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Research Article

Investigations of Prospective Elementary Mathematics Teachers' Lesson Plays on Multiplication of Fractions

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Abstract – The purpose of this study was to investigate the teaching images of prospective teachers for the multiplication of fractions in lesson plays. It was conducted at a public university in the southern region of Turkey and included 23 third-year prospective teachers (21 females and 2 males) enrolled in a mathematics teacher education program. A lesson play was used as a data collection tool, and 23 lesson plays were prepared by prospective teachers. The analysis of the data resulted in the classification of the lesson plays into three teaching images based on the presentation of mathematical ideas throughout the scenarios. This study found that prospective teachers struggle with teaching multiplication of fractions due to their lack of content knowledge. This deficiency resulted in rule-based explanations, random computational errors, and inappropriate tasks and models. Prospective teachers also exhibited difficulty in anticipating actual-like students' responses and constructing appropriate dialogues. These findings emphasize the need to provide more comprehensive mathematics content knowledge to prospective teachers to support their teaching the multiplication of fractions more effectively. Further research is needed to investigate how to support prospective teachers in developing their teaching images and strategies for teaching mathematics.

Keywords: Lesson play, multiplication of fractions, prospective teachers, teaching image.

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Introduction

The effectiveness of mathematics teaching is widely accepted to be of great importance (National Council of Teachers of Mathematics [NCTM], 2000); however, the factors that contribute to this effectiveness are still not well understood (Li et al., 2009). In order to ascertain the factors impacting the quality of teaching, researchers examined teachers' knowledge of mathematics teaching (Ball et al., 2008; Ma, 1999), classroom teaching practices (Rowland et al., 2005), teachers' perceptions and images of effective teaching (Crespo et al., 2011; Li et al., 2009), and the construction of lesson plans prior to teaching (Stigler & Hiebert, 2000). Building on Shulman's (1986) foundational concept of pedagogical content knowledge (PCK), Ball et al. (2008) proposed the Mathematical Knowledge for Teaching (MKT) framework to more specifically identify the domains of teacher knowledge necessary for effective mathematics instruction. This framework distinguishes between various types of content knowledge—such as Common Content Knowledge (CCK), Specialized Content Knowledge (SCK), and Knowledge of Content and Students (KCS)—which have informed the analytical focus of this study. Beyond identifying the specific domains of teacher knowledge, it is also important to explore how this knowledge is reflected in teachers' instructional planning and representations of teaching.

Schoenfeld (1998) has argued that the constructs of 'teacher lesson image' and 'teacher action plan' are more comprehensive representations of mathematics teaching in context, as they encompass the teacher's beliefs and knowledge of students, context, and subject matter. By analyzing lesson play thought to best represent their future teaching, and identifying clues about how the prospective teachers will teach, it is possible to gain insight into their teaching without having observed them in an actual context. Zazkis et al. (2009) suggested that by constructing a lesson play, future teachers can be more focused on specific teaching moments, encouraging them to envision these moments with greater clarity. Moreover, Crespo et al. (2011) suggested that the process of educating mathematics teachers is typically seen as including the requirement to question and expand prospective teachers' flawed images of teaching mathematics at the elementary level. The implications of research on the role of teachers' teaching images can contribute to efforts to overcome the challenges in students' understanding of fractions (Wright, 2008). However, the scope of research on prospective teachers' knowledge of fraction operations is limited (Newton, 2008), with a focus mainly on fraction division (Ball, 1990; Ma, 1999; Marchionda, 2006; Son & Crespo, 2009; Tirosh, 2000; Wright, 2008). In order to address this limitation, this study proposes to focus on the multiplication of fractions. Additionally, prospective teachers will be encouraged to consider their students' perspectives when creating lesson plans and tasks related to the multiplication of fractions, thus becoming aware of their teaching problems before entering their professional lives and contributing to the improvement of their problem-solving skills (Karaağaç & Köse, 2015). Therefore, to gain a fuller understanding of prospective teachers' images of mathematics teaching, this study employed lesson play in the context of teaching multiplication of fractions as a tool to investigate their teaching images.

Lesson Play and Teaching Image

There are various approaches for teachers to prepare lesson plans, one of which is "lesson play". This involves creating a detailed lesson plan that takes into account all the aspects of teaching practice, by constructing imaginary interactions between teacher and students, and amongst students (Zazkis et al., 2009). The use of lesson play in teacher education and professional development programs is becoming more widespread, providing researchers with an opportunity to examine prospective teachers' knowledge (Arnesen et al., 2017; Kontorovich & Zazkis, 2016; Kontorovich & Zazkis, 2017; Kontorovich, 2018; Zazkis et al., 2009). In this study, we consider that a lesson play is a detailed lesson plan that enables prospective teachers to include students in their teaching design and to prepare lesson plans using student knowledge. Lesson plays require prospective teachers to think about tasks and their solutions, to address students' thoughts and errors in an imaginary teaching, and to determine their own roles in this imaginary design. In this context, we developed an in-depth perspective on prospective teachers' teaching images. Schoenfeld (1998) described the teaching image as an image of the potential and possible outcomes related to a lesson, which includes the teacher's understanding of their students, what they may find difficult to understand, and how the teacher can handle this difficulty. He concluded that the idea of a lesson image was different from the idea of teacher planning. Moreover, teaching images are not only reflections of instructional design but also indicators closely linked to prospective teachers' beliefs and professional identities (Beijaard et al., 2004; Fives & Buehl, 2012). These images influence how teachers interpret students' thinking, choose representations, and make decisions during planning and instruction. In this study, we assumed that such teaching images would be revealed through the lesson plays constructed by prospective teachers, as these plays offer a structured space to externalize their pedagogical reasoning and instructional intentions.

Prospective Teachers' Knowledge of Multiplication of Fractions

Teachers have an important role to play in teaching fractions, which can be a difficult topic for students to understand (Hill & Ball, 2004). According to NCTM (2000), if teachers do not possess adequate content knowledge of fractions and students cannot learn fractions, they cannot progress to understanding algebra, which is a key subject in high school. The teacher is vital in assisting students to work with fractions and rational numbers in a flexible manner (NCTM, 2000).

