Strong Convergence of an explicit iteration method in uniformly convex Banach spaces

Ahmed A. Abdelhakim and R. A. Rashwan

Department of Mathematics, Assiut University, Assiut 71516, Egypt

Corresponding author E-mail: ahmed.abdelhakim@aun.edu.eg

Abstract

We obtain the necessary and sufficient conditions for the convergence of an explicit iterative procedure to a common fixed point of a finite family of non-self asymptotically quasi-nonexpansive type mappings in real Banach spaces. We also prove the strong convergence of this iterative method to a common fixed point of a finite family of non-self asymptotically quasi-nonexpansive in the intermediate sense mappings in uniformly convex Banach spaces. Our results mainly generalize and extend those obtained by Wang [L. Wang, Explicit iteration method for common fixed points of a finite family of nonself asymptotically nonexpansive mappings, Computers & Mathematics with applications, 53, (2007), 1012 - 1019.]

Keywords: common fixed point, iterative approximation, asymptotically quasi-nonexpansive in the intermediate sense, uniformly convex Banach spaces.

2010 Mathematics Subject Classification: 47H06, 47H09; 47J05; 47J25.

1. Introduction

Let $K$ be a nonempty subset of a real normed linear space $E$. A self-mapping $T : K \rightarrow K$ is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$ for every $x, y \in K$ and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that for every $n \geq 1$, $\| T^nx - T^ny \| \leq k_n \| x - y \|$ for all $x, y \in K$. If $F(T) = \{ x \in K : Tx = x \}$ is nonempty and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that $\| T^nx - y \| \leq k_n \| x - y \|$ for all $x \in K$, $y \in F(T)$ and every $n \geq 1$ then $T$ is called asymptotically quasi-nonexpansive. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [5] and the class forms an important generalization of that of nonexpansive mappings. It was proved in [5] that if $K$ is a nonempty closed convex subset of a real uniformly convex Banach space and $T$ is an asymptotically nonexpansive self-mapping on $K$, then $T$ has a fixed point.

Iterative methods for approximating fixed points of nonexpansive mappings have been studied by many authors (see for example [1], [2], [3], [4], [6], [8], [12], [14] and the references therein).

In Most of these papers, the well-known Mann iteration process [7],

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 1,$$

has been studied and the operator $T$ has been assumed to map $K$ into itself. The convexity of $K$ ensures that the sequence $\{x_n\}$ generated by (*) is well defined.

In 2001, Xu and Ori [25] introduced the following implicit iteration process for a finite family of nonexpansive self-mappings $\{T_i, i \in I\}$, where $I = \{1, 2, ..., N\}$.

For any initial point $x_0 \in K$,

$$x_n = \alpha_0x_{n-1} + (1 - \alpha_n)T_{\sigma_n}x_n, \quad n \geq 1,$$

where $\{ \alpha_n \}$ is a real sequence in $(0,1)$ and $T_{\sigma_n} = T_{\sigma_n(\text{mod}N)}$, the mod $N$ function takes values in $I$. They proved weak convergence of the above process to a common fixed point of the finite family of nonexpansive self-mappings. Later on, the implicit iteration method has been used to study the common fixed point of a finite family of strictly pseudocontractive self-mappings, asymptotically nonexpansive self-mappings or asymptotically quasi-nonexpansive self-mappings by some authors (see for example [10], [16] and [26], respectively). In 1991, Schu [15] introduced a modified iteration process to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space. More precisely, he proved the following theorem.

Email addresses: ahmed.abdelhakim@aun.edu.eg (Ahmed A. Abdelhakim), rashwan10@gmail.com (R. A. Rashwan)
Theorem 1.1. ([15]) Let $H$ be a Hilbert space, $K$ a nonempty closed convex and bounded subset of $H$. Let $T : K \to K$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $\lim_{n \to \infty} k_n = 1$ and $\sum_{n=1}^{\infty} (k_n^2 - 1) = \infty$. Let $\{a_n\}$ be a real sequence in $[0, 1]$ satisfying the condition $0 < a < 1 \leq \sum_{n=1}^{\infty} b_n = 1, n \geq 1$, for some constants $a$ and $b$. Then the sequence $\{x_n\}$ generated from $x_1 \in K$ by
\[ x_{n+1} = (1 - a_n)x_n + a_nT^n x_n, \quad n \geq 1, \]
converges strongly to some fixed point of $T$.

Since then, Schu’s iteration process has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert space or Banach space (see for example [9], [13], [12], [19]).

Recently, Wang [21] introduced the following concepts for nonself mappings:

Definition 1.2. [4] Let $K$ be a nonempty subset of a real normed space $E$. Let $P : E \to K$ be a nonexpansive retraction of $E$ onto $K$. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for every $n \geq 1$,
\[ \| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq k_n \| x - y \| \quad \text{for all } x, y \in K. \]

$T$ is said to be uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that for every $n \geq 1$,
\[ \| T(PT)^{n-1}x - T(PT)^{n-1}y \| \leq L \| x - y \| \quad \text{for all } x, y \in K. \]

It is easy to see that a nonself asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian. By studying the following iteration process
\[ x_1 \in K, \quad x_{n+1} = P((1 - a_n)x_n + a_nT(PT)^{n-1}x_n), \quad n \geq 1, \]

Recently, Wang [22] proved the following strong convergence theorems for common fixed points of two nonself asymptotically nonexpansive mappings as follows:

Theorem 1.3. ([22]) Let $K$ be a nonempty subset of a uniformly convex Banach space $E$. Suppose that $T_1, T_2 : K \to E$ are two nonself asymptotically nonexpansive mappings with sequences $\{k_n\}, \{\ell_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty, \sum_{n=1}^{\infty} (\ell_n - 1) < \infty$. From arbitrary $x_1 \in K$, let $\{x_n\}$ be defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n, \quad n \geq 1, \]
where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1 - \varepsilon]$ for some $\varepsilon > 0$. If one of $T_1$ and $T_2$ is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$ then $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

