\textbf{$\gamma$\,$b$-sets and $\gamma$\,$b$-continuous functions}

Aynur Keskin Kaymakcı

(Communicated by Ishak ALTUN)

\begin{abstract}
First of all we introduce a new class of sets is called $\gamma$\,$b$-sets and a new type of continuous functions is called $\gamma$\,$b$-continuous functions. Then, we investigate some properties and characterizations. Besides, $\gamma$\,$b$-sets and $\gamma$\,$b$-continuity are used to extend known results for $b$-open sets and $b$-continuity.
\end{abstract}

\textit{Keywords:} $\gamma$\,$b$-sets; $b$-open sets; $\gamma$\,$b$-continuity; $b$-continuity.

\textit{AMS Subject Classification (2010):} Primary: 54C08 ; Secondary: 54A10; 54A20.

\section{1. Introduction and Preliminaries}

It is well known that the notion of open sets and it's some modifications ( semi-open sets, $\alpha$-open sets, preopen sets, $b$-open sets, etc. ) are important topological notions in topological spaces. Of course, they are usefull tolls in several papers. Really, they are used to introduced related continuous functions, spaces, convergence etc. and investigated their some properties. Let $(X, \tau), (Y, \varphi), (Z, \psi)$ be topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset of $X$, we will denote $Cl(A)$ and $Int(A)$ of the closure $A$ and the interior of $A$, respectively. A subset $A$ of $X$ is called a semi-open [5] (resp. $\alpha$-set [9], pre-open set [7], $b$-open set [1] or $\gamma$-open set [4]) if $A \subset Cl(Int(A))$ (resp. $A \subset Int(Cl(Int(A)))$, $A \subset Int(Cl(A))$). The complement of a semi-open set (resp. $\alpha$-open set) is called semi-closed set (resp. $\alpha$-closed set). The family of all semi-open sets ( resp. $\alpha$-sets, pre-open sets, $b$-open sets ) in $X$ will be denoted by $SO(X, \tau)$ ( resp. $\alpha (X, \tau), PO(X, \tau), BO(X, \tau)$ ). A function $f : (X, \tau) \rightarrow (Y, \varphi)$ is called semi-continuous [5] ( resp. $\alpha$-continuous [8], pre-continuous [7], $b$-continuous [3] ) if $f^{-1}(V) \in SO(X, \tau)$ ( resp. $f^{-1}(V) \in \alpha (X, \tau), f^{-1}(V) \in PO(X, \tau), f^{-1}(V) \in BO(X, \tau)$ ) for each open set $V$ of $(Y, \varphi)$. A function $f : (X, \tau) \rightarrow (Y, \varphi)$ is called irresolute [2] ( resp. $\alpha$-irresolute [6], pre-irresolute [10], b-irresolute[3] ) if $f^{-1}(V) \in SO(X, \tau)$ ( resp. $f^{-1}(V) \in \alpha (X, \tau), f^{-1}(V) \in PO(X, \tau), f^{-1}(V) \in BO(X, \tau)$ ) for each semi-open set ( resp. $\alpha$-open set, pre-open set, $b$-open set ) $V$ of $(Y, \varphi)$. A function $f : X \times \tau \rightarrow (Y, \varphi)$ is called semi-open [5] ( resp. $\alpha$-open [8] ) if for every semi-open ( resp. $\alpha$-open ) set $U$ in $(X, \tau)$, $f(U)$ is semi-open ( resp. $\alpha$-open ) set in $(Y, \varphi)$. The notion of filters is another important in topology as well as some branches of mathematics.

In this paper, in first step we define a new class of sets is called $\gamma$\,$b$-sets and a new type of continuous functions is called $\gamma$\,$b$-continuous functions by using the notion of b-convergence of filters. We state that this term is used to investigated some characterizations related to b-continuous functions. Because of the notions $\gamma$\,$b$-sets and $\gamma$\,$b$-continuity are used to extend some well-known results for $b$-open sets and $b$-continuity, new introduce the notions $\gamma$\,$b$-sets and $\gamma$\,$b$-continuity.

