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Exact Traveling Wave Solutions of some Nonlinear Evolution Equations

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Abstract
In nonlinear sciences, it is important to obtain traveling wave solutions of nonlinear evolution equations to understand the phenomena they describe. In this study, we obtained the exact traveling wave solutions of the Liouville equation, two-dimensional Bratu equation, generalized heat conduction equation and coupled nonlinear Klein-Gordon equations by means of the trial equation method and the complete discrimination system. This method is reliable, effective and enables to get soliton, single-kink and compacton solutions of the generalized nonlinear evolution equations and systems of equations.

Keywords: *generalized heat conduction equation; coupled nonlinear Klein-Gordon equations; trial equation method.*

AMS Subject Classification (2010): *Primary: 35Q79; 35Q35; 35Q40; Secondary: 35A25.*

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1. Introduction

Nonlinear evolution equations have an important role in nonlinear sciences because of the applications in plasma physics, solid state physics, chemical physics, fluid mechanics, optical fibers, chemical kinematics, biology, and so on. The constructions of exact solutions to nonlinear differential equations have been investigated extensively using a large number of methods such as the inverse scattering method [1], the tanh method [2], Hirotas bilinear transformation [3], sine-cosine method [4], homogeneous balance method [5], (G'/G)-expansion method [6] and exp-function method [7]. Ma and Fuchssteiner proposed an effective approach to find exact solutions of nonlinear differential equations [8]. Their approach depends on expanding solutions of given nonlinear differential equations as solutions of solvable differential equations, specially, polynomial and rational functions. Liu [9-11] has proposed trial equation method to investigate the exact solutions of nonlinear evolution equations. To describe Liu's method, we can consider a differential equation of u and assume that its exact solution satisfies a solvable equation like $(u')^2 = F(u)$. So, when we find the function F, we can also obtain the exact solution of the differential equation. Liu has chosen $F(u)$ as a polynomial and rational function, Du [12] has chosen F as an irrational function and they have obtained some exact solutions of nonlinear differential equations. Y Liu proposed a new type of the trial equation method for nonlinear partial differential equations with variable coefficients [13]. Also, this method is studied by other authors [14-16]. In the present study, we have applied the complete discrimination system for polynomials [17] and the trial equation method to look for traveling wave solutions of nonlinear evolution

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equations. The equations that have this form are transformed into the nonlinear ordinary differential equations; it has been obtained the term of $(u')^2$. By this method, we have obtained some exact traveling wave solutions to the Liouville equation [18]:

$$u_{xt} + exp(u) = 0, \tag{1.1}$$

the two-dimensional Bratu equation [19]:

$$u_{xx} + u_{yy} + \lambda exp(su) = 0, \tag{1.2}$$

the generalized heat conduction equation [20]:

$$u_t - a(u^n)_{xx} - u + u^n = 0, \tag{1.3}$$

and the coupled nonlinear Klein-Gordon equations [21]:

$$u_{xx} - u_{tt} - u + 2u^3 + 2uv = 0, \tag{1.4}$$

$$v_x - v_t - 4uu_t = 0. \tag{1.5}$$

2. The trial equation method

The main steps of the modified trial equation method can be outlined as follows [22]:

Step1. Consider the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0. \tag{2.1}$$

Under the wave transformation

$$u(x,t) = u(\eta), \quad \eta = k(x - ct), \tag{2.2}$$

Eq.(2.1) becomes an ordinary differential equation in the form:

$$N(t, x, u, u', u'', \ldots) = 0, \tag{2.3}$$

where prime denotes differentiation with respect to η.

Step 2. Take trial equation

$$(u')^2 = \frac{F(u)}{G(u)} = \frac{\sum_{i=0}^{n} a_i u^i}{\sum_{j=0}^{l} b_j u^j}, \tag{2.4}$$

$$u'' = \frac{[F'(u)G(u) - F(u)G'(u)]\,F(u)}{G^3(u)}, \tag{2.5}$$

where $F(u)$ and $G(u)$ are polynomials. Substituting the Eq.(2.4) and Eq.(2.5) into equation Eq.(2.3) gives an equation of polynomial $\Gamma(u)$ of u:

$$\Gamma(u) = p_s u^s + \ldots + p_1 u + p_0 \tag{2.6}$$

If we use the balance procedure, we can get a relation between n and l.

Step 3. Letting the coefficients of $\Gamma(u)$ to zero will give a system algebraic equations:

$$p_i = 0, \quad i = 0, \ldots, s. \tag{2.7}$$

When we solve this system, we can determine $a_0, \ldots, a_n$ and $b_0, \ldots, b_l$.

Step 4. Rewrite the Eq.(2.4) by integral form

$$\pm(\eta - \eta_0) = \int \sqrt{\frac{G(u)}{F(u)}} du. \tag{2.8}$$

With the help of the complete discrimination system of polynomial, we can classify the roots of $F(u)$ and solve integral (2.8) by Mathematica programming. Thus we obtain the exact solution of the Eq.(2.1).