Theorem 1.4. ([22]) Let $K, E, T_1, T_2$ and $\{x_n\}$ be as in Theorem 1.2. If one of $T_1$ and $T_2$ is demicompact then $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

Definition 1.5. ([11]) Let $K$ be a nonempty subset of a real normed space $E$. Let $P : E \to K$ be a nonexpansive retraction of $E$ onto $K$. A nonself mapping $T : K \to E$ is called asymptotically nonexpansive in the intermediate sense if $T$ is uniformly continuous and
\[ \limsup_{n \to \infty} \sup_{x, y \in K} \{ \| T(PT)^{n-1}x - T(PT)^{n-1}y \| - \| x - y \| \} \leq 0. \] (1.1)

In 2007, Y. X. Tian, S. S. Chang and J. L. Huang [21] introduced the following concepts for nonself mappings:

Definition 1.6. [21] Let $E$ be a real Banach space, $C$ a nonempty nonexpansive retract of $E$ and $P$ the nonexpansive retraction from $E$ onto $C$. Let $T : C \to E$ be a non-self mapping.

(1) $T$ is said to be a nonself asymptotically quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that for every $n \geq 1$,
\[ \| T(PT)^{n-1}x - p \| \leq k_n \| x - p \| \quad \text{for all } x \in K, p \in F(T). \]

(2) $T$ is said to be a nonself asymptotically nonexpansive type mapping if
\[ \limsup_{n \to \infty} \{ \sup_{x, y \in K} \{ \| T(PT)^{n-1}x - T(PT)^{n-1}y \| - \| x - y \| \} \} \leq 0. \]

(3) $T$ is said to be a nonself asymptotically quasi-nonexpansive type mapping if $F(T) \neq \emptyset$ and
\[ \limsup_{n \to \infty} \{ \sup_{x \in K, q \in F(T)} \{ \| T(PT)^{n-1}x - q \| - \| x - q \| \} \} \leq 0. \]
Remark

(i) If \( T : C \to E \) is a nonself asymptotically nonexpansive mapping, then \( T \) is a nonself asymptotically nonexpansive type mapping.

(ii) If \( T : C \to E \) is a nonself asymptotically quasi-nonexpansive mapping, then \( T \) is a nonself asymptotically quasi-nonexpansive type mapping.

(iii) If \( F(T) \neq \emptyset \) and \( T : C \to E \) is a nonself asymptotically nonexpansive type mapping, then \( T \) is a nonself asymptotically quasi-nonexpansive type mapping.

Very recently, Lin Wang [23] constructed an explicit iteration scheme to approximate a common fixed point of a finite family of nonself asymptotically nonexpansive mappings \( \{ T_i : K \to E, i \in I \} \), where \( I \) denotes the set \( \{1, 2, \ldots, N\} \) and proved some strong convergence theorems for such mappings in uniformly convex Banach spaces as follows: From arbitrary \( x_0 \in K \),

\[
x_1 = P((1 - \alpha_1)x_0 + \alpha_1T_1(PT_1)^{m-1}x_0), \quad m \geq 1,
\]

\[
x_2 = P((1 - \alpha_2)x_1 + \alpha_2T_2(PT_2)^{m-1}x_1),
\]

\[
\vdots
\]

\[
x_N = P((1 - \alpha_N)x_{N-1} + \alpha_NT_N(PT_N)^{m-1}x_{N-1}),
\]

\[
x_{N+1} = P((1 - \alpha_{N+1})x_N + \alpha_{N+1}T_1(PT_1)^{m-1}x_N),
\]

\[
x_{N+2} = P((1 - \alpha_{N+2})x_{N+1} + \alpha_{N+2}T_2(PT_2)^{m-1}x_{N+1}),
\]

\[
\vdots
\]

\[
x_{2N} = P((1 - \alpha_{2N})x_{2N-1} + \alpha_{2N}T_N(PT_N)^{m-1}x_{2N-1}),
\]

\[
x_{2N+1} = P((1 - \alpha_{2N+1})x_{2N} + \alpha_{2N+1}T_1(PT_1)^{m-1}x_{2N}),
\]

\[
\vdots
\]

which can be rewritten in a compact form as follows

\[
x_n = P((1 - \alpha_n)x_{n-1} + \alpha_nT_n(PT_n)^{m-1}x_{n-1}), \quad n \geq 1, m \geq 1,
\]  \hspace{1cm} (1.2)

where \( n = (m - 1)N + i, T_n = T_{i \mod N}, T_i, i \in I, \{ \alpha_n \} \) is a real sequence in \([0,1]\).

Motivated and inspired by the previous facts, we extend the results obtained by Lin Wang [23] to the case of nonself asymptotically quasi-nonexpansive mappings and the case of nonself asymptotically quasi-nonexpansive mappings in the intermediate sense which is slightly more general than the class nonself asymptotically nonexpansive mappings in the intermediate sense introduced by S. Plubteing and R. Wangkeeree [11] as follows:

**Definition 1.7.** Let \( K \) be a nonempty subset of a real normed space \( E \). Let \( P : E \to K \) be a nonexpansive retraction of \( E \) onto \( K \). A nonself mapping \( T : K \to E \) with a nonempty fixed point set is called asymptotically quasi-nonexpansive in the intermediate sense if \( T \) is uniformly continuous and

\[
\limsup_{n \to \infty} \sup_{x \in K, y \in F(T)} \{ \| T(PT)^{m-1}x - y \| - \| x - y \| \} \leq 0.
\]  \hspace{1cm} (1.3)

Moreover, we discuss the necessary and sufficient condition for convergence of the explicit iterative scheme (1.1) to a common fixed point (assuming existence) of a finite family of nonself asymptotically quasi-nonexpansive type mappings in real Banach spaces.