Numerous studies have demonstrated that teachers and prospective teachers often lack the necessary pedagogical content knowledge for many topics, particularly fractions (Ball, 1990; Gökkurt et al., 2012; Ma, 1999; Newton, 2008; Tanışlı & Köse, 2013; Tirosh, 2000). In particular, these difficulties can be understood through the lens of MKT, where gaps in Specialized Content Knowledge (SCK)—such as explaining concepts and procedures—and Knowledge of Content and Students (KCS)—such as anticipating common misconceptions—are evident in how teachers and prospective teachers approach fraction operations (Ball et al., 2008). Results from investigations into the knowledge of teachers and prospective teachers on multiplying fractions showed that teachers and prospective teachers could only partially recognize student mistakes regarding fractions (Karaağaç & Köse, 2015) and they tend to prefer the technique of memorizing the rules to rectify student errors (Gökkurt et al., 2012). Furthermore, research has revealed that they have limited procedural knowledge when it comes to multiplication of fractions (Marchionda, 2006) and they usually look for the same denominators regardless of the operation in question (Newton, 2008). Bezuk and Armstrong (1993) found that middle school teachers struggled to explain their thought processes and draw representations when solving a problem using fraction multiplication. While teachers knew the most suitable approach, they had difficulty with the calculation itself. The data collection tools used in these investigations to assess the knowledge of teachers and prospective teachers are mostly based on open-ended written problems. As such, more research is required, making use of tools such as clinical interviews, scenario interviews (Jenkins et al., 2010), coaching, and microteaching (Cavin, 2007) to gain a better understanding and definition of the components of the prospective teachers' pedagogical content knowledge. The lesson play appears to be an effective means to assess the teacher's knowledge and draw in-depth inferences about that knowledge in a different way.

Method

This study aims to investigate the prospective teachers' teaching images for the multiplication of fractions in the lesson plays. Firstly, we defined the teaching images of each lesson play, and then analyzed each lesson play in the same teaching image group according to three components: the selection and ordering of the tasks, the estimated students' thought, and the use of models.

Setting and Participants

This investigation was conducted at a public university in southern Turkey. The study employed a convenience sampling method. Research participants included third-year prospective teachers (N=23; 21 female, 2 male) enrolled in a method course within mathematics teacher education programs. All third-year prospective mathematics teachers enrolled in the compulsory method course were included, as they were readily accessible and had prior experience in preparing lesson plays as part of the course requirements.

The scope of this method course was on topics related to the conceptual meaning of elementary mathematics concepts, and students' learning and teachers' teaching of these concepts. The course lasts for two semesters, with 14 weeks in each semester, and is compulsory for prospective teachers in the fifth and sixth semesters of their mathematics education program. Prospective teachers create their lesson plays in the second method course after learning multiplication with fractions. Prospective teachers learned 1. How students learn the fraction multiplication with a conceptual understanding, 2. Which forms of representation can be used in the teaching of the fraction multiplication, 3. Which errors students can make and what misconceptions they may have in this regard, 4. How the teacher can overcome these mistakes and errors, and 5. What types of questions teachers can ask in their lessons. Participation in the research was voluntary and written informed consent was obtained from all 23 students attending the course.

Research Instrument and Data Collection

The findings of this study were the results obtained from the lesson plays that the prospective teachers prepared for teaching the multiplication of fractions. Prospective teachers who individually prepared lesson plays designed teaching tasks about multiplying fractions and identified student thoughts such as possible errors, misconceptions, and alternative way of thinking or confusions that may arise in these tasks. Later, the researcher asked the prospective teachers to construct how they could follow these students' thoughts. While creating their scenarios, prospective teachers were asked to focus on the new teaching explanations the teacher will follow. The lesson plays prepared in this study were used as data collection tools. Each prospective teacher in the research prepared a scenario for teaching fraction multiplication and the data of the research consisted of 23 lesson plays. These lesson plays provided the researcher with insights into the prospective teachers' perspective of their students' knowledge and their possible thoughts in the lesson, and how the prospective teacher would respond to their students. In addition, the scenarios provided teaching images of prospective teachers occurred in their lesson plays.

Data Analysis

In this study, scenarios created by prospective teachers were first analyzed using the constant comparison method. In this analysis, data gathered from individuals who share similar experiences within a group is compared with each other (Boeije, 2002). In the lesson plays, the dialogues related to students' mathematical ideas and the ways teachers responded to those ideas were thematically coded and grouped based on their pedagogical functions. Throughout the analysis, each of the 23 lesson plays was coded as LP, with identifiers ranging from LP1 to LP23 corresponding to individual prospective teachers.

We used qualitative content analysis for each lesson play (Merriam, 1998), focusing on pedagogical dimensions informed by the frameworks of Ball et al. (2008) and Zazkis et al. (2009). Drawing on Zazkis et al. (2009), we used the idea of "Teaching Images" as an analytical model to classify the lesson plays. This model describes teachers' instructional orientations on a continuum from traditional to reform-oriented approaches. Accordingly, we identified three distinct teaching images in the lesson plays: (TI-1) Traditional, (TI-2) Transitional, and (TI-3) Reform-oriented. In the scenarios of TI-1, prospective teachers built mathematical ideas mainly around rule-based operational explanations and questions, while refusing to provide conceptual explanations. In TI-2, mathematical ideas were explained both procedurally and conceptually, but there was a lack of teacher questions and explanations that would link operational and conceptual explanations. Finally, in TI-3, prospective teachers included explanations based on conceptual knowledge, combined them with procedural knowledge, and asked productive questions to foster connections.

These images served as overarching categories for further analysis. Within each category, we examined (a) the presentation of mathematical ideas, (b) the selection and sequence of tasks, (c) the estimation of student thinking, and (d) the use of mathematical models, aligning with Ball et al.'s (2008) components of mathematical knowledge for teaching. The themes were distinguished based on recurring pedagogical patterns across lesson plays, such as the nature of mathematical explanations, the integration of student thinking, the complexity and sequencing of instructional tasks, and the purpose of using representations. Each theme was derived through iterative comparison and was evaluated for both frequency and pedagogical significance in line with the adopted analytical framework.

Lesson plays categorized according to “teaching images” were evaluated in the next process by assessing the selection and order of tasks, estimated students’ thoughts, and use of representations. Examining the selection and order of tasks included assessing whether: a. The selected tasks were suitable for teaching fractions multiplication, b. The content of the curriculum was integrated with the content of the selected tasks (including the multiplication of natural numbers, simple fractions, compound fractions, and integers), and c. The sequence of tasks progressed from simple to complex. When evaluating tasks for teaching fractions multiplication it was determined whether the tasks in the scenarios required multiplication in fractions. For example, in the scenario of the prospective teacher who prepared the LP21 scenario, the selection of a task requiring fraction division was deemed inappropriate.

Secondly, the number of student dialogues in each scenario was determined in order to examine the predictions of prospective teachers about students’ thoughts. After determining any differences between the teaching images of the lesson plays and the number of student dialogues, the contents of the teacher-student dialogues that emphasized student ideas were examined. During this review, mistakes, difficulties and misconceptions in the dialogues were identified. For example, difficulties and errors that occurred in TI-1 scenarios included: 1. Equalizing the denominator in fraction multiplication; 2. Generalizing the addition/subtraction algorithm to multiplication in fractions; 3. Not dividing the whole into equal parts; 4. Multiplying the numerator and denominator with the natural number in proper fraction multiplication by a natural number; and 5. Including random errors such as operational mistakes. Further, the tables included in the results provide a detailed explanation of other classifications for student thoughts that emerged in the lesson plays.

