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γb-sets and γb-continuous functions

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Abstract
First of all we introduce a new class of sets is called γb-sets and a new type of continuous functions is called γb-continuous functions. Then, we investigate some properties and characterizations. Besides, γb-sets and γb-continuity are used to extend known results for b-open sets and b-continuity.

Keywords: *γb-sets; b-open sets; γb-continuity; b-continuity.*

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1. Introduction and Preliminaries

It is well known that the notion of open sets and it's some modifications (semi-open sets, α-open sets, preopen sets, b-open sets, etc.) are important topological notions in topological spaces. Of course, they are usefull tolls in several papers. Really, they are used to introduced related continuous functions, spaces, convergence etc. and investigated their some properties. Let (X, τ), (Y, φ), (Z, ψ) be topological spaces on which no separation axioms are assumed unless explicitly stated. For A a subset of X, we will denote $Cl(A)$ and $Int(A)$ of the closure A and the interior of A, respectively. A subset A of X is called a semi-open [5] (resp. α-set [9], pre-open set [7], b-open set [1] or γ-open set [4]) if $A \subset Cl(Int(A))$ (resp. $A \subset Int(Cl(Int(A)))$, $A \subset Int(Cl(A))$). The complement of a semi-open set (resp. α-open set) is called semi-closed set (resp. α-closed set). The family of all semi-open sets (resp. α-sets, pre-open sets, b-open sets) in X will be denoted by $SO\,(X, \tau)$ (resp. $\alpha\,(X, \tau)$, $PO\,(X, \tau)$, $BO\,(X, \tau)$). A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called semi-continuous [5] (resp. α-continuous [8], pre-continuous [7], b-continuous [3]) if $f^{-1}(V) \in SO\,(X, \tau)$ (resp. $f^{-1}(V) \in \alpha\,(X, \tau)$, $f^{-1}(V) \in PO\,(X, \tau)$, $f^{-1}(V) \in BO\,(X, \tau)$) for each open set V of (Y, φ). A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called irresolute [2] (resp. α-irresolute [6], pre-irresolute [10], b-irresolute[3]) if $f^{-1}(V) \in SO\,(X, \tau)$ (resp. $f^{-1}(V) \in \alpha\,(X, \tau)$, $f^{-1}(V) \in PO\,(X, \tau)$, $f^{-1}(V) \in BO\,(X, \tau)$) for each semi-open set (resp. α-open set, pre-open set, b-open set) V of (Y, φ). A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called semi-open [5] (resp. α-open [8]) if for every semi-open (resp. α-open) set U in (X, τ), $f(U)$ is semi-open (resp. α-open) set in (Y, φ). The notion of filters is another important in topology as well as some branches of mathematics.

In this paper, in first step we define a new class of sets is called γb-sets and a new type of continuous functions is called γb-continuous functions by using the notion of b-convergence of filters. We state that this term is used to investigated some characterizations related to b-continuous functions. Because of the notions γb-sets and γb-continuity are used to extend some well-known results for b-open sets and b-continuity, new introduce the notions γb-sets and γb-continuity.

Recall that a subset $M(x)$ of space (X, τ) is called a b-$neighbourhood$ of a point $x \in X$ if there exists a *b-open* set S such that $x \in S \subset M(x)$.

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2. γb-sets

First of all, we define the notions of $b\text{-}neighbourhood\ filter$ at x any point of topological space and $b\text{-}convergence$ of filters.

Definition 2.1. Let (X, τ) be a topological space, $B(x) = \{A \in BO(X, \tau) \mid x \in A\}$ and let $B_x = \{A \subset X \mid \exists\ \mu \subset B(x)$ such that μ is finite and $\cap\mu \subset A\}$. Then, B_x is called the b-neighbourhood filter at x.

Definition 2.2. Let $\mathcal{F}$ is any filter on any topological space (X, τ). If $\mathcal{F}$ is finer than the b-neighbourhood filter at x, then $\mathcal{F}$ is called b-convergences to x.

Definition 2.3. Let (X, τ) be a topological space and A be a subset of (X, τ). If a filter $\mathcal{F}$ b-converges to x such that $x \in A$ and $A \in \mathcal{F}$, then A is called $\gamma b\text{-}set$ in (X, τ).