Recall that a subset $M(x)$ of space $(X, \tau)$ is called a $b$-neighbourhood of a point $x \in X$ if there exists a $b$-open set $S$ such that $x \in S \subset M(x)$.
2. γb-sets

First of all, we define the notions of \textit{b-neighbourhood filter} at \(x\) any point of topological space and \textit{b-convergence} of filters.

\textbf{Definition 2.1.} Let \((X, \tau)\) be a topological space, \(B(x) = \{A \in BO(X, \tau) \mid x \in A\}\) and let \(B_x = \{A \subset X \mid \exists \mu \subset B(x) \text{ such that } \mu \text{ is finite and } \cap \mu \subset A\}\). Then, \(B_x\) is called the \textit{b-neighbourhood filter at} \(x\).

\textbf{Definition 2.2.} Let \(F\) is any filter on any topological space \((X, \tau)\). If \(F\) is finer than the \(b\)-neighbourhood filter at \(x\), then \(F\) is called b-convergences to \(x\).

\textbf{Definition 2.3.} Let \((X, \tau)\) be a topological space and \(A\) be a subset of \((X, \tau)\). If a filter \(F\) \(b\)-converges to \(x\) such that \(x \in A\) and \(A \in F\), then \(A\) is called \(\gamma b\)-set in \((X, \tau)\).

We will denote by \(\gamma b(X, \tau)\) of the class of all \(\gamma b\)-set in \((X, \tau)\).

From the definitions of the notions \textit{b-neighbourhood filters} and \textit{\(\gamma b\)-sets}, we have the following proposition as a result.

\textbf{Proposition 2.1.} In any topological space, every \(b\)-open set is a \(\gamma b\)-set.

\textbf{Remark 2.1.} The converse of Proposition 2.1 is not true in generally as shown in the following example.

\textbf{Example 2.1.} Let \((\mathbb{R}, \tau)\) be usual space i.e. \(\mathbb{R}\) is real number set and \(\tau\) is usual topology. For each \(x \in X\), since both \((a, x)\) and \([x, b)\) are \(b\)-open sets containing \(x\) such that \(a < x < b\), then \(\{x\}\) is an element of \(B_x\). Consequently, for any filter \(F\) on \(\mathbb{R}\), if \(F\) b-converges to \(x\) then \(\{x\}\) is a \(\gamma b\)-set by using \(F\) contains \(B_x\). But, \(\{x\}\) is not \(b\)-open.

\textbf{Remark 2.2.} In a topological space \((X, \tau)\), \(\tau \subset \alpha (X, \tau) \subset BO(X, \tau) \subset \gamma b(X, \tau)\).

\textbf{Theorem 2.1.} Let \((X, \tau)\) be a topological space. The intersection of finitely many \(b\)-open subsets in \((X, \tau)\) is a \(\gamma b\)-set.

\textbf{Proof.} Let \(U\) and \(V\) be a \(b\)-open sets in \((X, \tau)\). For each \(x \in (U \cap V)\), we get \((U \cap V) \subset B_x\). Therefore, we have \((U \cap V) \in F\) for every filter \(F\) b-converges to \(x\) by using the notion of the converges of filters.

We define an operator by using the notion of \(\gamma b\)-sets.

\textbf{Definition 2.4.} Let \((X, \tau)\) be a topological space and \(A\) be a subset of \((X, \tau)\). The \(\gamma b\)-interior of \(A\) in \((X, \tau)\), denoted by \(\text{Int}_{\gamma b}(A)\), is the union of all \(\gamma b\)-sets contained in \(A\).

Now, we give a characterization of \(\gamma b\)-sets by using the notion of \(\gamma b\)-interior.

\textbf{Theorem 2.2.} Let \((X, \tau)\) be a topological space and \(A\) be a subset of \((X, \tau)\). Then, the following properties are hold:

1. \(\text{Int}_{\gamma b}(A) = \{x \in A \mid A \in B_x\}\),
2. \(A\) is \(\gamma b\)-set if and only if \(A = \text{Int}_{\gamma b}(A)\).

