



NUMERICAL SOLUTIONS OF TIME FRACTIONAL KORTEWEG–DE VRIES EQUATION AND ITS STABILITY ANALYSIS

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ABSTRACT. In this study, the fractional derivative and finite difference operators are analyzed. The time fractional KdV equation with initial condition is considered. Discretized equation is obtained with the help of finite difference operators and used Caputo formula. The inherent truncation errors in the method are defined and analyzed. Stability analysis is explored to demonstrate the accuracy of the method. While doing this analysis, considering conservation law, with the help of using the definition discovered by Lax-Wendroff, von Neumann stability analysis is applied. The numerical solutions of time fractional KdV equation are obtained by using finite difference method. The comparison between obtained numerical solutions and exact solution from existing literature is made. This comparison is highlighted with the graphs as well. Results are presented in tables using the Mathematica software package wherever it is needed.

1. INTRODUCTION

Nowadays, one of the developing conceptions is the fractional differential equations. This notion began to develop since 17th century with the help of several mathematicians' studies on differential and integration, like Leibniz, Euler, Lagrange, Abel, Liouville etc. [1, 2, 3]. $(0.5)^{th}$ order derivative was defined by Leibniz in the year 1695. Riemann-Liouville, Hadamard, Grunwald-Letnikov, Riesz and Caputo have given the integral inequalities to the literature. In 2006, by Kilbas, Srivastava, Trujillo and in 1993 by Samko, Kilbas, Marichev defined the fractional theory and different derivatives with developments [4, 5].

The exact solutions of the fractional differential equations may not be easily obtained, so we need numerical methods for fractional differential equations. One of them is finite different method and it is one of the most popular methods of

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numerical solution of partial differential equations. There are some studies about this method's stability analysis. B.F. Feng, in his study, examined Von Neumann's Stability analysis by linearizing Korteweg-de Vries (in short, KdV) equation.

In this study, classical partial differential equations have been extended to the fractional partial differential equations. There are many applications of this equation in the literature. The fractional partial differential equations have been used in applications such as fluid, flow, finance, hydrology and others [6, 21]. In this paper, we investigate finite difference numerical methods to solve the time fractional KdV equation of the form [22]

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} + 6u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{\partial^3 u(x, t)}{\partial x^3} = 0, \quad (1.1)$$

$u(x, 0) = u_0$, $a \leq x \leq b$ and $u(a, t) = u(b, t) = 0$, $0 < t \leq T$, where $0 < \alpha \leq 1$. Eq. (1.1) uses a Caputo fractional derivative of order α , defined by

$$\frac{\partial^\alpha f(x, t)}{\partial t^\alpha} = \frac{1}{\Gamma(m - \alpha)} \int_0^t \frac{\partial^m f(x, \xi)}{\partial \xi^m} (t - \xi)^{\alpha - m + 1} d\xi \quad (1.2)$$

where m is an integer that $m - 1 < \alpha \leq m$. The function $\Gamma(\cdot)$ is called as Gamma function.

2. ANALYSIS OF FINITE DIFFERENCE METHOD

Let us define some notations to describe the finite forward difference method. Δx is the spatial step, Δt is the time step, $x_i = a + i\Delta x$, $i = 0, 1, 2, \dots, N$ points are the coordinates of mesh and $N = \frac{b-a}{\Delta x}$, $t_j = j\Delta t$, $j = 0, 1, 2, \dots, M$ and $M = \frac{T}{\Delta t}$. The function $u(x, t)$ is the value of the solution at these grid points which are $u(x_i, t_j) \cong u_{i,j}$, where we denote by $u_{i,j}$ the numerical estimate of the exact value of $u(x, t)$ at the point (x_i, t_j) . Now, we define the difference operators as

$$H_t u_{i,j} = u_{i,j+1} - u_{i,j}, \quad (2.1)$$

$$H_x u_{i,j} = u_{i+1,j} - u_{i,j}, \quad (2.2)$$

$$H_{xx} u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}, \quad (2.3)$$

$$H_{xxx} u_{i,j} = u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j}. \quad (2.4)$$