We will denote by $\gamma b\,(X, \tau)$ of the class of all $\gamma b\text{-}set$ in (X, τ).

From the definitions of the notions $b\text{-}neighbourhood\ filters$ and $\gamma b\text{-}sets$, we have the following proposition as a result.

Proposition 2.1. *In any topological space, every b-open set is a $\gamma b\text{-}set$.*

Remark 2.1. The converse of Proposition 2.1 is not true in generally as shown in the following example.

Example 2.1. Let $(\mathbb{R}, \tau)$ be usual space i.e. $\mathbb{R}$ is real number set and τ is usual topology. For each $x \in X$, since both $(a, x]$ and $[x, b)$ are b-open sets containig x such that $a \prec x \prec b$, then $\{x\}$ is an element of B_x. Consequently, for any filter $\mathcal{F}$ on $\mathbb{R}$, if $\mathcal{F}$ b-converges to x then $\{x\}$ is a $\gamma b\text{-}set$ by using $\mathcal{F}$ contains B_x. But, $\{x\}$ is not b-open.

Remark 2.2. In a topological space (X, τ), $\tau \subset \alpha\,(X, \tau) \subset BO\,(X, \tau) \subset \gamma b\,(X, \tau)$.

Theorem 2.1. *Let (X, τ) be a topological space. The intersection of finitely many b-open subsets in (X, τ) is a $\gamma b\text{-}set$.*

Proof. Let U and V be a $b\text{-}open$ sets in (X, τ). For each $x \in (U \cap V)$, we get $(U \cap V) \in B_x$. Therefore, we have $(U \cap V) \in \mathcal{F}$ for every filter $\mathcal{F}$ $b\text{-}converges$ to x by using the notion of the converges of filters. □

We define an operator by using the notion of $\gamma b\text{-}sets$.

Definition 2.4. Let (X, τ) be a topological space and A be a subset of (X, τ). The γb-interior of A in (X, τ), denoted by $\mathrm{Int}_{\gamma b}(A)$, is the union of all $\gamma b\text{-}sets$ contained in A.

Now, we give a characterization of $\gamma b\text{-}sets$ by using the notion of γb-interior.

Theorem 2.2. *Let (X, τ) be a topological space and A be a subset of (X, τ). Then, the following properties are hold:*
(1) $Int_{\gamma b}(A) = \{x \in A \mid A \in B_x\}$,
(2) A is $\gamma b\text{-}set$ if and only if $A = Int_{\gamma b}(A)$.

Proof. (1) For each $x \in Int_{\gamma b}(A)$, there exists a $\gamma b\text{-}set$ U such that $x \in U$ and $U \subset A$. By using the notion of $\gamma b\text{-}set$, the subset U is in the $b\text{-}neighbourhood\ filter$ B_x. Since B_x is a filter, $A \in B_x$.

Conversely, let $x \in A$ and $A \in B_x$. Then, there exist $U_1, U_2, ..., U_n \in B(x)$ such that $U = U_1 \cap U_2 \cap ... \cap U_n \subset A$. By Theorem 2.1, U is a $\gamma b\text{-}set$ and $U \subset A$. This shows that $x \in Int_{\gamma b}(A)$.

(2) The proof is obvious. □

Theorem 2.3. *For any topological space (X, τ), the class of $\gamma b\,(X, \tau)$ of all γb-subsets in (X, τ) is a topology on X.*

Proof. Since $\varnothing$ and X are b-open, they are also $\gamma b\text{-}sets$. Let A, $B \in \gamma b\,(X, \tau)$, $x \in (A \cap B)$ and let $\mathcal{F}$ be a filter. Assume that the filter $\mathcal{F}$ b-converges to x. Therefore, $A \cap B$ is a $\gamma b\text{-}set$. For each $\alpha \in I$, let $A_\alpha \in \gamma b\,(X, \tau)$ and $U = \cup A_\alpha$. For each $x \in U$ and for a filter $\mathcal{F}$ b-converging to x there exists a subset A_α of U such that $x \in A_\alpha$, and since A_α is $\gamma b\text{-}set$, it is obvious that $A_\alpha \in \mathcal{F}$. Since $\mathcal{F}$ is filter, U is an element of the filter $\mathcal{F}$ and thus $U = \cup A_\alpha$ is $\gamma b\text{-}set$. □

In a topological space (X, τ), the class of all $\gamma b\text{-}sets$ induced by the topology τ will be denoted by $(X, \gamma_{b\tau})$.