\textbf{Proof.} (1) For each \(x \in \text{Int}_{\gamma b}(A)\), there exists a \(\gamma b\)-set \(U\) such that \(x \in U\) and \(U \subset A\). By using the notion of \(\gamma b\)-set, the subset \(U\) is in the \(b\)-neighbourhood filter \(B_x\). Since \(B_x\) is a filter, \(A \in B_x\).

Conversely, let \(x \in A\) and \(A \in B_x\). Then, there exist \(U_1, U_2, ..., U_n \in B(x)\) such that \(U = U_1 \cap U_2 \cap ... \cap U_n \subset A\). By Theorem 2.1, \(U\) is a \(\gamma b\)-set and \(U \subset A\). This shows that \(x \in \text{Int}_{\gamma b}(A)\).

(2) The proof is obvious.

\textbf{Theorem 2.3.} For any topological space \((X, \tau)\), the class of \(\gamma b(X, \tau)\) of all \(\gamma b\)-subsets in \((X, \tau)\) is a topology on \(X\).

\textbf{Proof.} Since \(\emptyset\) and \(X\) are \(b\)-open, they are also \(\gamma b\)-sets. Let \(A, B \in \gamma b(X, \tau)\), \(x \in (A \cap B)\) and let \(F\) be a filter. Assume that the filter \(F\) b-converges to \(x\). Therefore, \(A \cap B\) is a \(\gamma b\)-set. For each \(\alpha \in I\), let \(A_\alpha \in \gamma b(X, \tau)\) and \(U = \cup A_\alpha\). For each \(x \in U\) and for a filter \(F\) b-converging to \(x\) there exists a subset \(A_\alpha\) of \(U\) such that \(x \in A_\alpha\), and since \(A_\alpha\) is \(\gamma b\)-set, it is obvious that \(A_\alpha \in F\). Since \(F\) is filter, \(U\) is an element of the filter \(F\) and thus \(U = \cup A_\alpha\) is \(\gamma b\)-set.

In a topological space \((X, \tau)\), the class of all \(\gamma b\)-sets induced by the topology \(\tau\) will be denoted by \((X, \gamma b_\tau)\).

A subset \(A\) of \((X, \tau)\) is called a \(\gamma b\)-closed if the complement of \(A\) is a \(\gamma b\)-set. We will denote by \(\gamma bC(X, \tau)\) of the class of all \(\gamma b\)-closed in \((X, \tau)\). Therefore, we state the following by using Theorem 2.3:

1. \(\gamma bC(X, \tau)\) is closed arbitrary intersection,
2. \(\gamma bC(X, \tau)\) is closed finite union.

Now, we give a characterization of notion of \(\gamma b\)-closed by using the notion of \(b\)-convergency.
Theorem 2.4. Let $(X, \tau)$ be a topological space and $A$ be a subset of $(X, \tau)$. $A$ is $\gamma_b$-closed if and only if whenever $F$ $b$-converges to $x$ and $A \in F$, $x \in A$.

Definition 2.5. Let $(X, \tau)$ be a topological space. For $A$ a subset of $(X, \tau)$, $Cl_{\gamma_b}(A) = \{x \in X \mid A \cap U \neq \emptyset, \forall U \in B_x\}$ is called the $\gamma_b$-closure of a subset $A$ in $(X, \tau)$.

Now, we give some properties of $\gamma_b$-closure operation in the following theorem.

Theorem 2.5. Let $(X, \tau)$ be a topological space and $A$ be a subset of $(X, \tau)$. Then, the following properties hold:
1. $A \subset Cl_{\gamma_b}(A)$;
2. $A$ is $\gamma_b$-closed if and only $A = Cl_{\gamma_b}(A)$;
3. $Int_{\gamma_b}(A) = X - Cl_{\gamma_b}(X - A)$;
4. $Cl_{\gamma_b}(A) = X - Int_{\gamma_b}(A)$.

3. $\gamma_b$-continuous and $\gamma_b$-irresolute functions

In this section, we introduce new two types of continuity are called $\gamma_b$-continuity and $\gamma_b$-irresoluteness by using $\gamma_b$-sets. Then, we have obtained some characterizations of these notions. Furthermore, we define the notion of $\gamma_b$-open functions as a new modification of open functions. We have also given two characterizations of this function.

