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ON C-TOTALLY REAL SUBMANIFOLDS OF SASAKIAN SPACE FORMS

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Abstract. Let $\tilde{M}^{2n+1}(c)$ be $(2n+1)$-dimensional Sasakian space form of constant φ-sectional curvature c and M^n be an n-dimensional C-totally real minimal submanifold of $\tilde{M}^{2n+1}(c)$. If M^n is semi-parallel and the sectional curvature of M^n is greater than $\frac{(n-2)(c+3)}{4(2n-1)}$, then M^n is totally geodesic. Then we prove that a C-totally real minimal surface of a 5-dimensional Sasakian manifold $\tilde{M}(c)$ with constant φ-sectional curvature c, if M is semi-parallel surface, then M is totally geodesic or flat.

1. Introduction

In 1976, S. Yamaguchi, M. Kon and T. Ikawa [17] introduced the notion of a C-totally real submanifold of a Sasakian manifold and proved the following:

Theorem 1.1. *Let $M^{2n+1}(c)$ be a $(2n+1)$-dimensional Sasakian manifold with constant φ-sectional curvature c and M^n be an n-dimensional compact C-totally real minimal submanifold of $\tilde{M}^{2n+1}(c)$. If $\|\sigma\|^2 < \frac{n(n+1)(c+3)}{4(2n-1)}$ or, equivalently $\kappa > \frac{n^2(n-2)(c+3)}{2(2n-1)}$, then M is totally geodesic, where κ is the scalar curvature.*

Then S. Yamaguchi, M. Kon and Y. Miyahara [16] studied a C-totally real minimal surface of a 5-dimensional Sasakian manifold $\tilde{M}(c)$ with constant φ-sectional curvature c. They showed that if M is a complete nonnegative curved surface, then M is totally geodesic or flat. Then, A. Yildiz et al. [18] studied C-totally real pseudo-parallel submanifolds in Sasakian space forms.

Motivated by these results, in this paper we get the followings:

Theorem 1.2. *Let $\tilde{M}^{2n+1}(c)$ be a $(2n+1)$-dimensional Sasakian space form of constant φ-sectional curvature c and M^n be an n-dimensional C-totally real minimal submanifold of $\tilde{M}^{2n+1}(c)$. If M^n is semi-parallel and the sectional curvature of M^n is greater than $\frac{(n-2)(c+3)}{4(2n-1)}$, then M^n is totally geodesic.*

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Theorem 1.3. *Let M be a C-totally real minimal surface of a 5-dimensional Sasakian manifold $\tilde{M}(c)$ with constant φ-sectional curvature c. If M is semi-parallel surface, then M is totally geodesic or flat.*

2. **Preliminaries**

Let $f : M^n \longrightarrow \tilde{M}^{2n+1}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold M into $(2n+1)$-dimensional space form $\tilde{M}^{2n+1}(c)$. We denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}^{2n+1}(c)$ respectively, and $N(M)$ its normal bundle. Then for vector fields X, Y which are tangent to M, the second fundamental form σ is given by the formula $\sigma(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$. Furthermore, for $\xi \in N(M)$, $A_\xi : TM \longrightarrow TM$ will denote the Weingarten operator in the direction ξ, $A_\xi X = \nabla^\perp_X \xi - \tilde{\nabla}_X \xi$, where $\nabla^\perp$ denotes the normal connection of M. The second fundamental form σ and A_ξ are related by $\tilde{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$, where g is the induced metric of $\tilde{g}$ for any vector fields X, Y tangent to M. The mean curvature vector H of M is defined to be

$$H = \frac{1}{n} Tr(\sigma).$$

A submanifold M is said to be a *minimal* submanifold in $\tilde{M}$ if its mean curvature vector H is identically zero. Moreover, M is called a *totally geodesic* submanifold in $\tilde{M}$ if its second fundamental form σ is identically zero. The covariant derivative $\overline{\nabla}\sigma$ of σ is defined by

$$(\overline{\nabla}_X \sigma)(Y, Z) = \nabla^\perp_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.1}$$

where, $\overline{\nabla}\sigma$ is a normal bundle valued tensor of type $(0, 3)$ and is called the third fundamental form of M. The equation of Codazzi implies that $\overline{\nabla}\sigma$ is symmetric hence

$$(\overline{\nabla}_X \sigma)(Y, Z) = (\overline{\nabla}_Y \sigma)(X, Z) = (\overline{\nabla}_Z \sigma)(X, Y). \tag{2.2}$$

