MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES
4 (1) 164-174 (2016) ©MSAEN

Surfaces Family with Common Smarandache Geodesic Curve According to Bishop Frame in Euclidean Space

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(Communicated by Murat TOSUN)

Abstract
In this paper, we analyzed the problem of consructing a family of surfaces from a given some special Smarandache curves in Euclidean 3-space. Using the Bishop frame of the curve in Euclidean 3-space, we express the family of surfaces as a linear combination of the components of this frame, and derive the necessary and sufficient conditions for coefficents to satisfy both the geodesic and isoparametric requirements. Finally, examples are given to show the family of surfaces with common Smarandache geodesic curve.

Keywords: *keyword1; keyword2; keyword3; keyword4; keyword5; keyword6.*

AMS Subject Classification (2010): *Primary: 00A00 ; Secondary: 00B00; 00C00; 00D00; 00E00; 00F00.*

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1. Introduction

In differential geometry, there are many important consequences and properties of curves [1-3]. Researches follow labours about the curves. In the light of the existing studies, authors always introduce new curves. Special Smarandache curves are one of them. Special Smarandache curves have been investigated by some differential geometers [4-10]. This curve is defined as, a regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is Smarandache curve [4]. A.T. Ali has introduced some special Smarandache curves in the Euclidean space [5]. Special Smarandache curves according to Bishop Frame in Euclidean 3-space have been investigated by Çetin et al [6]. In addition, Special Smarandache curves according to Darboux Frame in Euclidean 3-space has introduced in [7]. They found some properties of these special curves and calculated normal curvature, geodesic curvature and geodesic torsion of these curves. Also, they investigate special Smarandache curves in Minkowski 3-space, [8]. Furthermore, they find some properties of these special curves and they calculate curvature and torsion of these curves. Special Smarandache curves such as -Smarandache curves according to Sabban frame in Euclidean unit sphere has introduced in [9]. Also, they give some characterization of Smarandache curves and illustrate examples of their results. On the Quaternionic Smarandache Curves in Euclidean 3-Spacehave been investigated in [10]. One of most significant curve on a surface is geodesic curve. Geodesics are important in the relativistic description of gravity. Einstein's principle of equivalence tells us that geodesics represent the paths of freely falling particles in a given space. (Freely falling in this context means moving only

Received : *05–August–2015,* ***Accepted :*** *22–March–2016*

This article is the written version of author's plenary talk delivered on September 03- August 31, 2015 at 4th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2015) in Athens, Greece.

under the influence of gravity, with no other forces involved). The geodesics principle states that the free trajectories are the geodesics of space. It plays a very important role in a geometric-relativity theory, since it means that the fundamental equation of dynamics is completely determined by the geometry of space, and therefore has not to be set as an independent equation. In architecture, some special curves have nice properties in terms of structural functionalityand manufacturing cost. One example is planar curves in vertical planes, whichcan be used as support elements. Another example is geodesic curves, [11]. Deng, B. , described methods to create patterns of special curves on surfaces, which find applications in design and realization of freeform architecture. He presented an evolution approach to generate a series of curves which are either geodesic or piecewise geodesic, starting from a given source curve on a surface.In [11], he investigated families of special curves (such as geodesics) on freeform surfaces, and propose computational tools to create such families. Also, he investigated patterns of special curves on surfaces, which find applications in design and realization of freeform architectural shapes (for details, see [11]). The concept of family of surfaces having a given characteristic curve was first introduced by Wang et.al. [12] in Euclidean 3-space. Kasap et.al. [13] generalized the work of Wang by introducing new types of marching-scale functions, coefficients of the Frenet frame appearing in the parametric representation of surfaces. With the inspiration of work of Wang, Li et.al.[14] changed the characteristic curve from geodesic to line of curvature and defined the surface pencil with a common line of curvature.

In this paper, we study the problem: given a curve (with Bishop frame), how to characterize those surfaces that posess this curve as a common isogeodesic and Smarandache curve in Euclidean 3-space. In section 2, we give some preliminary information about Smarandache curves in Euclidean 3-space and define isogeodesic curve. We express surfaces as a linear combination of the Bishop frame of the given curve and derive necessary and sufficient conditions on marching-scale functions to satisfy both isogeodesic and Smarandache requirements in Section 3. We illustrate the method by giving some examples.

2. Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [16]. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [17]). The Bishop frame is expressed as [16,18].

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} \tag{2.1}$$

Here, we shall call the set $\{T(s), N_1(s), N_2(s)\}$ as Bishop Frame and k_1 and k_2 as Bishop curvatures. The relation between Bishop Frame and Frenet Frame of curve $\alpha(s)$ is given as follows;

$$\begin{pmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2.2}$$

where

$$\begin{cases} \theta(s) = \arctan\left(\frac{k_2(s)}{k_1(s)}\right) \\ \tau(s) = -\frac{d\theta(s)}{ds} \\ \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)} \end{cases} \tag{2.3}$$

Here Bishop curvatures are defined by

$$\begin{cases} k_1(s) = \kappa(s)\cos\theta(s) \\ k_2(s) = \kappa(s)\sin\theta(s) \end{cases} \tag{2.4}$$

A curve on a surface is geodesic if and only if the normal vector to the curve is everywhere parallel to the local normal vector of the surface. Another criterion for a curve in a surface M to be geodesic is that its geodesic curvature vanishes.

An isoparametric curve $\alpha(s)$ is a curve on a surface $\Psi = \Psi(s,t)$ is that has a constant s or t-parameter value. In other words, there exist a parameter or such that $\alpha(s) = \Psi(s, t_0)$ or $\alpha(t) = \Psi(s_0, t)$.

Given a parametric curve $\alpha(s)$, we call $\alpha(s)$ an isogeodesic of a surface Ψ if it is both a geodesic and an isoparametric curve on Ψ.

Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 and $\{T(s), N_1(s), N_2(s)\}$ be its moving Bishop frame. Smarandache TN_1 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + N_1(s)\right),$

Smarandache TN_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(T(s) + N_2(s)\right),$

Smarandache N_1N_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{2}}\left(N_1(s) + N_2(s)\right),$

Smarandache TN_1N_2 curves are defined by

$\beta = \beta(s^*) = \frac{1}{\sqrt{3}}\left(T(s) + N_1(s) + N_2(s)\right)$, [5].

3. Surfaces with common Smarandache geodesic curve

Let $\varphi = \varphi(s, v)$ be a parametric surface. The surface is defined by a given curve $\alpha = \alpha(s)$ as follows:

$$\varphi(s, v) = \alpha(s) + [x(s, v)T(s) + y(s, v)N_1(s) + z(s, v)N_2(s)], L_1 \leq s \leq L_2, T_1 \leq v \leq T_2 \tag{3.1}$$

where $x(s, v)$, $y(s, v)$ and $z(s, v)$ are C^1 functions. The values of the marching-scale functions $x(s, v)$, $y(s, v)$ and $z(s, v)$ indicate, respectively; the extension-like, flexion-like and retortion-like effects, by the point unit through the time v, starting from $\alpha(s)$ and $\{T(s), N_1(s), N_2(s)\}$ is the Bishop frame associated with the curve $\alpha(s)$.

Our goal is to find the necessary and sufficient conditions for which the some special Smarandache curves of the unit space curve $\alpha(s)$ is an parametric curve and an geodesic curve on the surface $\varphi(s, v)$.

Firstly, since Smarandache curve of $\alpha(s)$ is an parametric curve on the surface $\varphi(s, v)$, there exists a parameter $v_0 \in [T_1, T_2]$ such that

$$x(s, v_0) = y(s, v_0) = z(s, v_0) = 0,\ L_1 \leq s \leq L_2\ ,\ T_1 \leq v \leq T_2\ . \tag{3.2}$$

Secondly,since Smarandache curve of $\alpha(s)$ is an geodesic curve on the surface $v_0 \in [T_1, T_2]$, there exist a parameter $v_0 \in [T_1, T_2]$ such that

$$n(s, v_0) \parallel N(s) \tag{3.3}$$

where n is a normal vector of $v_0 \in [T_1, T_2]$ and N is a normal vector of $\alpha(s)$.