Definition 3.1. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. A function $f : (X, \tau) \to (Y, \varphi)$ is called $\gamma_b$-continuous if the inverse image of each open set of $(Y, \varphi)$ is a $\gamma_b$-set in $(X, \tau)$.

Since the class of all $\gamma_b$-sets in a given topological space is also a topology, we get the following equivalent statements.

Theorem 3.1. Let $(X, \tau)$, $(Y, \varphi)$ be two topological spaces and $f : (X, \tau) \to (Y, \varphi)$ be a function. Then, the following statements are equivalent:
1. $f$ is $\gamma_b$-continuous;
2. The inverse image of each closed set in $(Y, \varphi)$ is $\gamma_b$-closed set in $(X, \tau)$;
3. $Cl_{\gamma_b}(f^{-1}(B)) \subset f^{-1}(Cl(B))$ for every $B \subset Y$;
4. $f(Cl_{\gamma_b}(A)) \subset Cl(f(A))$ for every $A \subset X$;
5. $f^{-1}(Int(B)) \subset Int_{\gamma_b}(f^{-1}(B))$ for every $B \subset Y$.

Theorem 3.2. Let $f : (X, \tau) \to (Y, \varphi)$ be a function between topological spaces $(X, \tau)$ and $(Y, \varphi)$. Then, the following statements are equivalent:
1. $f$ is $\gamma_b$-continuous at $x$;
2. If a filter $F$ $b$-converges to $x$, then $f(F)$ converges to $f(x)$;
3. For $x \in X$ and for each neighbourhood $V$ of $f(x)$, there exists a subset $V \subset B_x$ such that $f(U) \subset V$.

Proof. (1) $\implies$ (2). Let $V$ be any open neighbourhood of $f(x)$ in $(Y, \varphi)$. Since $f$ is $\gamma_b$-continuous at $x$, then $f^{-1}(V)$ is an element of $B_x$. Besides since $F$ is $b$-converges to $x$ and $f(F)$ is a filter, then we have $V \in f(F)$. Consequently, $f(F)$ converges to $f(x)$.

(2) $\implies$ (3). Let $V$ be any $\gamma_b$-neighbourhood of $f(x)$. By hypothesis, we have that $B_x$ $b$-converges to $x$. Hence, we have $B_{f(x)} \subset f(B_x)$ and then $V \in f(B_x)$. Therefore, there is subset $U \subset B_x$ such that $f(U) \subset V$.

(3) $\implies$ (1). This is obvious.

Now, we can easily show the following result.

Corollary 3.1. Let $f : (X, \tau) \to (Y, \varphi)$ be a function. If $f$ is $b$-continuous at $x \in X$, then whenever a filter $F$ $b$-converges to $x$ in $(X, \tau)$, $f(F)$ converges to $f(x)$ in $(Y, \varphi)$.

Remark 3.1. The converse of Corollary 3.1 is not true in generally as shown by the following example.

Example 3.1. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces such that $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$, $Y = \{a, b\}$ and $\varphi = \{Y, \emptyset, \{a\}\}$. A function $f : (X, \tau) \to (Y, \varphi)$ is defined $f(a) = f(c) = f(d) = b$ and $f(b) = a$. For $a \in Y$ and $\{a\} \in \varphi$, $f^{-1}(\{a\}) = \{b\}$. But, $\{b\}$ is not $b$-open in $(X, \tau)$. So, $f$ is not $b$-continuous.

Definition 3.2. Let $(X, \tau)$ and $(Y, \varphi)$ be two topological spaces. A function $f : (X, \tau) \to (Y, \varphi)$ is called $\gamma_b$-irresolute if the inverse image of each $\gamma_b$-set of $(Y, \varphi)$ is a $\gamma_b$-set in $(X, \tau)$.
The following theorems are obtained from Definition 3.2.