Finally, the use of models in their scenarios was examined in order to determine the types of models and the purposes for which they were employed. The models used in the scenarios in TI-2 are grouped into three distinct categories: area, set, and length. Furthermore, these models can be used for three key purposes: to support conceptual understanding in introductory parts of the topics, to validate the outcome of the operation, and as stand-alone tasks.

To enhance trustworthiness, both authors separately inspected the scripts and generated distinct sets of themes which were ordered according to an analytical model, with any discrepancies concerning the application of the framework being reconciled. Independently, both authors recognized and collected extracts that exemplified each theme. Afterwards, both authors collaborated to pick the most apt excerpts which encapsulate an identified theme to be included in the paper. The results were in accordance with the teaching images of prospective teachers that were revealed in lesson plays.

Results

The scripts prepared by the prospective teachers were classified based on the teaching images they followed in teaching the lesson. The data obtained from the lesson scripts were explained by supporting the quotations from the scripts created by the prospective teachers based on these classifications: (1) The presentation of mathematical ideas, i.e. teaching images (2) the selection and order of the task, (3) the estimated student thoughts, and (4) the use of the model were included in the analysis of the scripts.

Prospective Teachers' Teaching Images in Lesson Plays

The prospective teachers followed three different types of teaching images in their lesson scripts. These teaching images were named as Teaching Image-1 (TI-1), Teaching Image-2 (TI-2), and Teaching Image-3 (TI-3). The classification of teaching images, the codes for these classifications, and the frequency showing in which of these classifications the lesson plays (LP) are included were presented in Table 1.

Table 1 Explanations on Preferred Teaching Images in Lesson Plays

Category	The presentation of mathematical ideas	LP codes	Frequency
Teaching image 1	-Rule-oriented/ procedural explanations -Refusing conceptual explanations	LP14, LP6, LP17, LP23, LP20, LP5, LP15, LP10, LP11, LP21, LP22, LP1, LP2	13
Teaching image 2	-Procedural explanations -Conceptual explanations -Inadequate relationships between the procedural and conceptual knowledge	LP4, LP3, LP18, LP12, LP7, LP13	6
Teaching image 3	-Conceptual explanations -Procedural explanations -Relational explanations between procedural and conceptual knowledge	LP8, LP9, LP16, LP19	4

Teaching Image 1: Traditional Approaches

Scripts in which an instruction based on procedural knowledge and rules are followed are included in TI-1. There are 13 scripts in the TI-1 category. In these scripts, operational calculations are included as information or as a rule in student and teacher dialogues. In the scripts in TI-1, the dialogues created by the prospective teachers for teaching the subject proceeded with rule or knowledge-oriented explanations without establishing a relationship between the concepts. Instructional explanations in these types of scripts involved dialogues that alternated between teacher-led exposition and student responses. For example, in LP14, the first task in multiplying fractions is for multiplying a natural number by a fraction. The prospective teacher directly applied the multiplication algorithm in solving the question. S/he did not make any semantic associations with other student information regarding the meaning of multiplying by fractions in this multiplication. Below is the imaginary teaching explanation written by the prospective teacher in her script.

Teacher: Now let's look at the product of two fractions. When finding the specified fraction of a fraction, these two fractions are multiplied. When multiplying fractions, the numerators are multiplied with each other and written to the numerator of the product, and the denominators are multiplied with each and written to the denominator of the product.

Figure 1 The Part of an Imaginary Teaching Explanation From LP14

There are some errors in the selection and ordering of tasks in the teaching scripts in the TI-1 category. Although the tasks selected in five of the TI-1 scripts required multiplication of fractions, the order of processing did not proceed from simple to complex. For example, some of the lesson scripts included tasks involving the multiplication of two integers/composite fractions, without the task of multiplying mixed fractions with simple fractions. In addition to the inaccuracy of this ordering, there are also some errors and deficiencies in the selection of tasks. For example, in LP22, the prospective teacher started his/her script with a problem that required division of fractions.

The number of student dialogues and estimated student thoughts were also examined in the lesson scripts in the TI-1 category. There are at least 6 and at most 68 student dialogues in the scripts. Either less or more, the number of student dialogues does not make any difference in terms of instruction. All of the teaching in TI-1 is based on the traditional approach.

Student dialogues usually include re-voicing or short-answered expressions such as "yes", "no", "understood", "not understood". On the other hand, the examination of lesson plays in terms of estimated student thoughts showed that the scripts contained some conceptual difficulties and random errors. These difficulties and error types and their frequency values were given in Table 2.

Table 2 Student Errors and Misconceptions in TI-1 Scripts

Difficulty	f
Equalizing the denominator in fraction multiplication	2 (LP1, LP6)
Generalization of the addition/subtraction algorithm to multiplication in fractions	3 (LP5, LP15, LP22)
Not dividing the whole into equal parts	1 (LP23)
Multiplying the numerator and denominator with the natural number in simple fraction multiplication by a natural number	3 (LP17, LP15, LP10)
Random errors (misunderstanding the problem, operation error, incorrect use of terms, parentheses precedence)	9 (LP21, LP15, LP2)
Total	17

The overgeneralization of fractions addition/subtraction algorithms for fraction multiplication is included in three different scripts. The student dialogue (LP15) demonstrating this difficulty is as follows:

Student: I equated the denominators in the operation $\frac{1}{2} \times \frac{1}{3} = \frac{3}{6} \times \frac{2}{6} = \frac{6}{6}$ *teacher, I applied the operation.*

Figure 2 The Part of an Imaginary Student Explanation From LP15

An error has been identified in this case, which predicts that in the product of fractions only the numerators will be multiplied and the denominator kept the same after the denominator is equalized. Besides, equating the denominator in fraction multiplication was emphasized as a student difficulty in two different scripts categorized as TI-1 (LP6 and LP12), as also indicated in Table 2. One of the examples illustrating this difficulty is in the example below on LP6.

Student: Since $\frac{1}{2}$ of $\frac{2}{5}$ means its half, we expand $\frac{2}{5}$ through multiplying with 2; and we expand $\frac{1}{2}$ through multiplying with 5. Later, we carry out the multiplication.

Teacher: Is it correct what our friend said?

Students: No, wrong.

Teacher: Why?

Students: We don't expand.

Teacher: Yes, we don't expand because expansion was in the addition and subtraction operations of fractions. When multiplying fractions, the numerators are multiplied by themselves and written into the numerator, and the denominators are multiplied by themselves and written in the denominator.

Figure 3 The Part of an Imaginary Dialogue From LP6

This situation, which is considered as a difficulty, highlights the prospective teachers' own lack of knowledge. Prospective teacher regards it as a wrong answer and expresses information about algorithms as a rule. One of the predicted errors to be made by prospective teachers is that they state that when multiplying a whole number by a fraction, the integer will be multiplied by both the numerator and the denominator. Apart from that, other errors included in the scripts are random errors such as the ones based on misunderstanding of the problem and processing errors. The prospective teachers preferred to eliminate all of these errors by explaining the rule of the multiplication algorithm in the continuation of their scripts.