Theorem 3.1. *Smarandache N_1N_2 curve of the curve $\alpha(s)$ is isogeodic on a surface $\varphi(s, v)$ if and only if the following conditions are satisfied:*

$$\begin{cases} x(s, v_0) = y(s, v_0) = z(s, v_0) = 0, \\ k_1(s) + k_2(s) \neq 0 \\ \frac{\partial z}{\partial v}(s, v_0) \neq 0 \\ \frac{\partial y}{\partial v}(s, v_0) = -\tan\theta(s)\, \frac{\partial z}{\partial v}(s, v_0) \end{cases}$$

Proof. Let $\alpha(s)$ be a Smarandache N_1N_2 curve on surface $\varphi(s, v)$.From (3.1) , $\varphi(s, v)$, parametric surface is defined by a given Smarandache TN_1 curve of curve $\alpha(s)$ as follows:

$$\varphi(s, v) = \frac{1}{\sqrt{2}}\left(N_1(s) + N_2(s)\right) + [x(s, v)T(s) + y(s, v)N_1(s) + z(s, v)N_2(s)].$$

If Smarandache N_1N_2 curve is an parametric curve on this surface , then there exist a parameter $v = v_0$ such that , $\frac{1}{\sqrt{2}}\left(N_1(s) + N_2(s)\right) = \varphi(s, v_0)$, that is,

$$x(s, v_0) = y(s, v_0) = z(s, v_0) = 0 \tag{3.4}$$

The normal vector of $\varphi(s, v)$ can be written as

$$n(s, v) = \frac{\partial\varphi(s, v)}{ds} \times \frac{\partial\varphi(s, v)}{\partial v}$$

The normal vector can be expressed as

$$
\begin{aligned}
n(s,v) \quad = \quad & [\frac{\partial z(s,v)}{dv}(\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)-\frac{\partial y(s,v)}{dv}(\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\
& +[\frac{\partial x(s,v)}{dv}(\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))-\frac{\partial z(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{2}} \\
& +\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s)+[\frac{\partial y(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds} \\
& -k_1(s)y(s,v)-k_2(s)z(s,v))-\frac{\partial x(s,v)}{dv}(\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \qquad (3.5)
\end{aligned}
$$

Using (3.4), if we let

$$
\begin{cases}
\Phi_1(s,v_0)=0, \\
\Phi_2(s,v_0)=\left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial z}{dv}(s,v_0), \\
\Phi_3(s,v_0)=\left(\frac{k_1(s)+k_2(s)}{\sqrt{2}}\right)\frac{\partial y}{dv}(s,v_0).
\end{cases} \tag{3.6}
$$

we obtain

$$
n(s,v_0)=\Phi_2(s,v_0)N_1(s)+\Phi_3(s,v_0)N_2(s).
$$

From Eqn. (2.2), we get

$$
\begin{aligned}
n(s,v_0) \quad = \quad & (\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\
& +(\cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0))B(s).
\end{aligned}
$$

From the eqn. (3.2), we should have

$$
\begin{aligned}
\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0) \quad & \neq \quad 0 \\
\cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0) \quad & = \quad 0
\end{aligned} \tag{3.7}
$$

From (3.7), we have

$$
\Phi_3(s,v_0)=\tan\theta\,(s)\,\Phi_2(s,v_0) \tag{3.8}
$$

Using (3.8) and $\cos\theta\,(s)\,\Phi_2$(s,v_0)+$\sin\theta\,(s)\,\Phi_3$(s,v_0)≠0, we have

$$
\Phi_2(s,v_0)\neq 0 \tag{3.9}
$$

In eqn.(3.8), using (3.6) we obtain □

$$
\frac{\partial y}{\partial v}(s,v_0)=-\tan\theta\,(s)\,\frac{\partial z}{\partial v}(s,v_0) \tag{3.10}
$$

In eqn.(3.9), using (3.6) we obtain

$$
k_1(s)+k_2(s)\neq 0 \text{ and } \frac{\partial z}{\partial v}(s,v_0)\neq 0 \tag{3.11}
$$

Combining the conditions (3.4) ,(3.10) and (3.11) , we have found the necessary and sufficient conditions for the $\varphi(s,v)$ to have the Smarandache N_1N_2 curve of the curve is an isogeodesic·