Here, $\overline{\nabla}$ is called the van der Waerden-Bortolotti connection $\overline{\nabla} = \nabla \oplus \nabla^\perp$, where ∇ is the Levi-Civita connection and $\nabla^\perp$ is the normal connection of M^n. Given an isometric immersion $f : M \longrightarrow \tilde{M}$, if $\overline{\nabla}\sigma = 0$, then f is called *parallel* [10]. Then J. Deprez ([7], [8]) defined the immersion to be *semi-parallel* if

$$\bar{R}(X, Y) \cdot \sigma = 0.$$

The basic equations of *Gauss* and *Ricci* are defined by

$$\begin{aligned} g(R(X, Y)Z, W) &= \frac{c+3}{4}(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)) \\ &\quad + \sum_\alpha (g(A_\alpha X, W)(g(A_\alpha Y, Z) - g(A_\alpha X, Z)g(A_\alpha Y, W)), \end{aligned} \tag{2.3}$$

$$g(R^\perp(X, Y)\xi, \eta) = g([A_\xi, A_\eta]X, Y); \quad \xi, \eta \in N(M), \tag{2.4}$$

respectively. Where $R^\perp$ is the curvature operator of the normal connection defined by

$$R^\perp(X, Y)Z = \nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z.$$

An isometric immersion f (or the submanifold M) is said to have *flat normal connection* (or trivial normal connection) if $R^\perp = 0$. If M has flat normal connection then shortly we call it to be normally flat. The relation (2.4) shows that the triviality of the normal connection of M into space form $\mathbb{N}^{n+d}(c)$ (and more generally, for submanifolds in a locally conformally flat space) is equivalent to the fact that all second fundamental tensors are mutually commute, or that all second fundamental tensors are mutually diagonalizable [5].

The *sectional curvature* $K(X,Y)$ of M determined by an orthonormal pair X, Y is given by

$$K(X,Y) = \frac{c+3}{4} + \sum_{\alpha}(g(A_\alpha X, X)g(A_\alpha Y, Y) - g(A_\alpha X, Y)^2).$$

The second covariant derivative $\overline{\nabla}^2\sigma$ of σ is defined by

$$\begin{aligned} (\overline{\nabla}^2\sigma)(Z,W,X,Y) &= (\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W) \\ &= \nabla^{\perp}_X((\overline{\nabla}_Y\sigma)(Z,W)) - (\overline{\nabla}_Y\sigma)(\nabla_X Z, W) \\ &\quad -(\overline{\nabla}_X\sigma)(Z,\nabla_Y W) - (\overline{\nabla}_{\nabla_X Y}\sigma)(Z,W). \end{aligned} \tag{2.5}$$

Then we have

$$\begin{aligned} (\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W) - (\overline{\nabla}_Y\overline{\nabla}_X\sigma)(Z,W) &= (\bar{R}(X,Y)\cdot\sigma)(Z,W) \\ &= R^{\perp}(X,Y)\sigma(Z,W) - \sigma(R(X,Y)Z,W) \\ &\quad -\sigma(Z,R(X,Y)W). \end{aligned} \tag{2.6}$$

where $\bar{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$.