In the scripts in the TI-1 category, it was observed that the models were included in three different ways: (1) reaching the result of the operation without using symbolic representation, (2) showing the correctness of the result of the symbolic operation, and (3) a task independent of the operation. For example, an example of the models used to validate the outcome of the operation is in LP6 as described below. There is a problem in the script where 1/2 of 2/5 must be found. The problem was first solved symbolically in the script, then the accuracy of the result of the operation was explained with the model in Figure 4 below in GeoGebra.

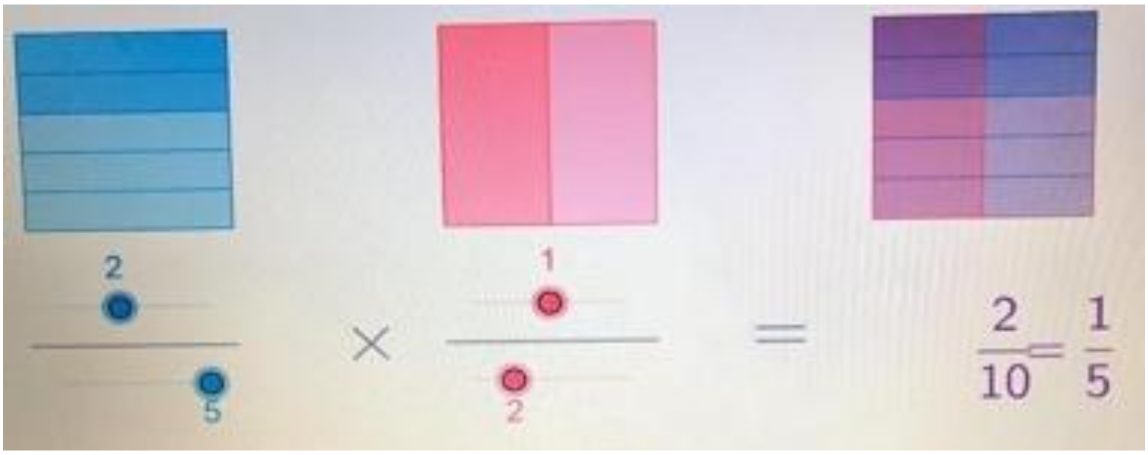


Figure 4 Verification of Multiplication in Fractions with the Model From LP6

An example of using models as a stand-alone task is included in LP10. In the script, the prospective teacher explained how to do 2/5×3/4=6/20 with transparent fraction cards as task. The model for this task in the script is shown in Figure 5.

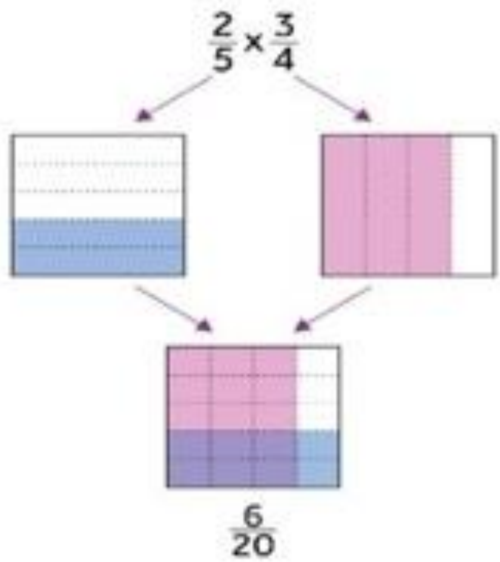


Figure 5 Model Used in Independent Task from LP10

The use of models in lesson plays in TI-1 was based on rules, as in symbolic operations. Models were used in the lesson plays through step-by-step explanations that resembled the process of performing operations. In addition, some models were used incorrectly. Figure 6 shows an example of LP1's model usage.

Teacher: And this is ¾ of ½ . Let's symbolize these two expressions.

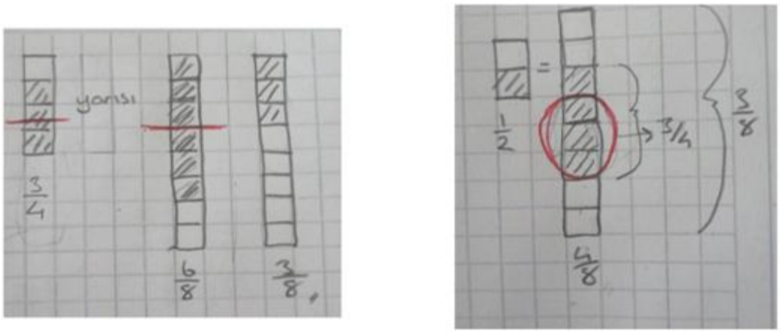


Figure 6 Example of Incorrect Model Use From LP1

In the task in LP1, the representations of fraction multiplication were performed on different wholes. In the script, the prospective teacher works by expanding the fraction, but while doing this, s/he works on different wholes. If the prospective teacher incorporated a dialogue between themselves and the student to address any difficulties that arose during the lesson play, it was concluded that the prospective teacher was cognizant of the issue and attempted to rectify it. On the other hand, if the prospective teacher included an erroneous expression in the imaginary teaching process as an instructional method, then it was inferred that the prospective teacher possessed a deficiency in their understanding. It is evident from Figure 6 that the prospective teacher has a misconception and lack of understanding regarding the model that they employed for teaching purposes in their imaginary teaching.

Teaching Image 2: Transitional Approaches

Six of the teaching scripts are in the TI-2 category. There are some dialogues for teacher inquiry in these scripts, but the inquiries are not completed with sufficient student dialogue or teacher questions. At the end of these inquiries, students are usually directed to procedural processing or rule application. In some scripts, the questions asked remained unanswered. For instance, for the operation 2/6 x 5 prospective teacher asked the students to think about the operation 5x 2/6 . This script section was presented in LP18.

Teacher: So, guys what would you think if it was $\frac{2}{6}$ x 5? What do we think, what do we do?

Student2: But, how is it possible tþ calculate $\frac{2}{6}$ of 5?

Teacher: Well, if we put it this way; for example we have 5 groups of $\frac{2}{6}$, and we want to make them one group. What should we do?

Figure 7 The Part of an Imaginary Dialogue From LP18

In the continuation of the script section above, the prospective teacher benefited from the model and modelled 5×2/6 product again. Since the result of the operation will be the same due to its commutative property, he calculated 5×2/6 and found the result of 2/6 x 5. Based on the meaning of the process, the prospective teacher tried to direct the students to the solution, but did not bring the continuation of these interrogating questions appropriately.

There are also scripts that conceptually correctly explain that the invariance feature in the TI-2 category does not change the result (LP12 and LP13). For example, a problem case is given in LP12 that requires the 6 × 2/3 operation. In the continuation of the script, conceptual explanations were made by making use of the models, and finally, the following dialogue took place.