Thus, partial derivatives are approximated through the finite difference operators as

$$\left. \frac{\partial u}{\partial x} \right|_{i,j} = \frac{H_x u_{i,j}}{\Delta x} + O(\Delta x^2), \quad (2.5)$$

$$\left. \frac{\partial^3 u}{\partial x^3} \right|_{i,j} = \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} + O(\Delta x^2). \quad (2.6)$$

According to the shifted Caputo definition [23],

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \cong \begin{cases} \frac{h^{-\alpha}}{\Gamma(2-\beta)} H_t u + \frac{h^{-\alpha}}{\Gamma(2-\beta)} \sum_{k=1}^i H_t u_{i,j-k} f(k), & j \geq 1 \\ \frac{h^{-\alpha}}{\Gamma(2-\beta)} H_t u_{i,0}, & j = 0 \end{cases} \quad (2.7)$$

There are many studies in the literature on fractional derivatives of Taylor Series. The generalized Taylor series which is in these studies has been awarded by Odibat [24].

$$\left. \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} \right|_{i,j} = \Gamma(\alpha + 1)(\Delta t)^{-\beta} H_t u_{i,j} + O(\Delta t^{2\alpha}). \quad (2.8)$$

In the finite difference method, substituting Eqs. (2.5), (2.6) and (2.7) into Eq. (1.1) can be written as indexed

$$u_{i+1,j} = \frac{6(\Delta x)^2 u_{i,j}^2 + \frac{1}{2}(2u_{i+1,j} + H_{xxx} u_{i,j}) + \vartheta \left[H_t u_{i,j} - \sum_{k=1}^j f(k)(H_t u_{i,j-k}) \right]}{-1 + 6(\Delta x)^2 u_{i,j}}, \quad (2.9)$$

where $\vartheta = \frac{(\Delta x)^3}{(\Delta t)^\alpha \Gamma(2-\alpha)}$, $f(k) = -k^{1-\alpha} + (1+k)^{1-\alpha}$ and the initial values $u_{i,0} = u_0(x_i)$.

3. CONSISTENCY ANALYSIS AND TRUNCATION ERROR

In this section, we investigate the consistency the Eq. (1.1) by the finite difference method. At first, Taylor series expansions can be given in the form as follows,

$$u_{i+1,j} = u_{i,j} + \Delta x \frac{\partial u}{\partial x} + (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3), \quad (3.1)$$

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t} + (\Delta t)^2 \frac{\partial^2 u}{\partial t^2} + O(\Delta t^3), \quad (3.2)$$

$$u_{i-1,j} = u_{i,j} - \Delta x \frac{\partial u}{\partial x} + (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} - O(\Delta x^3), \quad (3.3)$$

$$u_{i+2,j} = u_{i,j} + 2\Delta x \frac{\partial u}{\partial x} + (2\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3), \quad (3.4)$$

$$u_{i-2,j} = u_{i,j} - 2\Delta x \frac{\partial u}{\partial x} + (2\Delta x)^2 \frac{\partial^2 u}{\partial x^2} - O(\Delta x^3). \quad (3.5)$$

Now, let us define an operator L ,

$$L = \frac{\partial}{\partial t} + 6u \frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}. \quad (3.6)$$

The indexed form of operator L can be written as

$$L_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + 6u \frac{H_x u_{i,j}}{\Delta x} + \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3}. \quad (3.7)$$

If we substitute the indexed form Eqs. (3.1), (3.2), (3.3), (3.4), and Eq. (3.5) into the Eq. (3.7) and do some necessary manipulations, then the approach will be

$\Delta t \rightarrow 0$, and $\Delta x \rightarrow 0$. So the Eq. (3.7) will be the same as left hand side of the Eq. (1.1). This conclusion shows us that the Eq. (1.1) is consistent by finite difference method.