2. Preliminaries

Let \( E \) be a real normed linear space. The modulus of convexity of \( E \) is the function \( \delta_E : (0, 2] \to [0,1] \) defined by

\[
\delta_E(\varepsilon) = \inf \{ 1 - \frac{1}{2} \| x + y \| : \| x \| = \| y \| = 1, \| x - y \| = \varepsilon \}.
\]

\( E \) is uniformly convex if and only if \( \delta_E(\varepsilon) > 0 \) for every \( \varepsilon \in (0,2] \).

A subset \( K \) of \( E \) is said to be a retract of \( E \) if there exists a continuous map \( P : E \to K \) such that \( Px = x, x \in K \). Every closed convex subset of a uniformly convex Banach space is a retract. A map \( P : E \to E \) is said to be a retraction if \( P^2 = P \). It follows that if \( P \) is a retraction then \( Py = y \) for all \( y \in R(P) \), the range of \( P \).

A mapping \( T : K \to K \) is said to be semicompact if, for any bounded sequence \( \{x_n\} \) in \( K \) such that \( \| x_n - Tx_n \| \to 0 \) as \( n \to 0 \), there exists a subsequence \( \{x_{n_j}\} \), say, of \( \{x_n\} \) such that \( \{x_{n_j}\} \) converges strongly to some \( x' \) in \( K \). \( T \) is said to be completely continuous if, for any bounded sequence \( \{x_n\} \), there exists a subsequence \( \{T_{n_j}\} \), say, of \( \{Tx_n\} \) such that \( \{T_{n_j}\} \) converges strongly to some element of the range of the range of \( T \).

In what follows we shall use the following results.
Lemma 2.1. [19] Let \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\mu_n\}_{n=1}^{\infty} \) be sequences of nonnegative real numbers such that \( \lambda_{n+1} \leq \lambda_n + \mu_n \), \( n \geq 1 \) and \( \sum_{n=1}^{\infty} \mu_n < \infty \) then \( \lim_{n \to \infty} \lambda_n \) exists. Moreover, if there exists a subsequence \( \{\lambda_{n_k}\} \) of \( \{\lambda_n\} \) such that \( \lambda_{n_k} \to 0 \) as \( j \to \infty \) then \( \lambda_n \to 0 \) as \( n \to \infty \).

Lemma 2.2. [15] Let \( E \) be a real uniformly convex Banach space and \( 0 < \alpha \leq \beta < 1 \) for all positive integers \( n \geq 1 \). Suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) and each of \( \{x_n\} \) and \( \{y_n\} \) are two sequences such that

\[
\limsup_{n \to \infty} \|x_n\| \leq r, \quad \limsup_{n \to \infty} \|y_n\| \leq r \quad \text{and} \quad \limsup_{n \to \infty} \|t_n x_n + (1-t_n) y_n\| = r
\]

hold for some \( r \geq 0 \), then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

Lemma 2.3. [4] Let \( E \) be a real uniformly convex Banach space and \( K \) a nonempty closed convex subset of \( E \) and let \( T : K \to E \) be asymptotically nonexpansive mapping with a sequence \( \{k_n\} \subset [1, \infty) \) such that \( k_n \to 1 \) as \( n \to \infty \), then \( I - T \) is demiclosed at zero.

3. Main Results

3.1. Asymptotically quasi-nonexpansive type mappings

Theorem 3.1. Let \( K \) be a nonempty closed convex subset of a real Banach space \( E \) which is also a nonexpansive retract of \( E \) with a nonexpansive retraction \( P : E \to K \). Suppose that \( T_i : K \to E \), \( i \in I \) be \( N \) nonself asymptotically quasi-nonexpansive type mappings with a nonempty closed common fixed point set \( F = \bigcap_{i=1}^{N} F(T_i) \). Let \( \{x_n\}_{n=1}^{\infty} \) be the iterative sequence defined iteratively by (1.2) with the sequence \( \{\alpha_n\}_{n=1}^{\infty} \) satisfying that \( \sum_{n=1}^{\infty} \alpha_n < \infty \). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \), \( i \in I \) if and only if \( \lim_{n \to \infty} d(x_n, F) = 0 \), where \( d(x_n, F) \) is the distance from \( x_n \) to the set \( F \).

Proof. Necessity of the condition is obvious. Since \( x_n \to q \) as \( n \to \infty \), \( q \in F \), then \( \lim_{n \to \infty} d(x_n, F) = d(\lim_{n \to \infty} x_n, F) = d(q, F) = 0 \). Hence, \( \liminf_{n \to \infty} d(x_n, F) = 0 \).

Next, we prove sufficiency. Since \( T_i, i \in I \) are \( N \) nonself asymptotically quasi-nonexpansive type mappings, that is, for each \( i \in I \), \( F(T_i) \neq \emptyset \) and

\[
\limsup_{n \to \infty} \left\{ \sup_{x \in K, q \in F(T_i)} \left\| T_i((PT_i)^{n-1}x - q) - \| x - q \| \right\| \right\} \leq 0.
\]

Then given any \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that for all \( n \geq n_0 \),

\[
\sup_{x \in K, q \in F(T_i)} \left\{ \left\| T_i((PT_i)^{n-1}x - q) - \| x - q \| \right\| \right\} < \varepsilon, \quad i \in I.
\]

Since \( \{x_n\} \subset K \), then for any \( m \geq n_0 \) we have

\[
\| T_i((PT_i)^{n-1}x_n - q) - \| x_n - q \| < \varepsilon, \quad i \in I, \ n \geq 1.
\]

(3.1)

Hence for every \( x^* \in F \) and for any \( n \geq n_0 \), \( n \geq 1 \), it follows from (1.2) and (3.1) that

\[
\| x_n - x^* \| = \| P((1 - \alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{n-1}x_{n-1}) - x^* \| \leq \| (1 - \alpha_n)x_{n-1} + \alpha_n T_n(PT_n)^{n-1}x_{n-1} - x^* \| \leq (1 - \alpha_n) \| x_{n-1} - x^* \| + \alpha_n \| T_n(PT_n)^{n-1}x_{n-1} - x^* \| \leq (1 - \alpha_n) \| x_{n-1} - x^* \| + \alpha_n \| (PT_n)^{n-1}x_{n-1} - x^* \| + \alpha_n \| x_{n-1} - x^* \| \leq \| x_{n-1} - x^* \| + \alpha_n \| x_{n-1} - x^* \|.
\]