3. Applications

Example 1. We first consider the Liouville equation (1.1) that has applications in quantum physics. Under the transformation

$$u(x,t) = u(\eta), \quad \eta = (x - \ ct), \tag{3.1}$$

Eq.(1.1) change into the equation in the form:

$$-\, cu'' + \exp(u) = 0. \tag{3.2}$$

If we use the transformation

$$u = \ln v, \tag{3.3}$$

we obtain

$$-\, c(vv'' - v'^2) + v^3 = 0. \tag{3.4}$$

We can take trial equation as follows:

$$(v')^2 = F(v) = \sum_{i=0}^{n} a_i v^i. \tag{3.5}$$

If we substitute the Eq.(3.5) into Eq.(3.4) to balance the higher order terms v^3 and vv'' (or v'^2) in this equation, we get $n = 3$. When we use Eq.(3.5) in Eq.(3.4), we get a system of algebraic equations:

$$\begin{aligned} a_0 c &= 0 \\ a_1 c &= 0 \\ a_2 c - a_2 c &= 0 \\ a_3 c - 2 &= 0, \end{aligned} \tag{3.6}$$

and we find coefficients:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = a_2, \quad c = \frac{2}{a_3}. \tag{3.7}$$

When we substitute these coefficients in the integral form (2.8), we obtain

$$\pm\, (\eta - \eta_0) = \int \frac{1}{\sqrt{a_2 v^2 + a_3 v^3}} dv. \tag{3.8}$$

By the transformations

$$w = (a_3)^{\frac{1}{3}} v, \quad , d_2 = a_2 (a_3)^{\frac{-2}{3}}, \tag{3.9}$$

We can write the Eq. (3.8) as

$$\pm\, (a_3)^{\frac{1}{3}} (\eta - \eta_0) = \int \frac{1}{\sqrt{w^3 + d_2 w^2}} dw. \tag{3.10}$$

When

$$G(w) = w^3 + d_2 w^2 + d_1 w + d_0, \tag{3.11}$$

the values Δ and D_1 that constitutes the complete discrimination system [17] are

$$\Delta = -27 \left(2\frac{d_2^3}{27} + d_0 - \frac{d_1 d_2}{3} \right)^2 - 4 \left(d_1 - \frac{d_2^2}{3} \right)^3 \tag{3.12}$$

and

$$D_1 = d_1 - \frac{d_2^2}{3}. \tag{3.13}$$

As $d_0 = d_1 = 0$, we obtain $\Delta = 0$ and $D_1 = -\frac{d_2^2}{3}$.

Case 1. When $D_1 < 0$, if we integrate Eq. (3.10), we obtain traveling wave solutions to the Eq. (1.1) in the form:

$$u_1(x,t) = \ln\left[-\frac{1}{2}ca_2\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\mathrm{a_2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right)\right],\ a_2>0,\ c<0, \tag{3.14}$$

$$u_2(x,t) = \ln\left[-\frac{1}{2}ca_2\sec^2\left(\frac{1}{2}\sqrt{-\mathrm{a_2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right)\right],\ a_2<0,\ c>0. \tag{3.15}$$

Case 2. When $D_1 = 0$, if we integrate Eq. (3.10), we get the exact solution to Eq. (1.1) in the form:

$$u_3(x,t) = \ln\left[\frac{2c}{(x-ct-\eta_0)^2}\right],\ c>0. \tag{3.16}$$

Figures (1) and (2) show the graphics of the solutions (3.14) and (3.15).