A subset A of (X, τ) is called a $\gamma b\text{-}closed$ if the complement of A is a $\gamma b\text{-}set$. We will denote by $\gamma_b C\,(X, \tau)$ of the class of all $\gamma b\text{-}closed$ in (X, τ). Therefore, we state the following by using Theorem 2.3:

(1) $\gamma_b C\,(X, \tau)$ is closed arbitrary intersection,

(2) $\gamma_b C\,(X, \tau)$ is closed finite union.

Now, we give a characterization of notion of $\gamma_b\text{-}closed$ by using the notion of $b\text{-}convergency$.

Theorem 2.4. *Let (X, τ) be a topological space and A be a subset of (X, τ). A is γ_b-$closed$ if and only if whenever $\mathcal{F}$ b-convergens to x and $A \in \mathcal{F}$, $x \in A$.*

Definition 2.5. Let (X, τ) be a topological space. For A a subset of (X, τ), $Cl_{\gamma_b}(A) = \{x \in X \mid A \cap U \neq \varnothing, \forall U \in B_x\}$ is called the γ_b-closure of a subset A in (X, τ).

Now, we give the some properties of γ_b-closure operation in the following theorem.

Theorem 2.5. *Let (X, τ) be a topological space and A be a subset of (X, τ). Then, the following properties hold:*
(1) $A \subset Cl_{\gamma_b}(A)$;
(2) A is γ_b-$closed$ if and only $A = Cl_{\gamma_b}(A)$;
(3) $Int_{\gamma_b}(A) = X$-$Cl_{\gamma_b}(X$-$A)$;
(4) $Cl_{\gamma_b}(A) = X$- $Int_{\gamma_b}(A)$.

3. γ_b-continuous and γ_b-irresolute functions

In this section, we introduce new two types of continuity are called γ_b-$continuity$ and γ_b-$irresoluteness$ by using γ_b-sets. Then, we have obtained some characterizations of these notions. Furthermore, we define the notion of γ_b-open functions as a new modification of open functions. We have also given two characterizations of this function.

Definition 3.1. Let (X, τ) and (Y, φ) be two topological spaces. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called γ_b-continuous if the inverse image of each open set of (Y, φ) is a γ_b-set in (X, τ).

Since the class of all γ_b-sets in a given topological space is also a topology, we get the following equivalent statements.

Theorem 3.1. *Let (X, τ), (Y, φ) be two topological spaces and $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function. Then, the following statements are equivalent:*
(1) f is γ_b-continuous;
(2) The inverse image of each closed set in (Y, φ) is γ_b-closed set in (X, τ);
(3) $Cl_{\gamma_b}(f^{-1}(B)) \subset f^{-1}(Cl(B))$ for every $B \subset Y$;
(4) $f(Cl_{\gamma_b}(A)) \subset Cl(f(A))$ for every $A \subset X$;
(5) $f^{-1}(Int(B)) \subset Int_{\gamma_b}(f^{-1}(B))$ for every $B \subset Y$.

Theorem 3.2. *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function between topological spaces (X, τ) and (Y, φ). Then, the following statements are equivalent:*
(1) f is γ_b-continuous at x;
(2) If a filter $\mathcal{F}$ b-converges to x, then $f(\mathcal{F})$ converges to $f(x)$;
(3) For $x \in X$ and for each neighbourhood V of f(x), there exists a subset $V \in B_x$ such that $f(U) \subset V$.

Proof. (1)$\Longrightarrow$(2). Let V be any open neighbourhood of $f(x)$ in (Y, φ). Since f is γ_b-continuous at x, then $f^{-1}(V)$ is an element of B_x. Besides since $\mathcal{F}$ is b-converges to x and f($\mathcal{F}$) is a filter, then we have $V \in f(\mathcal{F})$. Consequently, $f(\mathcal{F})$ converges to $f(x)$.