Teacher: As we have seen in the models you have drawn in your notebook, since each of the six people will use $\frac{2}{3}$ of a paper, we can find the total amount of paper through multiplying 6 by $\frac{2}{3}$.

Figure 8 The Part of an Imaginary Teaching Explanation From LP12

Right after this problem situation, models and conceptual explanations of the 2/3 × 6 operation were given suitably. While ending the task, the same result was achieved in 6 × 2/3 and 2/3 × 6 operations, but a student dialogue emphasizing the semantic differences of both operations was included. The dialogue is below.

Student: Yes teacher, I understand very well. In other words, I saw that the expression 6 × $\frac{2}{3}$ refers to the sum of six $\frac{2}{3}$ expressions, and the expression $\frac{2}{3}$ × 6 refers to $\frac{2}{3}$ of 6, but the results are the same.

Figure 9 The Part of an Imaginary Student Explanation LP12

In some of the scripts, prospective teachers made conceptual explanations about the subject through student dialogues. In another part, teachers wanted to emphasize conceptual teaching, but they could not write their scripts appropriately.

Student2: Teacher, we will do 24×3/8 = 24/24×3/8

Teacher: But our whole is here 24 (whole). So we have a whole again, divided into 24 parts, and the operation wants 3/8 of it.

Teacher: Let's try to model with an example.

Teacher: Ali has 9 sticks. Two-thirds of the sticks are blue. How many blue sticks does Ali have?

Teacher: We have divided it into 3 equal parts, and we will take 2 of them. So what have we done here?

Student: We have divided it into 3 equal parts and got 2 of them. So 9:3=3; 3×2=6.

Teacher: Ayşe filled 5 glasses with 2/3 litres of milk in each. How much milk did Ayşe use? What do you think of this question guys? Let me give you a hint. We have 5 glasses, each of them contains 2/3 of milk. The operation asks us the total amount of milk.

Student: Hmmm... If we say we have 5 glasses, let's divide each glass into 3 equal parts and get two of them. At that time, we divided our glass into 3 equal parts 2/3+2/3+2/3+2/3+2/3=10/3. All 3 glasses were filled, and 1 part of 3 equal parts of the 4th glass was filled.

(Some students noticed that it was multiplied by 5.)

Student2: Teacher, then we can do 5×2/3 =. If the answer is 10/3, we multiply the numerator by the natural number when multiplying a natural number by a fraction.

Figure 10 The Part of an Imaginary Dialogue From LP4

In this task of LP4, no feedback was given to the students' incorrect answers and explanations were made through other examples. In this script, the prospective teacher tried to question the meaning of multiplication and wanted to associate the model with meaning. However, the questions asked in the introductory part of the task remained unanswered (24×3/8), and no explanation was given to overcome the difficulty experienced by the students in "multiplication of natural number with numerator and denominator in simple fraction multiplication by natural number". The prospective teacher brought the task to a conclusion by making inferences about the rules by the students in his script.

All tasks in TI-2 scripts are suitable for teaching fractions multiplication. None of the scripts implemented a task for multiplication of mixed/improper fractions with proper fractions, but there are tasks involving the product of two mixed fractions in the LP4 and LP7 scripts, and two improper fractions in LP3. In one of the scripts (LP3), the subject was introduced with the product of two proper fractions, without including a task on the product of natural numbers and proper fractions. In other TI-2 scripts, the tasks are ordered from simple to complex.

It is seen that the frequencies of the student dialogues in the scripts vary between 19 and 39. In addition, various difficulties and error types and their frequencies in predicted student thinking in TI-2 scripts are given in Table 3.

Table 3 Student Errors and Misconceptions in TI-2 Scripts

Difficulty	f
Multiplying the numerator and denominator with the natural number in proper fraction multiplication by a natural number	3 (LP4, LP12, LP13)
Generalizing the addition/subtraction algorithm for fractions to multiplication	1 (LP7)
Making same denominator in fraction multiplication	1 (LP3)
Splitting the whole into equal parts	1 (LP3)
Random errors (cross product)	2 (LP3, LP13)
Total	8

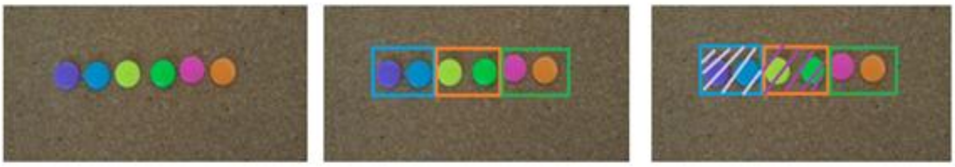
In some scripts, difficulties and errors were not corrected, and in the rest of the script, the difficulty was explained through other questions (LP4). In some of the TI-2 scripts, conceptual explanations were made in resolving the difficulties. For example, for the difficulty of multiplying the natural number by the numerator and denominator in the multiplication of a natural number by a proper fraction, the prospective teacher predicted that the students would be able to find the result by making use of the repeated addition meaning of the multiplication operation (LP12). That is, on the contrary, he refuted the student's erroneous conclusion by way of example. However, in the continuation of the script, the student who did the error did not continue his role and ended the script by explaining the rule of multiplication. The imaginary teacher dialogue that the prospective teacher wrote for the outcome in this script is as follows:

Teacher: As can be seen, in order to find the sum of a fraction as much as a natural number, the fraction is multiplied by this natural number. In this multiplication operation, the natural number and the numerator are multiplied, and the result is written into the numerator of the product, while the denominator remains unchanged.

Figure 11 The part of an imaginary teaching explanation LP12

In this script, the teacher carried out the teaching conceptually, but at the last point, s/he based the situation in a rule and brought the task to a conclusion.

It has been observed that the models in the TI-2 category are generally used effectively for one or more purposes within the same script. In the TI-2 scripts, models were used (1) to support conceptual understanding in the introductory parts of the topics, (2) to validate the outcome of the operation, and (3) as stand-alone tasks. An example of using a relational model to support conceptual understanding of the meaning of multiplication in fractions is in the script of LP13. In this script, fraction models are used to make sense of the 6x 2/3 operation. Below is the modelling section describing this usage:



Teacher: You see the 6 coloured pins, friends. In the question, he asked us to divide these 6 pins into 3 groups. But it wanted to take only 2 of these 3 groups. There are a total of 4 pins in these two groups. So, we find that 6x 2/3 = 4.

Figure 12 Fraction Models for Multiplication Operation

As seen in the example, models in TI-2 scripts were used not only by rote or as a separate method, but also in relation to meaning. The use of models to support conceptual understanding is one of the points where TI-2 scripts differ from TI-1 scripts. As in the TI-1 scripts, some modelling tasks are used to verify the result of the operation found by the multiplication algorithm (LP7). The model used for this purpose is given in Figure 13.