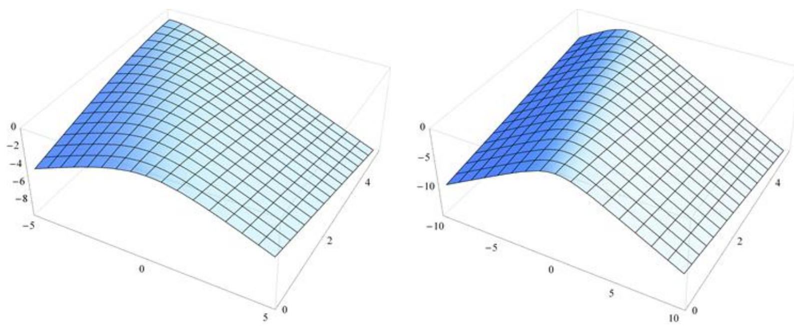

Figure 1. Graphics of the solution equation (3.14) for $a_2 = 1,\ c = -1.$

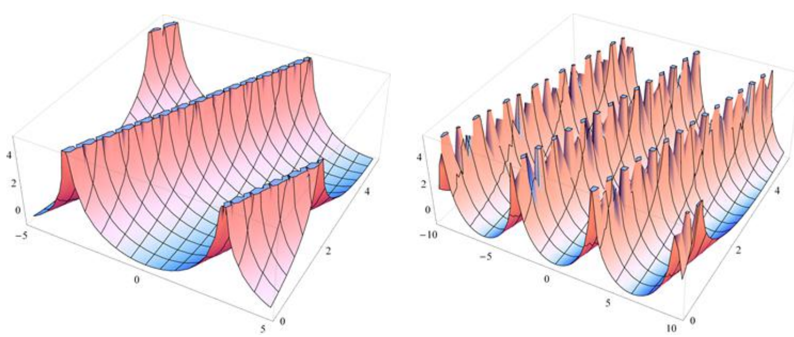

Figure 2. Graphics of the solution equation (3.15) for $a_2 = -1,\ c = 1.$

Example 2. We now consider the two-dimensional Bratu equation (1.2). Using the transformation

$$u(x,t) = u(\eta), \quad \eta = kx + wt, \tag{3.17}$$

and the transformation

$$u = (1/s)\ln v \tag{3.18}$$

Eq. (1.2) becomes the following ordinary differential equation:

$$(k^2+w^2)vv'' - (k^2+w^2)(v')^2 + \lambda s v^3 = 0. \tag{3.19}$$

From the balance procedure, we get $n - l = 3$. If we apply the steps of the modified trial equation method, we obtain the following results:

If we choose $n = 3$ and $l = 0$, then,

$$(u')^2 = \frac{a_0 + a_1 u + a_2 u^2 + a_3 u^3}{b_0}, \tag{3.20}$$

$$u'' = \frac{a_1 + 2a_2 u + 3a_3 u^2}{2b_0}. \tag{3.21}$$

When we use (3.20) and (3.21) in Eq.(3.19), we get a system of algebraic equations:

$$\begin{gathered} k^2 a_0 + w^2 a_0 = 0 \\ k^2 a_1 + w^2 a_1 = 0 \\ k^2 a_3 + w^2 a_3 + \lambda s b_0 = 0 \end{gathered} \tag{3.22}$$

Solving the system (3.22), we get the following coefficients:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = a_2, \quad a_3 = a_3, \quad b_0 = b_0, \quad w = \pm\sqrt{\frac{2\lambda s b_0}{a_3} - k^2}. \tag{3.23}$$

When we consider the coefficients (3.23), we obtain the solution to Eq.(1.2) in the form:

$$u(x,t) = \frac{1}{s} \ln \left[\frac{b_0}{a_3 \cosh^2 \left[\frac{1}{2} \left(\mathrm{kx} \pm \sqrt{\frac{2\lambda \mathrm{sb}_0}{\mathrm{a}_3} - \mathrm{k}^2}\mathrm{t} - \eta_0 \right) \right]} \right]. \tag{3.24}$$

This solution is a soliton solution according to some values of the parameters k and λ and arbitrary constants a_3 and b_0.

Example 3. Consider the generalized heat conduction equation (1.3) where $a > 0,\ n > 1$. Using the transformation

$$u(x,t) = u(\eta), \quad \eta = k(x - ct), \tag{3.25}$$

Eq. (1.3) becomes

$$-kcu' - ak^2(u^n)'' - u + u^n = 0, \tag{3.26}$$

or equivalently

$$-kcu' - ak^2 n(n-1)u^{n-2}(u')^2 - ak^2 n u^{n-1} u'' - u + u^n = 0. \tag{3.27}$$

If we use transformation $u(\eta) = v^{\frac{-1}{n-1}}(\eta)$ to obtain analytic solution, Eq.(3.27) becomes the following form:

$$-kc(n-1)v'v^2 + ak^2 n(1-2n)v'^2 + ak^2 n(n-1)vv'' + (n-1)^2 v^3 + (n-1)^2 v^2 = 0. \tag{3.28}$$

Choosing the trial equation in the form

$$v' = F(v) = a_0 + a_1 v + \ldots + a_n v^n \tag{3.29}$$

and substituting this equation in (3.28) and balancing the highest order nonlinear terms in Eq. (3.28), we obtain $n = 2$. Corresponding system of algebraic equations will be as follows:

$$\begin{gathered} a a_0^2 k^2 n - 2a a_0^2 k^2 n^2 = 0 \\ a a_0 a_1 k^2 n - 3a a_0 a_1 k^2 n^2 = 0 \\ a_0 ckn - a_0 ck - a a_1^2 k^2 n^2 - 2a a_0 a_2 k^2 n^2 + n^2 - 2n + 1 = 0 \\ a_1 ckn - a_1 ck - a a_1 a_2 k^2 n^2 - 2a a_1 a_2 k^2 n^2 - n^2 + 2n - 1 = 0 \\ a_2 ckn - a_2 ck - a a_2^2 k^2 n = 0. \end{gathered} \tag{3.30}$$