(2)$\Longrightarrow$(3). Let V be any γ_b-neighbourhood of $f(x)$. By hypothesis, we have that B_x b-converges to x. Hence, we have $B_{f(x)} \subset f(B_x)$ and then $V \in f(B_x)$. Therefore, there is subset $U \in B_x$ such that $f(U) \subset V$.

(3)$\Longrightarrow$(1). This is obvious. □

Now, we can easily show the following result.

Corollary 3.1. *Let $f : (X, \tau) \longrightarrow (Y, \varphi)$ be a function. If f is b-continuous at $x \in X$, then whenever a filter $\mathcal{F}$ b-converges to x in (X, τ), $f(\mathcal{F})$ converges to $f(x)$ in (Y, φ).*

Remark 3.1. The converse of Corollary 3.1 is not true in generally as shown in the following example.

Example 3.1. Let (X, τ) and (Y, φ) be two topological spaces such that $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{a, d\}, \{a, c\}, \{a, c, d\}\}$, $Y = \{a, b\}$ and $\varphi = \{Y, \varnothing, \{a\}\}$. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is defined $f(a) = f(c) = f(d) = b$ and $f(b) = a$. For $a \in Y$ and $\{a\} \in \varphi$, $f^{-1}(\{a\}) = \{b\}$. But, $\{b\}$ is not b-open in (X, τ). So, f is not b-continuous.

Definition 3.2. Let (X, τ) and (Y, φ) be two topological spaces. A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is called γ_b-irresolute if the inverse image of each γ_b-set of (Y, φ) is a γ_b-set in (X, τ).

The following theorems are obtained from Definition 3.2.

Theorem 3.3. *Let (X,τ), (Y,φ) be two topological spaces and $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function. Then, the following statements are equivalent:*

(1) f is γ_b-irresolute;

(2) The inverse image of each γ_b-closed set in (Y,φ) is a γ_b-closed set;

(3) $Cl_{\gamma_b\tau}(f^{-1}(V))\subset f^{-1}(Cl_{\gamma_b\varphi}(V))$ for every $V\subset Y$;

(4) $f(Cl_{\gamma_b\tau}(U))\subset Cl_{\gamma_b\varphi}(f(U))$ for every $U\subset X$;

(5) $f^{-1}(Int_{\gamma_b\varphi}(B))\subset Int_{\gamma_b\tau}(f^{-1}(B))$ for every $B\subset Y$.

Theorem 3.4. *Let (X,τ), (Y,φ) be two topological spaces and $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function. Then, the following statements are equivalent:*

(1) f is γ_b-irresolute;

(2) For $x\in X$ and for each $V\in B_{f(x)}$, there exists an element U in the b-neighbourhood filter B_x such that $f(U)\subset V$;

(3) For $x\in X$, if a filter $\mathcal{F}$ b-converges to x, then $f(\mathcal{F})$ b-converges to $f(x)$ in (Y,φ).

Proof. (1)$\Longrightarrow$(2). The proof is obvious.

(2)$\Longrightarrow$(3). Let V be an element of the b-neighbourhood filter of $B_{f(x)}$ and U be an element of B_x and let $\mathcal{F}$ be a filter on (X,τ) b-converging to x. Then $f(B_x)\subset f(\mathcal{F})$. Since U is an element of B_x and $\mathcal{F}$ is a filter, we have $V\in f(\mathcal{F})$. Consequently, $f(\mathcal{F})$ b-converges to $f(x)$.

(3)$\Longrightarrow$(1). Let V be any γ_b-set in (Y,φ) and assume that $f^{-1}(V)$ is not empty. For each $x\in f^{-1}(V)$, since the b-neighbourhood filter B_x b-converges to x. $f(B_x)$ b-converges to $f(x)$ by using the hypothesis. Also, since V is γ_b-set containing f(x) and $B_{f(x)}\subset f(B_x)$, we have $V\in f(B_x)$. If we take some γ_b-set U in B_x such that $f(U)\subset V$, then $U\subset f^{-1}(V)$ and hence $f^{-1}(V)$ is an element of B_x by using B_x is a filter. Consequently, $f^{-1}(V)$ is a γ_b-set in (X,τ) from Theorem 2.2(2). □

Corollary 3.2. *Let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function. If f is irresolute, then whenever a filter $\mathcal{F}$ b-converges to x in (X,τ), $f(\mathcal{F})$ b-converges to f(x) in (Y,φ).*