Theorem 3.1. *The truncation error of the finite difference method Eq. (1.1) to the KdV equation is $O((\Delta t)^{2\alpha} + (\Delta x)^2)$.*

Proof. Substituting Eqs. (2.5), (2.6) and (2.8) into Eq. (1.1) we arrive at

$$\Gamma(\alpha+1)(\Delta t)^{-\beta} H_t u_{i,j} + O(\Delta t^{2\alpha}) + 6u_{i,j} \left(\frac{H_x u_{i,j}}{\Delta x} + O(\Delta x^2) \right) + \left(\frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} + O(\Delta x^2) \right) = 0. \quad (3.8)$$

If the necessary corrections are made in Eq. (3.8), it becomes

$$\Gamma(\alpha+1)(\Delta t)^{-\beta} H_t u_{i,j} + u_{i,j} \frac{H_x u_{i,j}}{\Delta x} + \delta \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} + O(\Delta t^{2\alpha} + \Delta x^2) = 0. \quad (3.9)$$

Eq. (1.1) can be written as indexed

$$\Gamma(\alpha+1)(\Delta t)^{-\beta} H_t u_{i,j} + u_{i,j} \frac{H_x u_{i,j}}{\Delta x} + \delta \frac{H_{xxx} u_{i,j}}{2(\Delta x)^3} = 0. \quad (3.10)$$

The truncation error is $O(\Delta t^{2\alpha} + \Delta x^2)$. \square

4. LINEAR STABILITY ANALYSIS

In this section, we mainly study the stability for the finite difference method. To describe this method, we consider the first-order conservation equation

$$\frac{\partial u}{\partial t} + \tau \frac{\partial u}{\partial x} = 0, \quad (4.1)$$

where $u = u(x, t)$ is a physical function of the space variable x and time t . This equation is frequently encountered in applied mathematics. Lax and Wendroff studies using form Eq. (4.1) [25]. Substituting the Eq. (4.1) into Eq. (1.1) and choosing $\alpha = 1$, yields:

$$-\tau \frac{\partial u}{\partial x} + 6u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{\partial^3 u(x, t)}{\partial x^3} = 0, \quad (4.2)$$

$$\left(-\tau u + 3u^2 + \frac{\partial^2 u(x, t)}{\partial x^2} \right)_x = 0. \quad (4.3)$$

If we integrate the Eq. (4.3) with respect the variable x and choose the zero as an integration, we have

$$-\tau u + U + \frac{\partial^2 u(x, t)}{\partial x^2} = 0, \quad (4.4)$$

where $U = 3u^2$. The linear indexed form of the equation given Eq. (4.4) is as follow

$$-\tau u + U + \frac{H_{xx}}{(\Delta x)^2} = 0. \quad (4.5)$$

Theorem 4.1. *The finite difference method for the KdV equation is unconditionally linear stable.*

Proof. We consider Von Neumann’s Stability of the finite difference method for the KdV equation. Let

$$u_{i,j} = u(i\Delta x, j\Delta t) = u(p, q) = \lambda^q e^{I\xi p}, \quad \xi \in [-\pi, \pi], \quad (4.6)$$

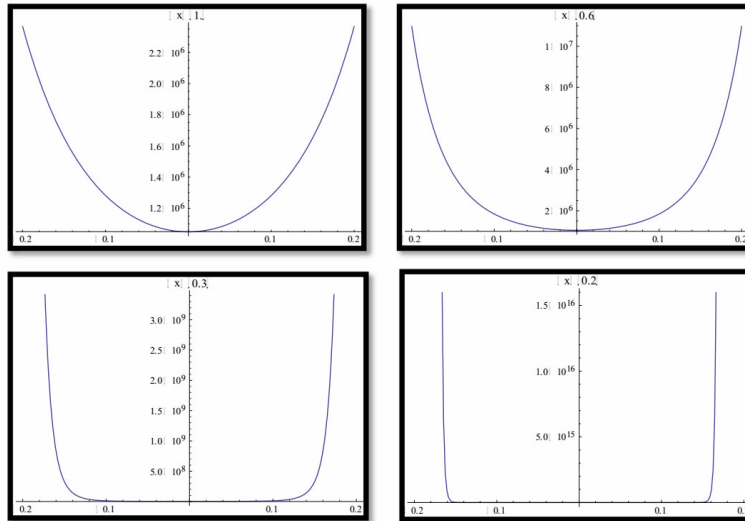
where $p = i\Delta x$, $q = j\Delta t$ and $I = \sqrt{-1}$. If we substitute the Eq. (4.6) into the Eq. (4.5) yields:

$$\lambda = \left[-\frac{U(\Delta x)^2}{-2 + \tau(\Delta x)^2 + 2Cos\xi} \right]^{\frac{1}{q}}. \quad (4.7)$$