3. C-totally real submanifolds of Sasakian space forms

Let $\tilde{M}$ be a $(2n+1)$-dimensional manifold and $\Gamma(\tilde{M})$ be the Lie algebra of vector fields on $\tilde{M}$. An *almost contact structure* on $\tilde{M}$ is defined by a $(1,1)$-tensor φ, a vector field ξ and a 1-form η on $\tilde{M}$ satisfy

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

where $X \in \Gamma(\tilde{M})$.Manifolds equipped with an almost contact structure are called *almost contact manifolds*. A Riemannian manifold $\tilde{M}$ with metric tensor g and almost contact structure (φ, ξ, η) such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

or equivalently

$$g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad \eta(X) = g(X, \xi),$$

for all $X, Y \in \Gamma(\tilde{M})$, is almost contact metric manifold. The existence of an almost contact metric structure on $\tilde{M}$ is equivalent with the existence of a reduction of the structurel group to $U(n) \times 1$, i.e., all the matrices of $O(2n+1)$ of the form

$$\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where A and B are real $(n \times n)$-matrices. The fundamental 2-form Ψ of an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$ is defined by

$$\Psi(X,Y) = g(\varphi X, Y),$$

for all $X, Y \in \Gamma(\tilde{M})$, and this form satisfies $\eta \wedge \Psi^n \neq 0$. When $\Psi = d\eta$ the associated structure is a contact structure and $\tilde{M}$ is an *almost Sasakian manifold.* We denote by $\tilde{\nabla}$ the Levi-Civita connection on $\tilde{M}$. Then we have [13]

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad \tilde{\nabla}_X \xi = -\varphi X,$$

for any vector fields X, Y tangent to $\tilde{M}$.

If moreover the structure is normal, that is, if $[\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] = -2d\eta(X, Y)\xi$, then the contact metric structure is called a *Sasakian structure* (normal contact metric structure) and $\tilde{M}^{2n+1}$ is called a *Sasakian manifold.* For more details and background, see the standard references ([2], [15].

A plane section in the tangent space $T_X\tilde{M}$ at $x \in \tilde{M}$ is called a *φ-section* if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature $K(X, \varphi X)$ with respect to a φ-section determined by a vector X is called a *φ-sectional curvature.* A Sasakian manifold with constant φ-sectional curvature c is a *Sasakian space form* and is denoted by $\tilde{M}(c)$. The curvature tensor of a Sasakian space form $\tilde{M}(c)$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z &= \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} \\ &\quad -\frac{1}{4}(c-1)\left\{\begin{array}{c}\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi \\ -g(X, Z)\eta(Y)\xi - g(\varphi Y, Z)\varphi X \\ +g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z\end{array}\right\}.\end{aligned}$$

Example 3.1. [2] Let $\mathbb{R}^{2n+1}$ be a Euclidean space with cartesian coordinates (x^i, y^i, z). Then a Sasakian structure on $\mathbb{R}^{2n+1}$ is defined by (φ, ξ, η, g) such that

$$\xi = 2\frac{\partial}{\partial z}, \qquad \eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), g = \frac{1}{4}[\eta \otimes \eta + \sum_{i=1}^{m}((dx^i)^2 + (dy^i)^2)],$$

and the tensor field φ is given by matrix

$$\begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y^j & 0 \end{pmatrix}.$$

With such a structure, $\mathbb{R}^{2n+1}$ is of constant φ-sectional curvature -3 and denoted by $\mathbb{R}^{2n+1}(-3)$.

Example 3.2. [1] For $\theta \in (0, \pi/2)$, the immersion

$$F(u, v, w, s, t) = 2(u, 0, w, 0, v\cos\theta, v\sin\theta, s\cos\theta, s\sin\theta, t),$$

defines a 5-dimensional submanifold M in $\mathbb{R}^9(-3)$. We consider on M the induced almost contact structure (φ, ξ, η, g), where $\varphi = (\sec\theta)T$, T being the tangential component of φ. It can be checked that

$$(\nabla_X \varphi)Y = \cos\theta(g(X, Y)\xi - \eta(Y)X),$$

for any vector fields X, Y tangent to M.

We remark that the immersion F in the Example 2 defines a 5-dimensional minimal submanifold M in a Sasakian space form $\mathbb{R}^9(-3)$.

A submanifold M of a Sasakian manifold $\tilde{M}$ is called a *C-totally real submanifold* if and only if $\varphi(T_xM) \subset T_x^\perp M$ for all $x \in M$ (T_xM and $T_x^\perp M$ are respectively the tangent space and normal space of M at x).