We can get the following two diagrams.

continuity $\longrightarrow$ α-continuity $\longrightarrow$ semi continuity
$\downarrow$ $\downarrow$
pre-continuity $\longrightarrow$ b-continuity $\longrightarrow$ γ_b-continuity

Diagram I

α-irresolute $\longrightarrow$ irresolute
$\downarrow$ $\downarrow$
pre-irresolute $\longrightarrow$ b-irresolute $\longrightarrow$ γ_b-irresolute

Diagram II

Definition 3.3. For two topological spaces (X,τ) and (Y,φ), let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function. f is called γ_b-open (resp. b-open [4]) if for every open set U in (X,τ), $f(U)$ is γ_b-set (resp. b-open set) in (Y,φ).

Now, we give a characterization of γ_b-openness.

Theorem 3.5. *Let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function between (X,τ) and (Y,φ) topological spaces. Then, f is γ_b-open if and only if $int(f^{-1}(B))\subset f^{-1}(int_{\gamma_b\varphi}(B))$ for each $B\subset Y$.*

Proof. $\Longrightarrow$ Let $B\subset Y$ and $x\in int(f^{-1}(B))$. Then, $f(int(f^{-1}(B)))$ is a γ_b-set containing $f(x)$. Since $f(int(f^{-1}(B)))\in B_{f(x)}$ and $B_{f(x)}$ is a filter, $B\in B_{f(x)}$. Therefore, we have $f(x)\in int_{\gamma_b\varphi}(B)$ and hence we have $x\in f^{-1}(int_{\gamma_b\varphi}(B))$.

$\Longleftarrow$ Let U be an open in (X,τ) and $y\in f(U)$. Then, $U\subset int(f^{-1}(f(U)))\subset f^{-1}(int_{\gamma_b\varphi}(f(A)))$. Let $x\in U$ be such that $f(x)=y$, then $x\in f^{-1}(int_{\gamma_b\varphi}(f(A)))$. So, $y\in int_{\gamma_b\varphi}(f(A))$ and hence by using Theorem 2.2(2), we have $f(A)$ is a γ_b-set. □

Corollary 3.3. *Every b-open function is γ_b-open.*

Remark 3.2. The converse of Corollary 3.3 is not true in generally as shown in the following example.

Example 3.2. Let $(\mathbb{R},\tau)$ be usual space i.e. $\mathbb{R}$ is real number set and τ is usual topology. Let $f:(\mathbb{R},\tau)\longrightarrow(\mathbb{R},\tau)$ be a function such that $f(x)=0$ for all $x\in\mathbb{R}$. Although f is γ_b-open, but is not b-open.

Theorem 3.6. *Let $f:(X,\tau)\longrightarrow(Y,\varphi)$ be a function between (X,τ) and (Y,φ) topological spaces. The function f is γ_b-open if and only if for each $x\in X$ and for each neighbourhood V of x, $f(V)$ is also an element of b-neighbourhood filter $B_{f(x)}$ in (Y,φ).*

Proof. ($\Longrightarrow$): Let V be a neighbourhood of x, then there exists an open set U such that $x\in U\subset V$. Since f is γ_b-open, $f(x)\in f(U)=int_{\gamma_b\varphi}(f(U))$ and hence $f(U)\in B_{f(x)}$. Since $B_{f(x)}$ is a filter, $f(V)\in B_{f(x)}$.

($\Longleftarrow$): Let $B\subset Y$ and $x\in int(f^{-1}(B))$ is an element of B_x and B_x is a filter, $f^{-1}(B)\in B_x$. By the hypothesis $f(f^{-1}(B))\in B_{f(x)}$, and since $B_{f(x)}$ is a filter, B is also an element of $B_{f(x)}$. According to Definition 4, $f(x)\in int_{\gamma_b\varphi}(f(B))$ and by using Theorem 3.5, we have the function f is γ_b-open. □

As similar to Diagram I and Diagram II, we have the following diagram for some open functions:

open function $\longrightarrow$ α-open function $\longrightarrow$ semi-open function
$\downarrow$ $\downarrow$
pre-open function $\longrightarrow$ b-open function $\longrightarrow$ γ_b-open function

Diagram III

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