According to the Von Neumann’s Stability analysis; if $|\lambda| \leq 1$, finite difference method for the KdV equation is stable.

$$|\lambda| \leq 1 \Leftrightarrow |\Delta x| = \sqrt{\frac{-2 + 2Cos\xi}{U - \tau}}. \quad (4.8)$$

For the Eq. (4.8) the stability depends on the constant τ . However, due to the nature of the method of finite difference, stability will be examined with respect to parameter h . For this reason, if we choose $\xi = \frac{\pi}{2}$, $U = 2$ and $\tau = 1$ in the Eq. (4.8) and have $-1 \leq \Delta x \leq 1$, then the finite difference method for the KdV equation is stable. By using the Eq. (4.7), neutral stability curve can be drawn [26] for the example Eq. (4.8).



The neutral stability curve is locally a parabola with minimum $(0, 0)$. As shown in the graphs, if we choose the Δx close to zero, finite difference methods for the KdV equation is stable. In other words, the finite difference algorithm is stable if

x_i	t_j	<i>NumericalSolution</i>	<i>ExactSolution</i>	<i>AbsoluteError</i>
0.00	0.02	0.500301	0.499950	3.51144×10^{-4}
0.02	0.02	0.500452	0.500000	4.51842×10^{-4}
0.04	0.02	0.500502	0.499950	5.52118×10^{-4}
0.06	0.02	0.500452	0.499800	6.51893×10^{-4}
0.08	0.02	0.500301	0.499550	7.51088×10^{-4}
0.10	0.02	0.500050	0.499201	8.49627×10^{-4}
0.12	0.02	0.499699	0.498752	9.47433×10^{-4}

TABLE 1. *Numerical and exact solutions of Eq. (1.1) and absolute errors when $\Delta x = 0.02$ and $0 \leq x \leq 1$*

the round-off errors are small enough. The finite difference algorithm is said to be stable if the round-off errors are small enough for all i as $j \rightarrow \infty$ [25]. \square

5. NUMERICAL EXAMPLE

We consider the fractional KdV equation of the form Eq. (1.1) with the initial condition as follow:

$$u_0(x) = \frac{1}{2} \operatorname{Sech}^2\left(\frac{x}{2}\right), -1 \leq x \leq 1. \quad (5.1)$$

In the following numerical experiments we choose $\alpha = 0.8$. The fractional KdV Eq. (1.1) together with the above initial condition is constructed [22] such that the exact solution is

$$u(x, t) = \frac{1}{2} \operatorname{Sech}^2\left(\frac{x-t}{2}\right). \quad (5.2)$$

The numerical solutions are obtained from the finite difference schemes discussed above considering Eq. (2.9). The numerical solutions in the interval $0 \leq x \leq 1$: and the numerical solutions in the interval $-1 \leq x < 0$:

x_i	t_j	<i>NumericalSolution</i>	<i>ExactSolution</i>	<i>AbsoluteError</i>
-0.02	0.02	0.500050	0.499800	2.50105×10^{-4}
-0.04	0.02	0.499699	0.499550	1.48806×10^{-4}
-0.06	0.02	0.499248	0.499201	4.73301×10^{-4}
-0.08	0.02	0.498698	0.498752	5.42414×10^{-4}
-0.10	0.02	0.498048	0.498204	1.55825×10^{-4}
-0.12	0.02	0.497301	0.497558	2.57338×10^{-4}

TABLE 2. *Numerical and exact solutions of Eq. (1.1) and absolute errors when $\Delta x = 0.02$ and $-1 \leq x < 0$*

We know that truncation error will be small if Δx and Δt choose sufficiently small. There are appointed values close to zero indicate that the truncation error

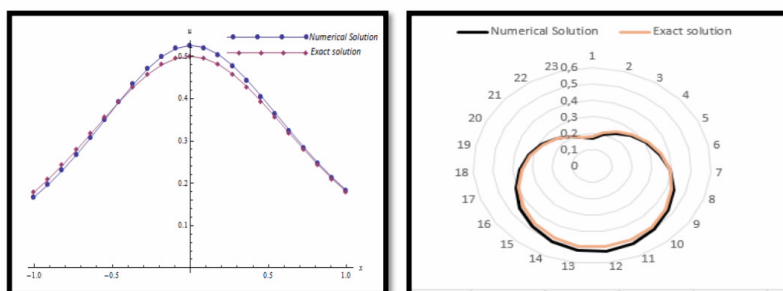


FIGURE 1. Numerical solution of Eq. (1.1) for finite difference and an expansion method