That is, we have

\[
\inf_{x^* \in F} \| x_{n-1} - x^* \| \leq \| x_0 - x^* \| + \alpha_n \| x_{n-1} - x^* \|.
\]

By arbitrariness of \( x^* \in F \), we get, upon taking infimum over \( x^* \in F \),

\[
\inf_{x^* \in F} \| x_{n-1} - x^* \| \leq \inf_{x^* \in F} \| x_0 - x^* \| + \alpha_n \| x_{n-1} - x^* \|
\]

so that

\[
d(x_{n+1}, F) \leq d(x_n, F) + \alpha_n \| x_{n+1} - x^* \|
\]

i.e., \( \beta_{n+1} \leq \beta_n + \mu_n \), \( n \geq 1 \), where \( \beta_n = d(x_n, F) \) and \( \mu_n = \alpha_n \| x_{n+1} - x^* \| \), \( n \geq 1 \). Clearly, \( \sum_{n=1}^{\infty} \mu_n < \infty \) by our assumption. Then \( \lim_{n \to \infty} d(x_n, F) \) exists, by Lemma 2.1. But \( \liminf_{n \to \infty} d(x_n, F) = 0 \), then \( \lim_{n \to \infty} d(x_n, F) = 0 \).

Now, for any \( x^* \in F \),

\[
\| x_{n+1} - x_n \| \leq \| x_{n+1} - x^* \| + \| x_n - x^* \|
\]

taking infimum on both sides over \( x^* \in F \), we obtain

\[
\| x_{n+1} - x_n \| \leq d(x_{n+1}, F) + d(x_n, F),
\]

letting \( n \to \infty \) on both sides of the above inequality yields that \( \lim_{n \to \infty} d(x_{n+1}, x_n) = 0 \), which shows that \( \{x_n\} \) is a Cauchy sequence. Since \( K \) is a closed subset of the real Banach space \( E \), then \( K \) is also complete. Hence there exists \( p \in K \) such that \( x_n \to p \) as \( n \to \infty \). Finally, we prove that \( p \in F \). Since \( \lim_{n \to \infty} d(x_n, F) = d(\lim_{n \to \infty} x_n, F) = d(p, F) = 0 \). Then \( p \in F \), but \( F \) is closed, then \( p \in F \) and the proof is complete.
Theorem 3.2. Let $K$ be a nonempty closed convex subset of a real Banach space $E$ which is also a nonexpansive retraction of $E$ with a nonexpansive retraction $P : E \rightarrow K$. Suppose that $T_i : K \rightarrow E$, $i \in I$ be $N$ continuous nonself asymptotically quasi-nonexpansive type mappings with a nonempty common fixed point set $F = \bigcap_{i \in I} F(T_i)$. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined iteratively by (1.2) with the sequence $\{\alpha_n\}_{n=1}^{\infty}$ satisfying that $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_i$, $i \in I$ if and only if $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, where $d(x_n, F)$ is the distance from $x_n$ to the set $F$.

We only need to show that $F$ is closed so that the conclusion of Theorem 3.2 follow from the conclusion of Theorem 3.1 immediately. Let $\{p_n\}$ be a sequence of elements of $F$, i.e., $T_i p_n = p_n$, $n \geq 1$, $i \in I$. Assume that $p_n \rightarrow p^*$ as $n \rightarrow \infty$. We claim that $p^* \in F$. Indeed, since for each $i \in I$, we have

$$
\| T_i p^* - p^* \| \leq \| T_i p^* - p_n \| + \| p_n - p^* \|
= \| T_i p^* - T_i p_n \| + \| p_n - p^* \|. 
$$

(3.2)

Since $T_i$ is continuous, $i \in I$, then letting $n \rightarrow \infty$ on both sides of (3.2) yields that

$$
\lim_{n \rightarrow \infty} \| T_i p^* - p^* \| = 0,
$$

which implies that $T_i p^* = p^*$, $i \in I$ and hence $p^* \in F$.

3.2. Asymptotically quasi-nonexpansive mappings

Lemma 3.3. Let $K$ be a nonempty closed convex subset of a normed linear space $E$ which is also a nonexpansive retraction of $E$ with a nonexpansive retraction $P$. Let $\{T_i : i \in I\}$ be $N$ nonself asymptotically quasi-nonexpansive mappings from $K$ to $E$ with sequences $\{\alpha_n^{(i)}\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \alpha_n^{(i)} < \infty$ and $\lim_{n \rightarrow \infty} \frac{\alpha_n^{(i)}}{\alpha_n^{(i)} + 1} = 1$ for all $i \in I$, respectively. Let $\{\alpha_0\}$ be a real sequence in $[0,1)$ and let $\{x_n\}$ be the sequence defined by (1.2). If $F = \bigcap_{i \in I} F(T_i) \neq \emptyset$, then $\lim_{n \rightarrow \infty} \| x_n - x^* \|$ exists for each $x^* \in F$.

Proof. For each positive integer $n$, put $k_n = \max_{i \in I} k_n^{(i)} = 1 + u_n$.