When ξ is tangent to M (e.g., when $m = n-1$), M is a C-totally real submanifold if and only if $\nabla_X \xi = 0$ for all $X \in TM$, where ∇ is the connection on M induced from the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$. For a C-totally real submanifold M^{m+1} in $\tilde{M}^{2(m+p)+1}$ with $m \geq 1$ it is impossible for M to be totally umbilical [11].

When ξ is normal to M then the submanifold M is automatically anti-invariant in $\tilde{M}$ with $m \leq n$ and also $\eta(X) = g(X, \xi) = 0$ for $X \in TM$ [14]. On the other hand, $\eta = 0$ defines a n-dimensional distribution on $\tilde{M}$, the so called contact distribution D. This distribution admits integral submanifolds up to (and including) dimension n. Moreover, it is proved that a manifold M immersed in $\tilde{M}$ is an integral submanifold of D if and only if $T_x M \subset D_x$ for $x \in M$ and $\varphi(T_x M) \subset T_x^{\perp} M$ [3]. The integral submanifols of the contact distribution of a Sasakian manifold are called C-totally real submanifolds. It is easy to see that the C-totally real submanifolds M of $\tilde{M}$ are the submanifolds with $\xi \in T^{\perp} M$.

Then we have known $\dim M \leq n$ and the following theorem has been proved [12]:

Theorem 3.1. *Let M be an m $(m \leq n)$ dimensional C-totally real submanifold in a Sasakian manifold $\tilde{M}^{2n+1}$ with structure tensors (φ, ξ, η, g). Then we have the following:*

(i): *The second fundamental form of ξ direction is identically zero.*
(ii): *If $X \in \chi(M)$, then $\varphi X \in \chi^{\perp}(M)$.*
(iii): *If $m = n$, then $A_{\varphi X}(Y) = A_{\varphi Y}(X)$, $X, Y \in \chi(M)$.*

Also, we need the followings:

Lemma 3.1. [15] *Let M be an n-dimensional C-totally real submanifold of a $(2n+1)$-dimensional Sasakian manifold $\tilde{M}$. If the second fundamental form of M is parallel, then M is totally geodesic in $\tilde{M}$.*

Lemma 3.2. [2] *If the sectional curvature of M^n is greater than δ, then*

$$\begin{aligned} \frac{1}{2}\Delta(\|\sigma\|^2) \quad \geq \quad & \left\|\overline{\nabla}\sigma\right\|^2 + (1+a)n\delta \|\sigma\|^2 - \frac{na(c+3)-(c-1)}{4} \|\sigma\|^2 \\ & + \frac{1-a}{2} \sum_{\alpha,\beta} tr(A_\alpha A_\beta - A_\beta A_\alpha)^2 + a \sum_{\alpha,\beta} tr(A_\alpha A_\beta)^2 \end{aligned} \tag{3.1}$$

for any real number $a \geq -1$.

Lemma 3.3. [2] $\left\|\overline{\nabla}\sigma\right\|^2 \geq \|\sigma\|^2$.

Proposition 3.1. [9] *If M is an n-dimensional C-totally real submanifold of a Sasakian space form $\tilde{M}(c)$, then the following conditions are equivalent: (i) M is minimal; (ii) the mean curvature vector H of M is parallel .*

4. **Proofs of the Theorems**

Let M be an n-dimensional C-totally real submanifold of $(2n+1)$-dimensional Sasakian space form $\tilde{M}^{2n+1}(c)$ of constant φ-sectional curvature c. We choose an orthonormal bases $\{e_1, e_2, ..., e_n, \varphi e_1 = e_{1^*}, ..., \varphi e_n = e_{n^*}, e_{(n+1)^*} = \xi\}$. Then for $1 \leq i, j \leq n$, $n+1 \leq \alpha \leq 2n+1$, the components of the second fundamental form σ are given by

$$\sigma_{ij}^{\alpha} = g(\sigma(e_i, e_j), e_\alpha). \tag{4.1}$$

Similarly, the components of the first and the second covariant derivative of σ are given by

$$\sigma^{\alpha}_{ijk} = g((\overline{\nabla}_{e_k}\sigma)(e_i, e_j), e_\alpha) = \overline{\nabla}^{\alpha}_{e_k}\sigma_{ij}, \tag{4.2}$$