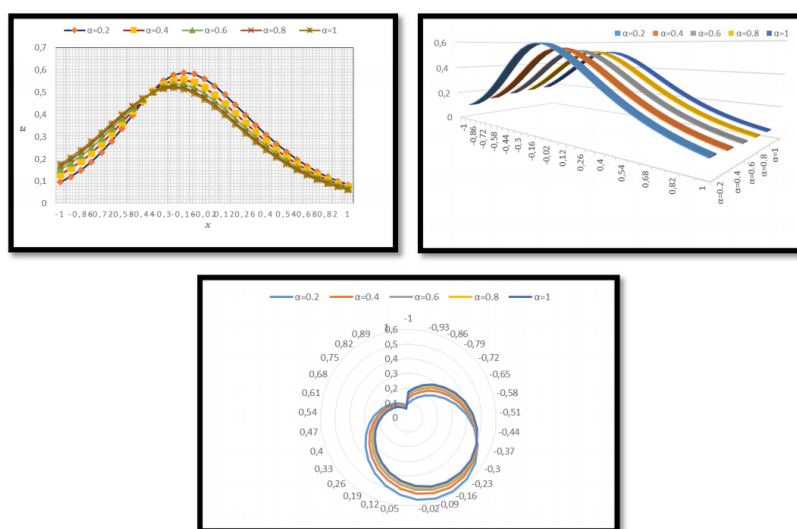


FIGURE 2. Exact solution of Eq. (1.1) for an expansion method and $0 < \alpha \leq 1$

becomes very small. The behavior of the numerical results of both numerical and exact solutions can be seen in the following graph by using value of $\Delta x = 0.2$. Considering the Eq. (2.9) which is obtained by using finite difference method, as can be observed in the graph, in the interval $-1 \leq x < -0.37$ the potential u increases with increasing the values of α . Nevertheless, the potential value u decreases with increasing the values of α at $-0.37 < x \leq 1$. We can see this situation in the figures as follow. The numerical solutions by using Eq. (2.9) and

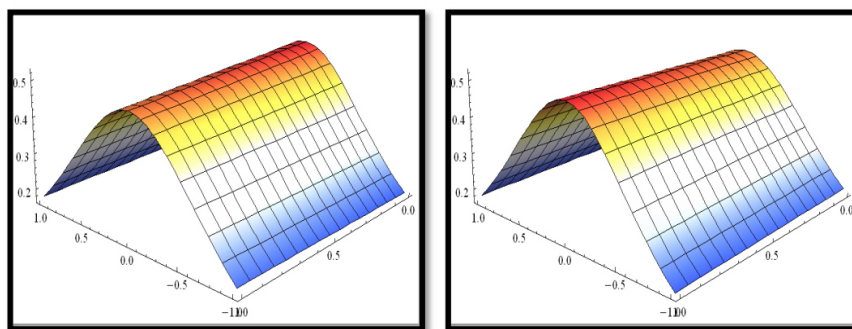


FIGURE 3. Exact solution (left) and Numerical solution (right) of Eq. (1.1)

the exact solution by using Eq. (5.2) are depicted in Fig. ?? We demonstrate how numerical solutions of the KdV equation are close to corresponding exact solution.

6. CONCLUSIONS

In this study, we considered the numerical solution of fractional dispersion equation by using Finite Difference Method. The method can be applied to many other nonlinear equations. What is more, this method is also computerizable, which allows us to perform complicate and tedious algebraic calculation on a computer. Fractional finite difference methods are useful to solve the fractional differential equations. In some way, these numerical methods have similar form with the classical equations. Some of them can be seen as the generalizations of the finite difference methods for the typical differential equations. The numerical method for solving the fractional reaction-dispersion equation has been described and demonstrated. Finally, we point out that, for given equation with initial values, the corresponding analytical and numerical solutions are obtained according to the recurrence Eq. (2.9) using Mathematica software package.

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