Thus, $1 \leq k_n \leq \sum_{i=1}^{\infty} k_n^{(i)} - (N - 1)$. Since for each $i \in I$, $\sum_{n=1}^{\infty} k_n^{(i)} - 1$ is less than $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, consequently $\sum_{n=1}^{\infty} u_n < \infty$. For any $x^* \in F$, $n = (m(n)-1)N + i(n)$, $i(n) \in I$, it follows from (1.2) that

$$
\| x_n - x^* \| = \| P \left[ (1 - \alpha_n)x_{n-1} + \alpha_n T_i (x_{n-1}) \right] \| \leq \| (1 - \alpha_n)x_{n-1} + \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n (1 + u_n) x_{n-1} - x^* \|
\leq \| (1 + u_n) x_{n-1} - x^* \|, 
$$

that is,

$$
\| x_n - x^* \| \leq \| x_{n-1} - x^* \| + u_n \| x_{n-1} - x^* \|. 
$$

(3.3)

Furthermore, we have

$$
\| x_n - x^* \| = \| x_{(m(n)-1)N+i(n)} - x^* \|
= \| P \left[ (1 - \alpha_n)x_{n-1} + \alpha_n T_i (x_{n-1}) \right] \| \leq \| (1 - \alpha_n)x_{n-1} + \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n T_i (x_{n-1}) - x^* \|
\leq \| (1 - \alpha_n)x_{n-1} - x^* \| + \| \alpha_n (1 + u_n) x_{n-1} - x^* \|
\leq \| (1 + u_n) x_{n-1} - x^* \|, 
$$

(3.4)

In addition, since $m = 1$, while $1 \leq n \leq N$, then

$$
\| x_1 - x^* \| \leq \| (1 - \alpha_1) x_0 + \alpha_1 T_1 (x_1) - x^* \|
\leq \| (1 - \alpha_1) x_0 - x^* \| + \| \alpha_1 T_1 (x_1) - x^* \|
\leq \| (1 - \alpha_1) x_0 - x^* \| + \| \alpha_1 (1 + u_1) x_0 - x^* \|
\leq \| (1 + u_1) x_0 - x^* \|, 
$$
\[ \| x_2 - x^* \| \leq \| (1 - \alpha_2) x_1 + \alpha_2 T_2 (PT_2)^{m(n)-1} x_1 - x^* \| \]
\[ \leq (1 - \alpha_2) \| x_1 - x^* \| + \alpha_2 \| T_2 (PT_2)^{m(n)-1} x_1 - x^* \| \]
\[ \leq (1 - \alpha_2) \| x_1 - x^* \| + \alpha_2 (1 + u_1) \| x_1 - x^* \| \]
\[ \leq (1 + u_1) \| x_1 - x^* \| \leq (1 + u_1)^2 \| x_0 - x^* \|. \]

hence,
\[ \| x_N - x^* \| \leq (1 + u_1)^N \| x_0 - x^* \|. \]

Similarly, we have
\[ \| x_{2N} - x^* \| \leq \| (1 - \alpha_{2N}) x_{2N-1} + \alpha_{2N} T_2 (PT_2)^{m(n)-1} x_{2N-1} - x^* \| \]
\[ \leq (1 - \alpha_{2N}) \| x_{2N-1} - x^* \| + \alpha_{2N} \| T_2 (PT_2)^{m(n)-1} x_{2N-1} - x^* \| \]
\[ \leq (1 - \alpha_{2N}) \| x_{2N-1} - x^* \| + \alpha_{2N} (1 + u_2) \| x_{2N-1} - x^* \| \]
\[ \leq (1 + u_2) \| x_{2N-1} - x^* \| \leq (1 + u_2)^N \| x_N - x^* \| \]
\[ \leq (1 + u_2)^N (1 + u_1)^N \| x_0 - x^* \|. \]

Therefore,
\[ \| x_{(m(n)-1)N} - x^* \| \leq (1 + u_1)^N (1 + u_2)^N \| x_{(m(n)-1)N} - x^* \|. \]

Finally (3.4) together with (3.5) imply that
\[ \| x_n - x^* \| \leq (1 + u_1)^{i(n)} \| x_{(m(n)-1)N} - x^* \| \]
\[ \leq (1 + u_1)^N (1 + u_2)^N \| x_{(m(n)-1)N} - x^* \|. \]

\[ i(n) \in I. \]

Thus
\[ \| x_n - x^* \| \leq (1 + u_1)^N (1 + u_2)^N \| x_{(m(n)-1)N} - x^* \| \leq (1 + u_1)^N (1 + u_2)^N \| x_0 - x^* \|. \]

Since \( 1 + x \leq e^x, x \geq 0 \), then
\[ \| x_n - x^* \| \leq e^{N_1 u_1} e^{N_2 u_2} \| x_{(m(n)-1)N} - x^* \| = e^{N \sum_{j=1}^{m(n)} u_j} \| x_0 - x^* \|. \]

But \( \sum_{j=1}^{m(n)} u_j < \infty \), then \( \{ x_n \} \) is a bounded sequence and there exists a constant \( M > 0 \) such that \( \| x_0 - x^* \| M, n \geq 0. \)

It follows, from (3.3), that
\[ \| x_n - x^* \| \leq \| x_{n-1} - x^* \| + u_n M. \]

Since \( n \to \infty \) is equivalent to \( m \to \infty \), it follows from Lemma 2.1 that \( \lim_{m \to \infty} \| x_n - x^* \| \) exists for any \( x^* \in E \).

The proof is complete. \( \square \)

**Lemma 3.4.** Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \) which is also a nonexpansive retract of \( E \) with a nonexpansive retraction \( P \). Let \( T_i, i \in I \) be \( N \) nonself asymptotically quasi-nonexpansive mappings from \( K \) to \( E \) with sequences \( \{ k_i(n) \} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} k_i(n) < \infty \) and \( \lim_{m \to \infty} k_i(m) = 1 \) for all \( i \in I \), respectively and suppose that \( T_i \) are uniformly \( L_i \)-Lipschitzian with the uniform Lipschitz constants \( L_i > 0, i \in I \), respectively. Let \( x_n \) be the sequence defined by (1.2) where \( \{ \alpha_n \} \) is a real sequence in \( [\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). If \( F \cap \bigcap_{i \in I} F(T_i) = \emptyset \), then \( \lim_{m \to \infty} \| x_n - T_{\alpha_n} x_n \| = 0 \) for each \( i \in I \).}

**Proof.** Lemma 3.3 asserts that \( \lim_{m \to \infty} \| x_n - x^* \| \) exists for each \( x^* \in E \). We may assume that, for some \( x^* \in E \), \( \lim_{m \to \infty} \| x_n - x^* \| = c \) for some \( c \geq 0 \). If \( c = 0 \), we are done. So let \( c > 0 \). Set \( n = (m(n)-1)N + i(n), i(n) \in I \).