and

$$\begin{aligned} \sigma^{\alpha}_{ijkl} &= g((\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma)(e_i, e_j), e_\alpha) \\ &= \overline{\nabla}^{\alpha}_{e_l}\sigma_{ijk} \\ &= \overline{\nabla}_{e_l}\overline{\nabla}^{\alpha}_{e_k}\sigma_{ij}, \end{aligned} \tag{4.3}$$

respectively. If f is semi-parallel, then by definition, the condition

$$\bar{R}(e_l, e_k)\cdot\sigma = 0. \tag{4.4}$$

By (2.6), we have

$$(\bar{R}(e_l, e_k)\cdot\sigma)(e_i, e_j) = (\overline{\nabla}_{e_l}\overline{\nabla}_{e_k}\sigma)(e_i, e_j) - (\overline{\nabla}_{e_k}\overline{\nabla}_{e_l}\sigma)(e_i, e_j). \tag{4.5}$$

Making use of (4.1), (4.3), (4.5), the semi-parallelity condition (4.4) turns into

$$\sigma^{\alpha}_{ijkl} - \sigma^{\alpha}_{ijlk} = 0. \tag{4.6}$$

Recall that the Laplacian $\Delta\sigma^{\alpha}_{ij}$ of σ^{α}_{ij} is defined by

$$\Delta\sigma^{\alpha}_{ij} = \sum_{i,j,k=1}^{n}\sigma^{\alpha}_{ijkk}. \tag{4.7}$$

Then we obtain

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p}\sigma^{\alpha}_{ij}\sigma^{\alpha}_{ijkk} + \left\|\overline{\nabla}\sigma\right\|^2, \tag{4.8}$$

where

$$\|\sigma\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p}(\sigma^{\alpha}_{ij})^2, \tag{4.9}$$

and

$$\left\|\overline{\nabla}\sigma\right\|^2 = \sum_{i,j,k=1}^{n}\sum_{\alpha=1}^{p}(\sigma^{\alpha}_{ijkk})^2, \tag{4.10}$$

are the square of the length of the second and the third fundamental forms of M^n, respectively. In addition, making use of (4.1) and (4.3), we obtain

$$\begin{aligned} \sigma^{\alpha}_{ij}\sigma^{\alpha}_{ijkk} &= g(\sigma(e_i, e_j), e_\alpha)g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i, e_j), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i, e_j)g(\sigma(e_i, e_j), e_\alpha), e_\alpha) \\ &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i, e_j), \sigma(e_i, e_j)). \end{aligned} \tag{4.11}$$

Due to (4.11), the equation (4.8) becomes

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i, e_j), \sigma(e_i, e_j)) + \left\|\overline{\nabla}\sigma\right\|^2. \tag{4.12}$$

Further, by the use of (4.4) and (4.5), we get

$$\begin{aligned} g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_k}\sigma)(e_i, e_j), \sigma(e_i, e_j) &= g((\overline{\nabla}_{e_k}\overline{\nabla}_{e_i}\sigma)(e_k, e_j), \sigma(e_i, e_j)) \\ &= g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_k}\sigma)(e_j, e_k), \sigma(e_i, e_j)) \end{aligned} \tag{4.13}$$

Substituting (4.13) into (4.12), we have

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j,k=1}^{n} [g((\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}\sigma)(e_k,e_k),\sigma(e_i,e_j)) + \left\|\overline{\nabla}\sigma\right\|^2 \tag{4.14}$$

Furthermore, by the definition

$$\|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i,e_j),\sigma(e_i,e_j)), \tag{4.15}$$

$$H^\alpha = \sum_{k=1}^{n} \sigma_{kk}^\alpha, \tag{4.16}$$

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=1}^{p}(H^\alpha)^2. \tag{4.17}$$

After some calculations, we get

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \sum_{i,j=1}^{n}\sum_{\alpha=1}^{p}\sigma_{ij}^\alpha(\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}H^\alpha) + \left\|\overline{\nabla}\sigma\right\|^2. \tag{4.18}$$

