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THE SCHIFFER'S THEOREM RE-VISITED

FARUK UÇAR AND YUSUF AVCI

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Abstract. In this paper, we consider Schiffer's differential equation for the functions in the class of normalized analytic and univalent functions which maximize the second and the third coefficients.

1. Introduction

The Bieberbach conjecture [1] stated that the Maclaurin coefficients a_n of an analytic and univalent function must satisfy $|a_n| \leq n$ for all $n = 1, 2, 3, ...$ Bieberbach [1] in 1916 proved that $|a_2| \leq 2$ for every $f \in S$. In 1923 Löwner [5] introduced the parametric method, so-called the Loewner differential equation, in the geometric function theory to solve the famous Bieberbach problem about obtaining sharp estimates of Maclaurin coefficients of normalized univalent functions in the unit disk. Loewner himself used his techniques for slit mappings to prove the conjecture for the third coefficient that is $|a_3| \leq 3$ for every f in S. Later, Schiffer [7] and many other mathematicians developed some variational methods for analytic and univalent functions. The proof of the Bieberbach conjecture was given in 1985 by Louis de Branges [2]. The Löwner differential equation has led to some inequalities for univalent functions that played an important role in the solution of the Bieberbach conjecture by de Branges.

It is well-known that, the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots + nz^n + \cdots$$

which maps $\mathbb{D}$ onto $\mathbb{C}- \left(-\infty, -\frac{1}{4}\right]$, and its rotations

$$k_\theta(z) = \frac{z}{(1-e^{i\theta}z)^2}$$

provide the solution to many extremal problems for the class S.

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As a usual notation S denotes the class of functions $f(z) = z + a_2 z^2 + \cdots$, analytic and univalent in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In 1909, Koebe [4] proved that S is compact in the topology of uniform convergence on compact subsets of $\mathbb{D}$, and therefore variational problems of the form $\operatorname{Re}\phi(f) = \max$ must have solution in S if ϕ is a continuous functional S. Over seven decades ago, Schiffer [7] showed that an extremal function f for $\phi(f) = \operatorname{Re} a_n$ must satisfy the differential equation

$$\left(\frac{zf'(z)}{f(z)}\right)^2 A\left(f(z)\right) = B(z) \qquad (z \in \mathbb{D}) \tag{1.1}$$

where A and B are rational functions and $B(e^{i\theta}) \geq 0$.

Let $\zeta \neq 0$. We denote by $S(\zeta)$, the class of the functions which are satisfy $f(0) = 0$, $f'(0) = 1$ and $f(\zeta) = \infty$, univalent and meromorphic in the unit disc $\mathbb{D}$. Given a function $f \in S$ we define a function

$$F(z,\zeta) = \frac{f(z)f(\zeta)}{f(\zeta) - f(z)} \tag{1.2}$$

for $z, \zeta \in \mathbb{D}$ and $\zeta \neq 0$. For a constant $\zeta \in \mathbb{D}$, the function $F \in S(\zeta)$ and has Taylor expansion

$$F(z,\zeta) = z + A_2(\zeta)z^2 + \cdots + A_n(\zeta)z^n + \cdots \tag{1.3}$$

for all $|z| < |\zeta|$. On the other hand, for a function $f \in S$, about the $z = 0$ we have

$$F(z,\zeta) = f(z) + [f(z)]^2/f(\zeta) + \cdots . \tag{1.4}$$

Comparing (1.3) and (1.4), we find that

$$A_2(z) = a_2 + \frac{1}{f(z)} \tag{1.5}$$

which shall be used in the sequel.

2. Applications of The Schiffer's Theorem

Theorem 2.1. *If $f \in S$, then $|a_2| \leq 2$, with equality if and only if f is a rotation of the Koebe function.*

Proof: Let $f \in S$ be an extremal function for the problem $\max\limits_{f \in S} \operatorname{Re} a_n$, and Γ be its omitted set. By using the theory of Schiffer's boundary variation

$$f^*(z) = f(z) + \frac{\varepsilon\,[f(z)]^2}{w^2\,[f(z) - w]} + O(\varepsilon^2) \tag{2.1}$$

be a variation of f with respect to a point $w \in \partial\Gamma$. Setting $w = f(\zeta)$ in (1.2) and simple calculation gives

$$a_2^* = a_2 + \frac{\varepsilon}{[f(\zeta)]^2}\{A_2(\zeta) - a_2\} + O(\varepsilon^2) \tag{2.2}$$

Since f is extremal, we conclude that

$$\operatorname{Re}\left\{\frac{\varepsilon}{[f(\zeta)]^2}\{A_2(\zeta) - a_2\} + O(\varepsilon^2)\right\} \geq 0. \tag{2.3}$$