Since
\[ \| x_{n+1} - x^* \| = \| P((1 - \alpha_{n+1}) x_n + \alpha_{n+1} T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n) - x^* \| \]
\[ \leq \| (1 - \alpha_{n+1}) (x_n - x^*) + \alpha_{n+1} (T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n - x^*) \| \]
\[ \leq (1 + u_{n+1}) \| x_n - x^* \|. \]

Taking \( \liminf \) on both sides of (3.7), we obtain
\[ \lim_{m \to \infty} \| (1 - \alpha_{n+1}) (x_n - x^*) + \alpha_{n+1} (T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n - x^*) \| \geq c. \]

Also,
\[ \| (1 - \alpha_{n+1}) (x_n - x^*) + \alpha_{n+1} (T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n - x^*) \| \leq (1 + u_{n+1}) \| x_n - x^* \|, \]

which on taking \( \limsup \) on both sides yields that
\[ \lim_{m \to \infty} \| (1 - \alpha_{n+1}) (x_n - x^*) + \alpha_{n+1} (T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n - x^*) \| \leq \limsup_{m \to \infty} (1 + u_{n+1}) \| x_n - x^* \| = c. \]

Inequalities (3.8) and (3.9) imply
\[ \lim_{m \to \infty} \| (1 - \alpha_{n+1}) (x_n - x^*) + \alpha_{n+1} (T_{\alpha_{n+1}} (PT_{\alpha_{n+1}})^{m(n)-1} x_n - x^*) \| = c. \]
Since $\lim_{n \to +\infty} \| x_n - x^* \| = c$ and $\limsup_{n \to +\infty} \| T_{n+1} (PT_{n+1})^{m(n)} x_n - x^* \| \leq c$, it follows from Lemma 2.2 that

$$\lim_{n \to +\infty} \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \| = 0. \quad (3.11)$$

Since

$$\| x_{n+1} - x_n \| \leq \alpha_{n+1} \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \|$$

then, by (3.11), we have

$$\lim_{n \to +\infty} \| x_{n+1} - x_n \| = 0.$$  

By induction, we have

$$\lim_{n \to +\infty} \| x_{n+r} - x_n \| = 0 \quad (3.12)$$

for any positive integer $r$.

Let $L = \max_{i \in I} \{ L_i \}$. When $n > N \ (m \geq 2)$, we have

$$\| x_n - T_{n+1} x_n \| \leq \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \| + \| T_{n+1} (PT_{n+1})^{m(n)-1} x_n - T_{n+1} x_n \|$$

$$\leq \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \| + L \| T_{n+1} (PT_{n+1})^{m(n)-2} x_n - x_n \|$$

$$\leq \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \| + L \| x_n - T_{n+1} (PT_{n+1})^{m(n)-2} x_n - x_n \| +$$

$$\| T_{n+1} (PT_{n+1} - N) (PT_{n+1} - N)^{m(n)-2} x_n - N \| + \| T_{n+1} (PT_{n+1} - N)^{m(n)-2} x_n - N \|$$

Hence

$$\| x_n - T_{n+1} x_n \| \leq \| x_n - T_{n+1} (PT_{n+1})^{m(n)-1} x_n \| + L \{ (1 + L) \| x_n - x_n - N + \} \| x_n - T_{n+1} (PT_{n+1} - N)^{m(n)-2} x_n - N \| \} \quad (3.13)$$

Noticing that $n = (m(n) - 1)N + i(n), i(n) \in I$, we have $n - N = (m(n) - 1)N + i(n) - N = (m(n) - 2)N + i(n) = (m(n) - 1)N + i(n - N), \quad \text{thus } m(n - N) = m(n) - 1$ and $i(n - N) = i(n), n \geq 1$. Hence

$$\| x_n - T_{n+1 - N} (PT_{n+1 - N} - N)^{m(n)-2} x_n - N \| = \| x_n - T_{n+1 - N} (PT_{n+1 - N} - N)^{m(n)-1} x_n - N \| .$$

Using (3.11), we get

$$\lim_{n \to +\infty} \| x_n - T_{n+1 - N} (PT_{n+1 - N} - N)^{m(n)-2} x_n - N \| = 0. \quad (3.14)$$

Using (3.12) and (3.14), it follows from (3.13) that

$$\lim_{n \to +\infty} \| x_n - T_{n+1} x_n \| = 0. \quad (3.15)$$

Furthermore, for each $i \in I$

$$\| x_n - T_{n+i} x_n \| \leq \| x_n - x_{n+i-1} \| + \| x_{n+i-1} - T_{n+i} x_{n+i-1} \| + \| T_{n+i} x_{n+i-1} - T_{n+i} x_n \|$$

$$\leq (1 + L) \| x_n - x_{n+i-1} \| + \| x_{n+i-1} - T_{n+i} x_{n+i-1} \| .$$

Using (3.12) and (3.15), we obtain

$$\lim_{n \to +\infty} \| x_n - T_{n+i} x_n \| = 0, \quad i \in I.$$

Thus

$$\lim_{n \to +\infty} \| x_n - T_i x_n \| = 0, \quad i \in I.$$

This completes the proof.
3.3. Asymptotically quasi-nonexpansive in the intermediate sense mappings

**Lemma 3.5.** Let $K$ be a nonempty closed convex subset of a normed linear space $E$ which is also a nonexpansive retraction of $E$ with a nonexpansive retraction $P$. Let $\{T_i: i \in I\}$ be $N$ nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from $K$ to $E$ with a nonempty common fixed point set $F = \bigcap_{i=1}^N F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x \in K, x' \in F} \|T_i(T_i)^m x - x'\| - \|x - x'\|, 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. If $\{x_n\}$ is the sequence defined by (1.2), then $\lim_{n \to \infty} \|x_n - x^*\|$ exists for each $x^* \in F$.