**Theorem 3.3.** Let \((X, \tau), (Y, \varphi)\) be two topological spaces and \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function. Then, the following statements are equivalent:

1. \(f\) is \(\gamma_b\)-irresolute;
2. The inverse image of each \(\gamma_b\)-closed set in \((Y, \varphi)\) is a \(\gamma_b\)-closed set;
3. \(\text{Cl}_{\gamma_b}(f^{-1}(V)) \subset f^{-1}(\text{Cl}_{\gamma_b}(V))\) for every \(V \subset Y\);
4. \(f(\text{Int}_{\gamma_b}(U)) \subset \text{Int}_{\gamma_b}(f(U))\) for every \(U \subset X\);
5. \(f^{-1}(\text{Int}_{\gamma_b}(B)) \subset \text{Int}_{\gamma_b}(f^{-1}(B))\) for every \(B \subset Y\).

**Theorem 3.4.** Let \((X, \tau), (Y, \varphi)\) be two topological spaces and \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function. Then, the following statements are equivalent:

1. \(f\) is \(\gamma_b\)-irresolute;
2. For \(x \in X\) and for each \(V \in B_{f(x)}\), there exists an element \(U \in \text{b-neighbourhood filter} B_x\) such that \(f(U) \subset V\);
3. For \(x \in X\), if a filter \(\mathcal{F}\) b-converges to \(x\), then \(f(\mathcal{F})\) b-converges to \(f(x)\) in \((Y, \varphi)\).

**Proof.** (1)\(\implies\)(2). The proof is obvious.

(2)\(\implies\)(3). Let \(V\) be an element of the b-neighbourhood filter of \(B_{f(x)}\) and \(U\) be an element of \(B_x\) and let \(\mathcal{F}\) be a filter on \((X, \tau)\) b-converging to \(x\). Then \(f(B_x) \subset f(\mathcal{F})\). Since \(U\) is an element of \(B_x\) and \(\mathcal{F}\) is a filter, we have \(V \in f(\mathcal{F})\). Consequently, \(f(\mathcal{F})\) b-converges to \(f(x)\).

(3)\(\implies\)(1). Let \(V\) be any \(\gamma_b\)-set in \((Y, \varphi)\) and assume that \(f^{-1}(V)\) is not empty. For each \(x \in f^{-1}(V)\), since the b-neighbourhood filter \(B_x\) b-converges to \(x\), \(f(B_x)\) b-converges to \(f(x)\) by using the hypothesis. Also, since \(V\) is \(\gamma_b\)-set containing \(f(x)\) and \(B_{f(x)} \subset f(B_x)\), we have \(V \in f(B_x)\). If we take some \(\gamma_b\)-set \(U\) in \(B_x\) such that \(f(U) \subset V\), then \(U \subset f^{-1}(V)\) and hence \(f^{-1}(V)\) is an element of \(B_x\) by using \(B_x\) is a filter. Consequently, \(f^{-1}(V)\) is a \(\gamma_b\)-set in \((X, \tau)\) from Theorem 2.2(2).

**Corollary 3.2.** Let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function. If \(f\) is irresolute, then whenever a filter \(\mathcal{F}\) b-converges to \(x\) in \((X, \tau)\), \(f(\mathcal{F})\) b-converges to \(f(x)\) in \((Y, \varphi)\).

We can get the following two diagrams.

- **Diagram I**
  
  \[
  \text{continuity} \rightarrow \alpha\text{-continuity} \rightarrow \text{semi continuity} \quad \downarrow \quad \downarrow \\
  \text{pre-continuity} \rightarrow \text{b-continuity} \rightarrow \gamma_b\text{-continuity}
  \]

- **Diagram II**
  
  \[
  \alpha\text{-irresolute} \rightarrow \text{irresolute} \quad \downarrow \quad \downarrow \\
  \text{pre-irresolute} \rightarrow \text{b-irresolute} \rightarrow \gamma_b\text{-irresolute}
  \]

**Definition 3.3.** For two topological spaces \((X, \tau)\) and \((Y, \varphi)\), let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function. \(f\) is called \(\gamma_b\)-open ( resp. \(b\)-open [4]) if for every open set \(U\) in \((X, \tau)\), \(f(U)\) is \(\gamma_b\)-set ( resp. \(b\)-open set ) in \((Y, \varphi)\).

Now, we give a characterization of \(\gamma_b\)-openness.