Thus by Schiffer's theorem, we have

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \{A_2(z) - a_2\} > 0 \tag{2.4}$$

on $|z| = 1$. Substituting (1.5) in (2.4), we find that

$$\frac{[zf'(z)]^2}{[f(z)]^3} > 0 \tag{2.5}$$

on $|z| = 1$. If we write

$$\frac{zf'(z)}{[f(z)]^{3/2}} = T(\theta) \tag{2.6}$$

where θ is real, then it follows from (2.6) that

$$-\frac{2}{\sqrt{f}} = i\int T(\theta)d\theta$$

and $\frac{1}{f}$ is real and negative on the circle $|z| = 1$. The function $G(z) = \dfrac{1}{f(z)}$ is analytic in $|z| < 1$ apart from a simple pole at $z = 0$. Therefore it can be continued analytically by the Schwarz reflection principle;

$$G(\frac{1}{\bar{z}}) = \overline{G(z)}. \tag{2.7}$$

For all $f \in S$, the function $A_2(z)$ is a meromorphic function and has a simple pole at $z = 0$. Then

$$a_2 + \frac{1}{f(z)} = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots \tag{2.8}$$

or

$$G(z) = \frac{1}{z} - a_2 + b_1 z + b_2 z^2 + \cdots \tag{2.9}$$

Using (2.7) we deduce that a_2 is real and $b_1 = 1$, $b_n = 0$ for all $n \geq 2$. Thus we find that

$$\frac{1}{f(z)} = \frac{1}{z} - a_2 + z$$

or equivalently

$$f(z) = \frac{z}{1 - a_2 z + z^2}. \tag{2.10}$$

If we write

$$f(z) = \frac{z}{(1 - e^{i\alpha}z)(1 - e^{-i\alpha}z)}$$

then we have

$$f(z) = \frac{z}{1 - 2\cos\alpha z + z^2} = z + 2\cos\alpha \cdot z^2 + \cdots$$

and

$$|a_2| = |2\cos\alpha| \leq 2.$$

Therefore a_2 attains its maximum value at $\alpha = 0$ and we deduce that (2.10) is equivalent to well-known Koebe function

$$f(z) = \frac{z}{(1-z)^2}.$$

Now we pose the extremal problem finding $\max_{f \in S} \operatorname{Re} a_3$.

Theorem 2.2. *If $f \in S$, then $|a_3| \leq 3$, with equality if and only if f is a rotation of the Koebe function.*

Proof: Let $f \in S$ be an extremal function for the problem $\max_{f\in S} \operatorname{Re} a_3$, and Γ be its omitted set. Then

$$f^*(z) = f(z) + \frac{\varepsilon\,[f(z)]^2}{w^2\,[f(z)-w]} + O(\varepsilon^2) \tag{2.11}$$

be a variation of f with respect to a point $w \in \Gamma$. Setting $w = f(\zeta)$ in (1.2) and simple calculation gives

$$a_3^* = a_3 + \frac{\varepsilon}{[f(\zeta)]^2}\,\{a_3 - A_3(z)\} + O(\varepsilon^2) \tag{2.12}$$

Since f is extremal, we conclude that

$$\operatorname{Re}\left\{\frac{\varepsilon}{[f(\zeta)]^2}\,\{A_3(z)-a_3\} + O(\varepsilon^2)\right\} \geq 0. \tag{2.13}$$

Thus by Schiffer's theorem, we have

$$\left[\frac{zf'(z)}{f(z)}\right]^2 \{A_3(z)-a_3\} > 0 \tag{2.14}$$

on $|z| = 1$. Comparing (1.3) and (1.4), we find that

$$A_3(z) - a_3 = \frac{2a_2}{f(\zeta)} + \frac{1}{[f(\zeta)]^2} \tag{2.15}$$

Substituting (2.15) in (2.14), we find that

$$\frac{[zf'(z)]^2}{[f(z)]^4}\,[2a_2 f(z) + 1] > 0 \tag{2.16}$$

on $|z| = 1$. If we write

$$\frac{zf'(z)}{[f(z)]^2}\sqrt{2a_2 f(z)+1} = T(\theta) \tag{2.17}$$

we conclude from (2.17) that the function

$$H(z) = -\frac{\sqrt{2a_2 f(z)+1}}{f(z)} + a_2 \log\frac{\sqrt{2a_2 f(z)+1}-1}{\sqrt{2a_2 f(z)+1}+1} \tag{2.18}$$

is imaginary on the circle $|z| = 1$. H is analytic in $|z| < 1$ apart from simple poles at 0 and ∞. Thus H is a function of the form

$$H(z) = \frac{c_{-1}}{z} + c_0 + c_1 z + a_2 \log z. \tag{2.19}$$

for some constants c_{-1}, c_0, c_1 and a_2. On the other hand, since H is imaginary on the circle, it satisfies the following relation

$$\overline{H(z)} = -H(\frac{1}{\bar{z}}) \tag{2.20}$$

on $|z| = 1$. Therefore, we have

$$c_{-1} = -\bar{c}_1, \bar{c}_0 = -c_0, c_1 = -\bar{c}_{-1} \text{ and } a_2 = \overline{a_2}$$

by (2.20). Multiplying both side of (2.18) by z and taking limit as $z \to 0$, we find that $c_{-1} = 1$ and $c_1 = -1$. A simple calculation shows that $c_0 = a_2 \log \frac{a_2}{2}$. Hence, this is only possible for $a_2 = 2$. It follows that the maximal function is the Koebe function.

3. Conclusions

The authors beleive that the Schiffer's differential equation can be used to prove that the Koebe function is also the maximal function for the fourth coefficient. This we will try to do next.

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Department of Mathematics, Marmara University TR-34722, Istanbul-TURKEY
E-mail address: fucar@marmara.edu.tr

Bahçeşehir University, Department of Mathematic Engineering,TR-34353 Beşiktaş, Istanbul, Turkey
E-mail address: yusuf.avci@eng.bahcesehir.edu.tr