**Proof.** For any $x^* \in F$, we have

$$\|x_n - x^*\| = \|P'(1 - \alpha_n)x_{n-1} + \alpha_n T_n^m x_{n-1} - x^*\|
\leq \|P'(1 - \alpha_n)|x_{n-1} + \alpha_n T_n^m x_{n-1} - x^*\|
\leq (1 - \alpha_n) \|x_{n-1} - x^*\| + \alpha_n \|T_n^m x_{n-1} - x^*\|
\leq (1 - \alpha_n) \|x_{n-1} - x^*\| + \alpha_n (G_m^{(n)} + \|x_{n-1} - x^*\|).$$

Thus

$$\|x_n - x^*\| \leq \|x_{n-1} - x^*\| + G_m^{(n)}.$$ 

Since $\sum_{m=1}^{\infty} G_m^{(n)} < \infty$, $n \geq 1$ and $n \to \infty$ is equivalent to $m \to \infty$, then applying Lemma 2.1 implies that $\lim_{n \to \infty} \|x_n - x^*\|$ exists for each $x^* \in F$. The proof is complete.

**Lemma 3.6.** Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space which is also a nonexpansive retraction of $E$ with a nonexpansive retraction $P$. Let $\{T_i: i \in I\}$ be $N$ nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from $K$ to $E$ with a nonempty common fixed point set $F = \bigcap_{i=1}^N F(T_i)$. For each $i \in I$, put $G_m^{(i)} = \max\{\sup_{x \in K, x' \in F} \|T_i(T_i)^m x - x'\| - \|x - x'\|, 0\}$ so that $\sum_{m=1}^{\infty} G_m^{(i)} < \infty$, $i \in I$. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. Then $\lim_{n \to \infty} \|x_n - x^*\| = 0$ for each $x^* \in F$.

**Proof.** It follows from Lemma 3.5 that $\lim_{n \to \infty} \|x_n - x^*\|$ exists for each $x^* \in F$. Assume that $\lim_{n \to \infty} \|x_n - x^*\| = c$, $x^* \in F$ for some $c \geq 0$. If $c = 0$, we are done. So let $c > 0$. Set $n = (m(n) - 1)N + i(n), i(n) \in I$. Since

$$\|x_{n+1} - x^*\| = \|P'(1 - \alpha_{n+1})x_n + \alpha_{n+1} T_{n+1}^m x_n - x^*\|
\leq \|P'(1 - \alpha_{n+1})|x_n - x^*\| + \alpha_{n+1} \|T_{n+1}^m x_n - x^*\|
\leq (1 - \alpha_{n+1}) \|x_n - x^*\| + \alpha_{n+1} (G_m^{(n+1)} + \|x_n - x^*\|).$$

Taking lim inf on both sides of (3.16), we obtain

$$\liminf_{n \to \infty} \|(1 - \alpha_{n+1})(x_n - x^*) + \alpha_{n+1} T_{n+1}^m (x_n - x^*)\| \geq c. \quad (3.17)$$

In addition,

$$\| (1 - \alpha_{n+1})(x_n - x^*) + \alpha_{n+1} T_{n+1}^m (x_n - x^*) \| \leq \|x_n - x^*\| + \alpha_{n+1} G_m^{(n+1)}.$$

Hence

$$\| (1 - \alpha_{n+1})(x_n - x^*) + \alpha_{n+1} T_{n+1}^m (x_n - x^*) \| \leq \|x_n - x^*\| + G_m^{(n+1)}.$$ 

which on taking lim sup on both sides yields that

$$\limsup_{n \to \infty} \| (1 - \alpha_{n+1})(x_n - x^*) + \alpha_{n+1} T_{n+1}^m (x_n - x^*) \| \leq \limsup_{n \to \infty} \|x_n - x^*\| + \limsup_{n \to \infty} G_m^{(n+1)} = c. \quad (3.18)$$

Inequalities (3.17) and (3.18) imply

$$\lim_{n \to \infty} \| (1 - \alpha_{n+1})(x_n - x^*) + \alpha_{n+1} T_{n+1}^m (x_n - x^*) \| = c. \quad (3.19)$$

Since $\lim_{n \to \infty} \|x_n - x^*\| = c$ and $\limsup_{n \to \infty} \|T_{n+1}^m (x_n - x^*)\| \leq c$, it follows from Lemma 2.2 that

$$\lim_{n \to \infty} \|x_n - T_{n+1}^m x_n\| = 0. \quad (3.20)$$

Since

$$\|x_{n+1} - x_n\| \leq \alpha_{n+1} \|x_n - T_{n+1}^m x_n\|$$

then, by (3.20), we have

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.$$
By induction, we have
\[
\lim_{n \to \infty} \| x_{n+r} - x_n \| = 0 \tag{3.21}
\]
for any positive integer \( r \).