Using minimallity condition, the equation (4.18) reduces to

$$\frac{1}{2}\Delta(\|\sigma\|^2) = \left\|\overline{\nabla}\sigma\right\|^2. \tag{4.19}$$

On the other hand Blair [2] shown that

$$\frac{1}{2}\Delta(\|\sigma\|^2) \geq \left\|\overline{\nabla}\sigma\right\|^2 + (1+a)n\delta\|\sigma\|^2 - \frac{(na-1)(c+3)}{4}\|\sigma\|^2 -(1-a)\|\sigma\|^4 + \sum_{i=1}^{n}(trA_i^2)^2, \tag{4.20}$$

for $-1 \leq a \leq 1$. Hence using (4.19) in (4.20), we have

$$0 \geqslant (1+a)n\delta\|\sigma\|^2 - \frac{(na-1)(c+3)}{4}\|\sigma\|^2 -(1-a)\|\sigma\|^4 + \sum_{i=1}^{n}(trA_i^2)^2$$

for $-1 \leq a \leq 1$. Moreover one can easily show that

$$\sum_{\alpha,\beta=1}^{n}(trA_\alpha A_\beta)^2 \geqslant \frac{1}{n}\|\sigma\|^4.$$

Thus we have

$$0 \geqslant (1+a)n\delta\|\sigma\|^2 - \frac{(na-1)(c+3)}{4}\|\sigma\|^2 +(\frac{1}{n}-(1-a))\|\sigma\|^4. \tag{4.21}$$

Setting $a = 1 - \frac{1}{n}$ in (4.21), we obtain

$$0 \geqslant [(2n-1)\delta - \frac{(n-2)(c+3)}{4}]\,\|\sigma\|^2 .$$

If $\delta > \frac{(n-2)(c+3)}{4(2n-1)}$ then $\|\sigma\|^2 = 0$, i.e. $\sigma = 0$. This completes the proof of Theorem 2.

Now we assume that $\tilde{M}(c)$ is a 5-dimensional Sasakian manifold with constant φ-sectional curvature c and M is a C-totally real minimal surface of $\tilde{M}(c)$. Now we take a frame e_1, e_2 for $T_p(M)$ and a frame $\varphi e_1, \varphi e_2, \xi$ for $T_p(M)^{\perp}$. Then the second fundamental form can be expressed as:

$$A_{\varphi e_1} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad A_{\varphi e_2} = \begin{pmatrix} 0 & -b \\ -b & 0 \end{pmatrix}, \quad A_{\xi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.22}$$

From (2.3) and (2.4), we obtain

$$\begin{aligned} R(e_1, e_2)e_1 &= (-\frac{c+3}{4} + 2b^2)e_2, \\ R(e_1, e_2)e_2 &= (\frac{c+3}{4} - 2b^2)e_1, \\ R^{\perp}(e_1, e_2)\varphi e_1 &= 2b^2\varphi e_2, \\ R^{\perp}(e_1, e_2)\varphi e_2 &= -2b^2\varphi e_1. \end{aligned} \tag{4.23}$$

Moreover, by the Gauss equation, we have

$$2b^2 = \frac{c+3}{4} - \gamma, \tag{4.24}$$

where γ denotes the Gauss curvature of M. If M is semi-parallel surface, then we obtain

$$\begin{aligned} (R(e_1, e_2)\cdot\sigma)(e_1, e_1) &= 6b^3 - \frac{b(c+3)}{2} = 0, \\ (R(e_1, e_2)\cdot\sigma)(e_2, e_2) &= -6b^3 + \frac{b(c+3)}{2} = 0, \\ (R(e_1, e_2)\cdot\sigma)(e_1, e_2) &= 6b^3 - \frac{b(c+3)}{2} = 0, \end{aligned} \tag{4.25}$$

which give that

$$(b^2 - \gamma)b = 0. \tag{4.26}$$

Now we have two cases:

Case i) If $\gamma = b^2 > 0$, then from [16], we can say that M is totally geodesic or,

Case ii) If $b = 0$, then we can say that M is flat.

This completes the proof of Theorem 1.3.

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