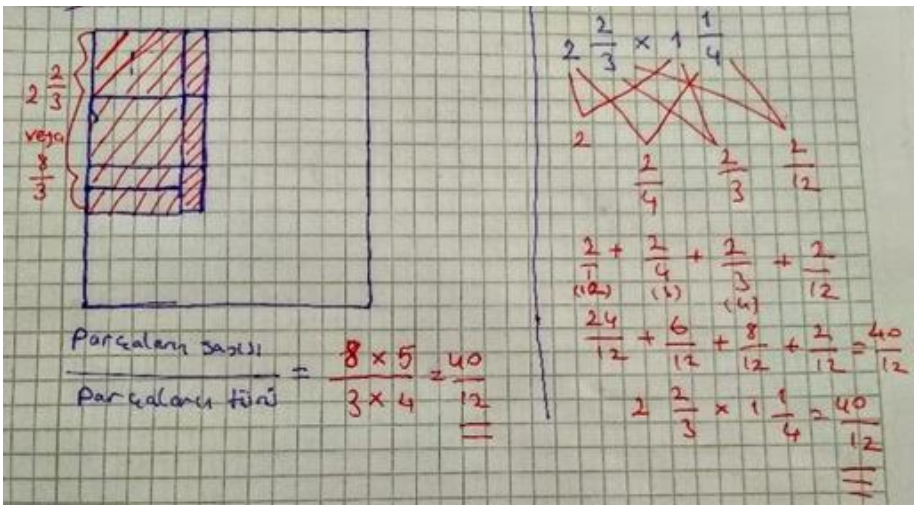


Figure 13 The Part of an Imaginary Teaching Explanation From LP7

The prospective teacher wrote a script in which s/he wanted the result of the operation 2 2/3×1 1/4 to be found in different ways in LP7. S/he emphasizes that one of these ways should be the area model. However, operational calculation and the model were not effectively correlated, only the results were compared. In TI-2, as in TI-1, models were also presented as a separate task. For example, procedural explanations for the use of fraction cards are made in LP3. However, there are also examples of relational use of models in LP3. In this respect, the TI-1 and TI-2 scripts differ from each other.

Teaching Image 3: Reform Oriented Approaches

There are 4 lesson plays in the TI-3 category. The imaginary dialogues in the scripts in this category are based on inquiries and conceptual explanations; concept-model associations are carried out effectively. Different meanings of fractions, different models, their associations, and estimated student difficulties are included in the scripts. Appropriate inquiries were made to overcome student difficulties. For example, in LP9, the meaning of the fraction was questioned in the script, and area and length models were included. Below is a section on finding the product of 4/5 x 1/3 using counting objects.

Student: If we think that we have 5 marbles, how do we find $\frac{1}{3}$ of it?

Teacher: ...how many objects do you want to work with?

Student: Can it be 10?

Teacher: Let's try. You have 10 countable objects. You will find. $\frac{4}{5}$ first.

Student: It is divided into 2. If we take 4 of 2 pieces, it will be 8 pieces.

Teacher: Now you need to find $\frac{1}{3}$ of these 8 parts.

Student: It is not divided equally.

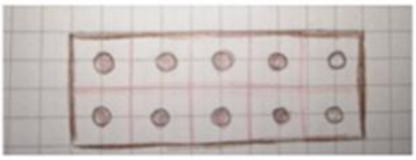


Figure 14 The Part of an Imaginary Dialogue From LP9.

In this part of LP9, the teacher made the students question their own answers, without answering the students' answers as true or false. The student first plans to model using 10 objects for the product of 4/5×1/3. However, as a result of the teacher's questioning, the student realized that it would be more accurate to determine the number of objects by finding the common multiple of 5 and 3. The main goal of teaching in all TI-3 scripts is to provide a conceptual understanding of multiplication in fractions. Scripts sometimes include estimated student difficulties or misconceptions. These challenges were used to deepen the teaching, emphasizing conceptual understanding and actively encouraging students to explore and reason about the underlying principles of multiplication with fractions.

In TI-3 scripts, the selection of tasks is suitable for teaching fractions multiplication. In all TI-3 scenarios, natural number multiplications are introduced as a reminder of previous learning of students and are followed by the teaching of proper fractions. Again, in all scripts, the product of two proper fractions takes place appropriately. However, as seen in TI-1 and TI-2, there is no task involving the product of mixed/improper fractions in 2 of the 4 scripts in the TI-3 group. Despite these shortcomings, the tasks preferred in all TI-3 scripts are arranged from simple to complex, that is, appropriately.

Frequencies of student dialogues in lesson plays in TI-3 are between 19-77. Some of the teacher-student dialogues in these scripts are based on questioning and students' exploration of concepts. In some places, there are conceptual explanations in teacher and student dialogues. Some of the student dialogues included estimated student difficulties and errors. The student difficulties included in the TI-3 scripts are summarized in Table 4.

Table 4 Student Difficulties Mentioned in TI-3 Scripts

Difficulty	f
Working on different wholes	4 (LP8, LP16[2], LP19)
Inability to divide the whole into equal parts	1 (LP16)
Random errors (inappropriate use of the model, not understanding the question, etc.)	2 (LP19, LP16, LP9)
Total	7

In TI-3 scripts, unlike TI-1 and TI-2, there are difficulties that may be encountered while using the models and associating the concept with the model. In the scripts, it is foreseen that these difficulties are eliminated with appropriate questioning. In the script below (LP9), an estimated student error that can be made while showing 1/2 of 4/5 was included.

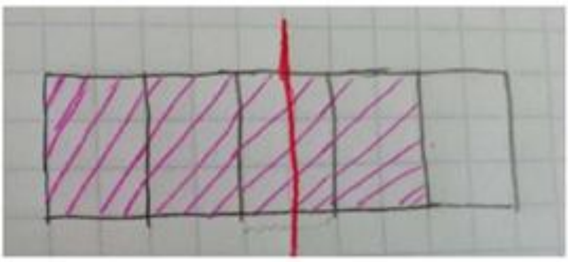


Figure 15 A Hypothetical Student Error in Representing One-half of Four-fifths

It was assumed that the student could reach the result by taking half of the whole, not the shaded area while modelling. In another hypothetical student difficulty, it was predicted that students could work on different fraction wholes. A script was written for students so that they can find 3/5 of 2/7. The script section (LP8) regarding the difficulty that may be encountered in solving this problem is given below.

Student: Teacher, I first modelled the fraction $\frac{2}{7}$ *and then the fraction* $\frac{3}{5}$

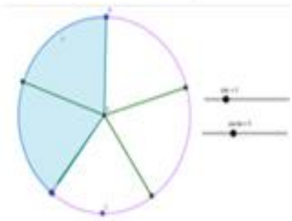
Student: Thus, our total shaded area consists of 3+2 =5 and our whole consist of 5+7=12 pieces. The answer is $\frac{5}{12}$.