From the system (3.30), we obtain the following coefficients:

$$a_0 = 0, \quad a_1 = \pm\frac{n-1}{kn\sqrt{a}}, \quad a_2 = \pm\frac{1-n}{kn\sqrt{a}}, \quad c = \pm\sqrt{a}. \tag{3.31}$$

When we integrate the Eq.(3.29) according to the complete discrimination system and as $\Delta = a_1^2 - 4a_0a_2 > 0$, we obtain the solution to Eq.(1.3) as follows:

$$u_{1,2}(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\pm\frac{\mathrm{n}-1}{2\mathrm{n}\sqrt{\mathrm{a}}}\left(\mathrm{x} \pm \sqrt{\mathrm{a}}\mathrm{t} - \eta_0\right)\right)\right]^{\frac{-1}{n-1}}, \tag{3.32}$$

$$u_{3,4}(x,t) = \left[\frac{1}{2} + \frac{1}{2}\coth\left(\pm\frac{\mathrm{n}-1}{2\mathrm{n}\sqrt{\mathrm{a}}}\left(\mathrm{x} \pm \sqrt{\mathrm{a}}\mathrm{t} - \eta_0\right)\right)\right]^{\frac{-1}{n-1}}. \tag{3.33}$$

Example 4. Consider the coupled nonlinear Klein-Gordon equations (1.4) and (1.5). Klein-Gordon equations have important applications in quantum mechanics. Using the transformations

$$u(x,t) = u(\eta), \quad v(x,t) = v(\eta), \quad \eta = x - ct, \tag{3.34}$$

Eq.(1.4) and (1.5) becomes the following ordinary equations of u and v:

$$(1-c^2)u'' - u + 2u^3 + 2uv = 0, \tag{3.35}$$

$$(1+c)v' + 4cuu' = 0. \tag{3.36}$$

When we integrate the Eq. (3.36) according to the η, we obtain

$$v = -\frac{2c}{1+c}u^2. \tag{3.37}$$

If we substitute (3.37) into the equation (3.35) and integrate this equation, we obtain the following ordinary differential equation:

$$(1-c^2)u'' - u - \frac{2(c-1)}{c+1}u^3 = 0. \tag{3.38}$$

Balancing the highest order nonlinear terms in this equation, we get $n = 2$, so we can choose the trial equation as follows:

$$u' = a_2u^2 + a_1u + a_0. \tag{3.39}$$

Corresponding system of algebraic equations is:

$$\begin{gathered} a_0a_1 + ca_0a_1 - c^2a_0a_1 - c^3a_0a_1 = 0 \\ a_1^2 + ca_1^2 - c^2a_1^2 - c^3a_1^2 + 2ca_0a_2 - 2c^3a_0a_2 - 1 - c = 0 \\ 3a_1a_2 + 3ca_1a_2 - 3c^2a_1a_2 - 3c^3a_1a_2 = 0 \\ 2a_2^2 + 2ca_2^2 - 2c^2a_2^2 - 2c^3a_2^2 + 2 - 2c = 0. \end{gathered} \tag{3.40}$$

Solving this system with the help of Mathematica programming, we obtain the following coefficients:

$$a_0 = \frac{\sqrt{-(1+c)^2}}{2-2c^2}, \quad a_1 = 0, \quad a_2 = \frac{1}{\sqrt{-(1+c)^2}}. \tag{3.41}$$

Thus, we obtain

$$\begin{aligned} u_1(x,t) &= \pm\frac{c+1}{\sqrt{2-2c^2}}\tanh\left[\frac{1}{\sqrt{2\mathrm{c}^2-2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right], \\ v_1(x,t) &= \pm\frac{c(c+1)}{1-c}\tanh^2\left[\frac{1}{\sqrt{2\mathrm{c}^2-2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right], \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} u_2(x,t) &= \pm\frac{c+1}{\sqrt{2-2c^2}}\coth\left[\frac{1}{\sqrt{2\mathrm{c}^2-2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right], \\ v_2(x,t) &= \pm\frac{c(c+1)}{1-c}\coth^2\left[\frac{1}{\sqrt{2\mathrm{c}^2-2}}(\mathrm{x}-\mathrm{ct}-\eta_0)\right], \end{aligned} \tag{3.43}$$

which are the solitary wave and kink soliton solutions of the nonlinear coupled Klein-Gordon equations.

4. Conclusion

The trial equation method is studied for the nonlinear partial differential equations. By means of the method, we have obtained some traveling wave solutions to the Liouville equation, two-dimensional Bratu equation, generalized heat conduction equation and coupled nonlinear Klein-Gordon equations. Results show that the method is effective and reliable and gives several solution functions such as soliton solutions and single kink solutions.

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