**Theorem 3.5.** Let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function between \((X, \tau)\) and \((Y, \varphi)\) topological spaces. Then, \(f\) is \(\gamma_b\)-open if and only if \(\text{int}(f^{-1}(B)) \subset f^{-1}(\text{int}_{\gamma_b}(B))\) for each \(B \subset Y\).

**Proof.** \(\implies\) Let \(B \subset Y\) and \(x \in \text{int}(f^{-1}(B))\). Then, \(f(\text{int}(f^{-1}(B)))\) is a \(\gamma_b\)-set containing \(f(x)\). Since \(f(\text{int}(f^{-1}(B))) \subset B_{f(x)}\) and \(B_{f(x)}\) is a filter, \(B \in B_{f(x)}\). Therefore, we have \(f(x) \in \text{int}_{\gamma_b}(B)\) and hence we have \(x \in f^{-1}(\text{int}_{\gamma_b}(B))\).

\(\implies\) Let \(U\) be an open in \((X, \tau)\) and \(y \in f(U)\). Then, \(U \subset \text{int}(f^{-1}(f(U))) \subset f^{-1}(\text{int}_{\gamma_b}(f(A)))\). Let \(x \in U\) be such that \(f(x) = y\), then \(x \in f^{-1}(\text{int}_{\gamma_b}(f(A)))\). So, \(y \in \text{int}_{\gamma_b}(f(A))\) and hence by using Theorem 2.2(2), we have \(f(A)\) is a \(\gamma_b\)-set. \(\Box\)
Corollary 3.3. Every b-open function is \( \gamma_b \)-open.

Remark 3.2. The converse of Corollary 3.3 is not true in generally as shown in the following example.

Example 3.2. Let \((\mathbb{R}, \tau)\) be usual space i.e. \(\mathbb{R}\) is real number set and \(\tau\) is usual topology. Let \(f : (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)\) be a function such that \(f(x) = 0\) for all \(x \in \mathbb{R}\). Although \(f\) is \(\gamma_b\)-open, but is not b-open.

Theorem 3.6. Let \(f : (X, \tau) \rightarrow (Y, \varphi)\) be a function between \((X, \tau)\) and \((Y, \varphi)\) topological spaces. The function \(f\) is \(\gamma_b\)-open if and only if for each \(x \in X\) and for each neighbourhood \(V\) of \(x\), \(f(V)\) is also an element of b-neighbourhood filter \(B_{f(x)}\) in \((Y, \varphi)\).

Proof. \((\Rightarrow)\): Let \(V\) be a neighbourhood of \(x\), then there exists an open set \(U\) such that \(x \in U \subseteq V\). Since \(f\) is \(\gamma_b\)-open, \(f(x) \in f(U) = \text{int}_{\gamma_b \varphi}(f(U))\) and hence \(f(U) \in B_{f(x)}\). Since \(B_{f(x)}\) is a filter, \(f(V) \in B_{f(x)}\).

\((\Leftarrow)\): Let \(B \subseteq Y\) and \(x \in \text{int}(f^{-1}(B))\) is an element of \(B_x\) and \(B_x\) is a filter, \(f^{-1}(B) \subseteq B_x\). By the hypothesis \(f(f^{-1}(B)) \subseteq B_{f(x)}\), and since \(B_{f(x)}\) is a filter, \(B\) is also an element of \(B_{f(x)}\). According to Definition 4, \(f(x) \in \text{int}_{\gamma_b \varphi}(f(B))\) and by using Theorem 3.5, we have the function \(f\) is \(\gamma_b\)-open.

As similar to Diagram I and Diagram II, we have the following diagram for some open functions:

- open function \(\rightarrow\) \(\alpha\)-open function \(\rightarrow\) semi-open function
- ↓
- ↓
- pre-open function \(\rightarrow\) b-open function \(\rightarrow\) \(\gamma_b\)-open function

Diagram III

References


Affiliations

AYNUR KESKIN KAYMAKCI
ADDRESS: Selcuk University, Faculty of Sciences, Department of Mathematics, 42030, Campus, Konya/TURKEY
E-MAIL: akeskin@selcuk.edu.tr