Figure 16 A Hypothetical Student Error in Finding Three-fifths of Two-sevenths (LP8)

In this estimated error, it was foreseen that the students would be able to do an addition by working on the same unit fractions, not the same whole. In addition, it was predicted that in this difficulty, the student would be able to add, not multiply, to determine a different fraction of a fraction. In LP8, the prospective teacher wrote his script based on teacher questions and conceptual explanations. At the end of the above section, the student's difficulty was tried to be resolved by agreeing on the idea that "the same whole should be worked on". Difficulties in TI-3 scripts are usually "How do we draw this?", "Do you think this drawing is correct?", "Why did we divide by 2?" and "What does the shaded area mean to me?" It has been scripted in such a way that it can be corrected with questions such as -by asking the students to give reasons for the paths they have followed and by making them think about the errors.

Unlike other teaching images, TI-3 scripts include more model types (area models, set models, length models). Prospective teachers who followed this teaching image used the models relationally to support conceptual understanding. In the example of LP19 below, s/he wanted students to demonstrate their understanding of the ratio of the number of pages Efe has already read to the total number of pages by using a fraction model. S/he associated this model with the contextual information of the problem to help students conceptualize the ratio, and used the model to visualize the value (2/3) expressed by the fraction.

Student: (The student drew the model below.)



Teacher: What do the shaded area and unshaded parts mean to me?

Student: The shaded pieces represent the pages that Efe read, and the remaining pieces represent the unread pages.

Figure 17 The Part of an Imaginary Dialogue of Fraction From LP19

In another TI-3 script (LP16), each of the fraction models (set, length, area) was used in appropriate contexts. The explanations of the imaginary teacher in LP16 are intended to relate the model both operationally and conceptually. Examples of models used in this script are shown in Figure 18, below.

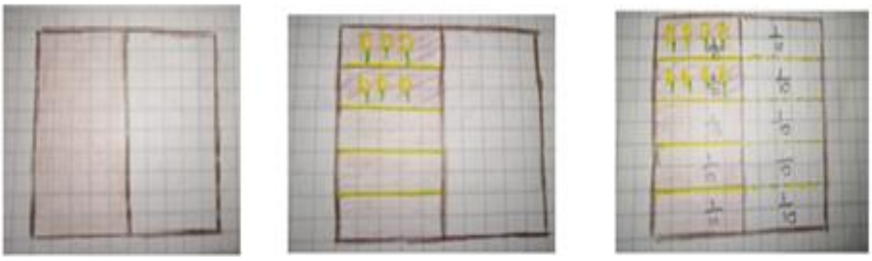


Figure 18 The Use of Fraction Model to Make Relations Both Operationally and Conceptually

In the continuation of the script of LP16, it was predicted that the students could realize the multiplication algorithm in fractions, from the results obtained on the models.

Conclusions and Discussion

In this study, we examined the teaching images of prospective teachers regarding the multiplication of fractions in their lesson plays. The lesson plays were classified into three teaching images (TI-1, TI-2, and TI-3) based on the presentation of mathematical ideas throughout the scenarios.

TI-1 scenarios were operation-oriented and based on procedures, knowledge, definitions, and rules. It was observed that these scenarios did not support conceptual learning, and the imaginary answers or explanations were based on rules and information rather than on associated concepts, operations, and representations. These findings are consistent with those of Zazkis et al.'s (2013) study, which found that prospective teachers often base their teaching on rules rather than concepts.

In TI-2 scenarios, the focus was on conceptual teaching, and prospective teachers attempted to connect operations and the meaning of the concept through appropriate inquiry. However, due to their lack of content knowledge, they could not perform questioning and make connections effectively. Despite questioning the conceptual meaning of the concept in some parts of the scenarios, the relationship between the meaning of the concept and the procedures could not be established properly, and the focus was on the rule.

TI-3 scenarios were conceptually oriented, and prospective teachers used conceptual explanations and inquiries based on different meanings of fractions and representations. They were able to connect procedures, meaning, and models, indicating that they had a better understanding of the multiplication of fractions. However, most of the lesson plays did not have conceptual explanation dialogs. Our findings are in line with Toluk-Uçar's (2011) research, which found that prospective teachers' explanations of operations in fractions are generally at the operational level and involve repetition of rules, and that fewer prospective teachers can provide conceptual explanations. This highlights the need to improve the content knowledge of prospective teachers to support their ability to teach the multiplication of fractions effectively.

It has been observed that most prospective teachers use models to verify computations, rather than to support students' understanding of concepts or to use them as part of a larger instructional strategy. This result indicates that prospective teachers have deficiencies in content knowledge. The scenarios developed by prospective teachers offer insights into their likely classroom practices. Results show that prospective teachers struggle to predict typical student mistakes and often make random computational errors. Additionally, prospective teachers ask mostly questions that require only one correct answer due to their inadequate content knowledge. These problems are typically addressed with conventional and non-conceptual explanations in subsequent instructional moves.

A promising finding of this study is that prospective teachers included many student dialogues in their lesson plays, which indicates their intention to focus on student thinking. However, in most of these dialogues, students simply repeated what the teacher said or gave short, confirmatory responses. This showed that prospective teachers had difficulty imagining realistic student contributions and common misconceptions. Similar findings are reported in the study by Fernández et al. (2023), in which prospective teachers were able to notice students' mathematical ideas but struggled to respond in ways that support deeper learning. Likewise, Santagata et al. (2021) highlighted that teacher education programs often focus more on recognizing student thinking than on how to respond pedagogically. In our study, although prospective teachers included student errors in their scenarios, they mostly corrected them using rule-based and procedural explanations. This suggests that their ability to respond meaningfully to student thinking needs further support. Similar tendencies were observed in previous research, where prospective teachers preferred using rules and procedures rather than conceptual explanations to address student errors (Gökkurt et al., 2012; Toluk-Uçar, 2011).

Moreover, the types of student dialogues varied across the three scenario types. In TI-1 scenarios, student statements mostly consisted of short answers or rule confirmations. In TI-2 scenarios, dialogues focused on procedures and definitions. In contrast, TI-3 scenarios contained inquiry-based and conceptually rich dialogues, although these appeared scripted rather than realistic. These differences reflect the level of conceptual understanding demonstrated in each teaching image. However, even in the most conceptually oriented scenarios, student contributions seemed more like reflections of the prospective teacher's own knowledge than authentic classroom discourse (Zazkis, 2017).

In addition to challenges in representing student thinking, prospective teachers also experienced difficulties in designing instructional sequences. In TI-1 scenarios, inappropriate task selection was observed. For instance, LP22 began with a division task instead of multiplication, indicating a misunderstanding of content structure. In TI-2 scenarios, tasks were sometimes not ordered from simple to complex, leading to ineffective instructional progression. Only the TI-3 scenarios demonstrated appropriate task selection and sequencing. Moreover, all scenarios included tasks with proper fractions only; none addressed mixed fraction multiplication, even though it is an essential part of the curriculum. These findings further highlight gaps in prospective teachers' Specialized Content Knowledge (SCK)—such as selecting and sequencing appropriate tasks—and Knowledge of Content and Students (KCS)—such as anticipating typical student difficulties and designing appropriate instructional moves (Ball et al., 2008). Together, these results underline the importance of supporting prospective teachers in both anticipating student thinking and designing instructional tasks that promote conceptual understanding.