Now, we have
\[
\| x_n - T_{n+1}x_n \| \leq \| x_n - x_{n+N} \| + \| x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} \| + \\
\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| + \\
\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n - T_{n+1}x_n \|. 
\]

Since \( n = (m(n) - 1)N + i(n), i(n) \in I \), then \( n+N = (m(n) - 1)N + i(n) + N = m(n)N + i(n) = (m(n+N) - 1)N + i(n+N) \), thus \( m(n+N) = m(n) + 1, i(n+N) = i(n) \) and \( T_{n+1} = T_{n+1+N} = T_{i(n+1)} \), \( n \geq 1 \). Hence
\[
\| x_n - T_{n+1}x_n \| \leq \| x_n - x_{n+N} \| + \| x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} \| + \\
\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| + \\
\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n - T_{n+1}x_n \|. 
\]

But (3.20) implies that
\[
\| PT_{n+1}(PT_{n+1})^{m(n)-1}x_n - x_n \| \leq \| PT_{n+1}(PT_{n+1})^{m(n)-1}x_n - x_n \| \to 0, \quad n \to \infty, 
\]
since \( T_{n+1} \) are uniformly continuous, then
\[
\| T_{n+1}(PT_{n+1})^{m(n)-1}x_n - T_{n+1}x_n \| = \| T_{n+1}PT_{n+1}(PT_{n+1})^{m(n)-1}x_n - T_{n+1}x_n \| \to 0, \quad n \to \infty. 
\]

Also, uniform continuity of \( T_{n+1} \) and (3.21) yield
\[
\| T_{n+1}(PT_{n+1})^{m(n+N)-1}x_{n+N} - T_{n+1}(PT_{n+1})^{m(n+N)-1}x_n \| \to 0, \quad n \to \infty. 
\]

Finally, using (3.20), (3.21), (3.23) and (3.24), it follows from (3.22) that
\[
\lim_{n \to \infty} \| x_n - T_{n+1}x_n \| = 0. 
\]

Furthermore, for each \( i \in I \)
\[
\| x_n - T_{i+1}x_n \| \leq \| x_n - x_{i+1} \| + \| x_{i+1} - T_{i+1}x_{i+1} \| + \| T_{i+1}x_{i+1} - T_{i+1}x_n \|, 
\]
using (3.21), (3.25) and uniform continuity of \( T_{i+1} \), we get
\[
\lim_{n \to \infty} \| x_n - T_{i+1}x_n \| = 0, \quad i \in I. 
\]

Thus
\[
\lim_{n \to \infty} \| x_n - T_{i}x_n \| = 0, \quad i \in I. 
\]

The proof is complete. \( \square \)

Now, we are in a position to state our main theorems

**Theorem 3.7.** Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \) which is also a nonexpansive retract of \( E \) with a nonexpansive retraction \( P \). Let \( T_i, i \in I \) be \( N \) nonexpansive asymptotically quasi-nonexpansive mappings from \( K \) to \( E \) with sequences \( \{k^{(i)}_n\} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} (k^{(i)}_n - 1) < \infty \) and \( \lim_{n \to \infty} k^{(i)}_n = 1 \) for all \( i \in I \), respectively. Suppose that \( T_i \) are uniformly \( L_i \)-Lipschitzian with the uniform Lipschitz constants \( L_i > 0, i \in I \), respectively. Let \( \{x_n\} \) be the sequence defined by (1.2) where \( \{a_n\} \) is a real sequence in \([\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). If \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and if one of the mappings \( T_i, i \in I \) is completely continuous, then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( T_i, i \in I \).

**Theorem 3.8.** Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \) which is also a nonexpansive retract of \( E \) with a nonexpansive retraction \( P \). Let \( \{T_i : i \in I\} \) be \( N \) nonexpansive asymptotically quasi-nonexpansive in the intermediate sense mappings from \( K \) to \( E \) with a nonexpansive common fixed point set \( F = \bigcap_{i=1}^{N} F(T_i) \). For each \( i \in I \), let \( G_m^{(i)} = \max_{x \in K, x \neq F} \{ \| T_i(PT_i)^{m-1}x - x^* \| - \| x - x^* \| \} \}
so that \( \sum_{m=1}^{\infty} G_m^{(i)} < \infty, i \in I \). Let \( \{x_n\} \) be the sequence defined by (1.2) where \( \{a_n\} \) is a real sequence in \([\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). If one of the mappings \( T_i, i \in I \) is completely continuous, then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( T_i, i \in I \).

**Proof.** The proof of theorems 3.7 and 3.8 follows from the proof of Theorem 3.4 in [23]. \( \square \)

**Theorem 3.9.** Let \( K \) be a nonempty closed convex subset of a real uniformly convex Banach space \( E \) which is also a nonexpansive retract of \( E \) with a nonexpansive retraction \( P \). Let \( T_i, i \in I \) be \( N \) nonexpansive asymptotically quasi-nonexpansive mappings from \( K \) to \( E \) with sequences \( \{k^{(i)}_n\} \subset [1, \infty) \) such that \( \sum_{n=1}^{\infty} (k^{(i)}_n - 1) < \infty \) and \( \lim_{n \to \infty} k^{(i)}_n = 1 \) for all \( i \in I \), respectively. Suppose that \( T_i \) are uniformly \( L_i \)-Lipschitzian with the uniform Lipschitz constants \( L_i > 0, i \in I \), respectively. Let \( \{x_n\} \) be the sequence defined by (1.2) where \( \{a_n\} \) is a real sequence in \([\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). If \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and one of the mappings \( T_i, i \in I \) is demicoercive then \( \{x_n\} \) converges strongly to a common fixed point of the mappings \( T_i, i \in I \).
Theorem 3.10. Let $K$ be a nonempty closed convex subset of a real uniformly convex Banach space $E$ which is also a nonexpansive retract of $E$ with a nonexpansive retraction $P$. Let $\{T_i : i \in I\}$ be $N$ nonself asymptotically quasi-nonexpansive in the intermediate sense mappings from $K$ to $E$ with a nonempty common fixed point set $F = \bigcap_{i=1}^{N} F(T_i)$. For each $i \in I$, put $G_m(i) = \max\{\sup_{x \in K} \{\| T_i(P(T_i)x - x\| - \| x - x^* \|\}\}, 0\}$ so that $\sum_{m=1}^{\infty} G_m(i) < \infty$, $i \in I$. Let $\{x_n\}$ be the sequence defined by (1.2) where $\{\alpha_n\}$ is a real sequence in $[\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. If one of the mappings $T_i$, $i \in I$ is demicompact then $\{x_n\}$ converges strongly to a common fixed point of the mappings $T_i$, $i \in I$.

Proof. The proof of theorems 3.9 and 3.10 follows from the proof of Theorem 3.5 in [23].

References