representation

$$\varphi(u,v) = \left([v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix}, [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix}, [v]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}\begin{bmatrix} u^2 \\ u \\ 1 \end{bmatrix} \right). \tag{5}$$

Also as biquadratic Bézier surface it has the following matrix form

$$\varphi(u,v) = \begin{bmatrix} [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} S_x \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} S_y \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T, \\ [v]\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} S_z \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}[u]^T \end{bmatrix} \tag{6}$$

where

$$S_x=\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix},S_y=\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix},S_z=\begin{bmatrix}S_{0z}&S_{3z}&S_{6z}\\S_{1z}&S_{4z}&S_{7z}\\S_{2z}&S_{5z}&S_{8z}\end{bmatrix}.$$

Using the equality of both left sides we get

$$\begin{bmatrix}[v]\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\\[v]\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\\[v]\begin{bmatrix}0&0&0\\0&0&0\\a^2&0&0\end{bmatrix}[u]^T=[v]\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{00}&S_{01}&S_{02}\\S_{10}&S_{11}&S_{12}\\S_{20}&S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}[u]^T\end{bmatrix}.$$

If we solve these matrix equations we have

$$\begin{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\\\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\\\begin{bmatrix}0&0&0\\0&0&0\\a^2&0&0\end{bmatrix}=\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\begin{bmatrix}S_{00}&S_{01}&S_{02}\\S_{10}&S_{11}&S_{12}\\S_{20}&S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}1&-2&1\\-2&2&0\\1&0&0\end{bmatrix}\end{bmatrix}$$

and

$$\begin{bmatrix}\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}=\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}\\\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}=\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}\\\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\a^2&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&\frac{1}{2}&1\\1&1&1\end{bmatrix}=\begin{bmatrix}S_{0z}&S_{3z}&S_{6z}\\S_{1z}&S_{4z}&S_{7z}\\S_{2z}&S_{5z}&S_{8z}\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}S_{0x}&S_{3x}&S_{6x}\\S_{1x}&S_{4x}&S_{7x}\\S_{2x}&S_{5x}&S_{8x}\end{bmatrix}=\begin{bmatrix}0&\frac{1}{2}&1\\0&\frac{1}{2}&1\\0&\frac{1}{2}&1\end{bmatrix}$$

$$\begin{bmatrix}S_{0y}&S_{3y}&S_{6y}\\S_{1y}&S_{4y}&S_{7y}\\S_{2y}&S_{5y}&S_{8y}\end{bmatrix}=\begin{bmatrix}0&0&0\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\1&1&1\end{bmatrix}.$$

$$\begin{bmatrix}S_0&S_{3z}&S_{6z}\\S_z&S_{4z}&S_{7z}\\S_{2z}&S_{5z}&S_{8z}\end{bmatrix}=\begin{bmatrix}0&0&a\\0&0&a\\0&0&a\end{bmatrix}$$

Example 4: Lets find the control points of the hyperbolic paraboloid $z = 16x^2$ as biquadratic Bézier surface. First we can write as

$$\varphi(u,v) = (u, v, 16u^2)$$

By replacing by $a = 4$ as in the following way, the control points are

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 4 \\ 1 & \frac{1}{2} & 4 \\ 1 & 1 & 4 \end{bmatrix}.$$

CONCLUSION

Biquadratic Bézier surfaces serve as a valuable tool in surface modeling, providing a smooth and computationally efficient method for creating surfaces in various applications, including CAD, animation, and 3D modeling. By using the matrix representations and the ability to calculate control points for special surfaces like parabolic cylinders, elliptic paraboloids, and hyperbolic paraboloids, biquadratic Bézier surfaces offer a flexible, intermediate solution for surface design. They are especially useful when a balance between computational efficiency and surface precision is required.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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