The use of models in scenarios was handled in different ways. In TI-1 scenarios, models were used to show the operational process, while in TI-2 scenarios they were used to verify the result of the computation and explain its meaning. In TI-3 scenarios, area models, length models, and set models were appropriately included. Prospective teachers were found to have deficiencies in modelling fraction multiplication correctly (Osmanoğlu & Özgeldi, 2018). Other studies have corroborated this finding, stating that area models are generally used in fraction multiplication, with less space given to set and length models (Osmanoğlu & Özgeldi, 2018). In all scenarios, the deficiencies in the prospective teachers' content knowledge caused incomplete/inaccurate mathematical language to be included in the dialogues involving models. These challenges in using and explaining models can be interpreted as a lack of Specialized Content Knowledge (SCK) and Knowledge of Content and Teaching (KCT) as defined in the MKT framework (Ball et al., 2008). While prospective teachers may be familiar with specific representations, they often struggle to choose appropriate models and explain their use effectively in instruction.

One reason why teachers are hesitant to adopt concept-oriented teaching is because it requires a greater level of teacher knowledge. When the teacher only gives rules in the lesson and the students learn the algorithm to find the result of multiplication in fractions, the teacher considers this sufficient. This eliminates the need for the teacher to increase their pedagogical content knowledge. In both rule-based and concept-oriented scenarios, it is clear that the deficiencies in the prospective teachers' content knowledge directly affects the teaching. Previous studies have noted that the knowledge of prospective teachers regarding multiplication in fractions is inadequate (Işıksal & Çakıroğlu, 2011). This study further reveals that the mistakes in the questions asked by the prospective teachers in the examples they gave to the students and in the instructional design (inappropriate example selection, incorrect use of representations, or inability to explain the representation conceptually) are caused by insufficient pedagogical content knowledge/mathematics teaching knowledge.

Finally, it is important to reflect on the role and limitations of lesson plays as a research tool. Although lesson plays offer a valuable opportunity to explore how prospective teachers envision teaching practices, they also have certain limitations. As Zazkis et al. (2009) note, lesson plays reflect imagined classroom interactions rather than actual teaching, which may limit their ability to capture spontaneous decisions or real-time responses to student thinking. Moreover, since prospective teachers write these scripts outside of real classroom settings, the pedagogical moves and student dialogues they include may not fully reflect authentic classroom dynamics. This limits the generalizability of the findings and suggests that the results should be interpreted with caution. On the other hand, lesson plays allow researchers to gain insights into participants' instructional thinking in a structured yet flexible way, without the time and setting limitations of classroom observation. In this study, they served as a valuable tool for analyzing how prospective teachers plan instruction, anticipate student ideas, and use representations. Still, future research may benefit from combining lesson plays with classroom-based data—such as video recordings or stimulated recall interviews—to triangulate findings and strengthen validity.

Recommendations for Future Research and Practice

Although the lesson play utilized in this research provided a comprehensive perspective on prospective teachers' teaching images, requesting them to compose a general scenario about a shared objective has not disclosed specific aspects of their teacher knowledge. For example, insights into the content knowledge of a prospective teacher who engages in process-oriented teaching are infrequent. Therefore, rather than requiring prospective teachers to create an entire lesson play, a more effective approach would be to request they compose scenes with focused and explicit objectives, including a rubric for assessment, to better reveal their content knowledge.

Writing a lesson play requires a focus on imaginary interactions, and one effective way to create a scenario is to focus on "different student questions" (Zazkis et al., 2009). Involving actors in the process and naming the imaginary students who ask these questions can also help develop prospective teachers. However, in this study, few prospective teachers named the students or followed their thoughts. Therefore, it is important for future research to investigate lesson plays that follow students' thoughts individually. Additionally, future studies should explore how watching videos of actual teacher-student interactions changes prospective teachers' lesson plays. In conclusion, writing a lesson play encourages prospective teachers to think like both a teacher and a student, making it an effective way to prepare them for actual teaching.

Compliance with Ethical Standards

Disclosure of potential conflicts of interest

The author declares that there are no conflicts of interest, financial or otherwise, that could have influenced the research presented in this manuscript.

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CRediT author statement

Both authors contributed equally to all stages of the study, including conceptualization, methodology, data collection, analysis, and writing.

Research involving Human Participants and/or Animals

This study was approved by the Ethics Committee of Burdur Mehmet Akif Ersoy University (Meeting Date: April 7, 2021; Meeting No: 2021/04; Decision No: GO 2021/161). Participation was voluntary, and informed consent was obtained from all participants. All procedures were conducted in accordance with institutional and national ethical standards.

İlköğretim Matematik Öğretmeni Adaylarının Kesirlerin Çarpımı Konusundaki Ders Senaryolarının İncelenmesi

Özet:

Bu çalışmanın amacı, öğretmen adaylarının ders senaryolarında kesirlerin çarpımı konusundaki öğretim imajlarını incelemektir. Bu araştırma Türkiye'nin güney bölgesinde bulunan bir devlet üniversitesinde yürütülmüş ve araştırmaya matematik öğretmenliği programına kayıtlı 23 üçüncü sınıf öğretmen adayı (21 kadın ve 2 erkek) katılmıştır. Veri toplama aracı olarak ders senaryoları kullanılmış ve öğretmen adayları tarafından 23 farklı ders senaryosu hazırlanmıştır. Verilerin analizi sonucunda, senaryolar boyunca matematiksel fikirlerin sunumuna dayalı olarak ders senaryoları üç öğretim imajı altında sınıflandırılmıştır. Bu çalışma, öğretmen adaylarının kesirlerin çarpımını öğretmede alan bilgilerindeki eksiklikleri nedeniyle zorlandıklarını ortaya koymuştur. Bu eksiklikler "kurala dayalı açıklamalar", "işlem hataları" ve "amaçla örtüşmeyen görev ve model seçimi" şeklinde ortaya çıkmıştır. Ayrıca öğretmen adayları, öğrencilerin gerçekçi tepkilerini öngörmede ve uygun diyaloglar oluşturmada da zorluklar yaşamıştır. Bu bulgular, öğretmen adaylarının kesirlerin çarpımını daha etkili bir şekilde öğretebilmeleri için daha kapsamlı matematik alan bilgisi sağlanmasının gerekliliğini vurgulamaktadır. Öğretmen adaylarının öğretim imajlarının ve matematik öğretim stratejilerinin nasıl desteklenebileceğinin araştırılması önerilmektedir.

Anahtar kelimeler: Ders senaryosu, kesirlerin çarpımı, öğretmen adayları, öğretim imajı.

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