Let $\alpha(s)$ be a Smarandache TN_1 curve on a surface $\varphi(s,v)$. Thus, from (3.1) ,φ(s,v) parametric surfaces is defined by a given Smarandache TN_1 curve of $\alpha(s)$ as follows:

$$
\varphi(s,v)=\frac{1}{\sqrt{2}}\,(T(s)+N_1(s))+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].
$$

If Smarandache TN_1 curve is an parametric curve on this surface , then there exist a parameter $v=v_0$ such that , $\frac{1}{\sqrt{2}}\,(T(s)+N_1(s))=\varphi(s,v_0)$, that is,

$$
x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \tag{3.12}
$$

The normal vector can be expressed as

$$\begin{aligned} n(s,v) &= [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\ &-\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\ &+[\frac{\partial x(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))-\frac{\partial z(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds} \\ &-k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s)+[\frac{\partial y(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds} \\ &-k_1(s)y(s,v)-k_2(s)z(s,v)) \\ &-\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.13}$$

Using (3.12), if we let

$$\begin{cases} \Phi_1(s,v_0)=\frac{k_1(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)-\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0), \\ \Phi_2(s,v_0)=\frac{k_2(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)+\frac{k_2(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0), \\ \Phi_3(s,v_0)=-\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0)-\frac{k_1(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0). \end{cases} \tag{3.14}$$

we obtain

$$n(s,v_0)=\Phi_1(s,v_0)T(s)+\Phi_2(s,v_0)N_1(s)+\Phi_3(s,v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s,v_0) &= \Phi_1(s,v_0)T(s)+(\cos\theta(s)\,\Phi_2(s,v_0)+\sin\theta(s)\,\Phi_3(s,v_0))N(s) \\ &+(\cos\theta(s)\,\Phi_3(s,v_0)-\sin\theta(s)\,\Phi_2(s,v_0))B(s). \end{aligned}$$

We know that $\alpha(s)$ is a geodesic curve if and only if

$$\begin{cases} \Phi_1(s,v_0)=0, \\ \cos\theta(s)\,\Phi_2(s,v_0)+\sin\theta(s)\,\Phi_3(s,v_0)\neq 0 \\ \cos\theta(s)\,\Phi_3(s,v_0)-\sin\theta(s)\,\Phi_2(s,v_0)=0 \end{cases} \tag{3.15}$$

From eqns. (3.6) and (3.7) ,we obtain

$$\tan\theta(s)=-\frac{k_1(s)}{k_2(s)}. \tag{3.16}$$

Using eqns. (2.4) and (3.16), we have

$$\frac{1}{\cos\theta(s)\sin\theta(s)}=0 \tag{3.17}$$

that is not possible.Thus , Smarandache TN_1 curve of unit speed curve $\alpha=\alpha(s)$, is not geodesic on the surface $\varphi(s,v)$.

Let $\alpha(s)$ be a Smarandache TN_2 curve on a surface $\varphi(s,v)$. Thus, from (3.1) ,φ(s,v) parametric surfaces is defined by a given Smarandache TN_2 curve of $\alpha(s)$ as follows:

$$\varphi(s,v)=\frac{1}{\sqrt{2}}(T(s)+N_2(s))+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].$$

If Smarandache TN_2 curve is an parametric curve on this surface , then there exist a parameter $v=v_0$ such that , $\frac{1}{\sqrt{2}}(T(s)+N_2(s))=\varphi(s,v_0)$, that is,

$$x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \tag{3.18}$$

The normal vector can be expressed as

$$
\begin{aligned}
n(s,v) \quad = \quad & [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\
& -\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\
& +[\frac{\partial x(s,v)}{dv}(\frac{k_2(s)}{\sqrt{2}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))-\frac{\partial z(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds} \\
& -k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s)+[\frac{\partial y(s,v)}{dv}(-\frac{k_2(s)}{\sqrt{2}}+\frac{\partial x(s,v)}{ds} \\
& -k_1(s)y(s,v)-k_2(s)z(s,v))-\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{2}}+\frac{\partial y(s,v)}{ds} \\
& +k_1(s)x(s,v))]N_2(s)
\end{aligned} \tag{3.19}
$$

Using (3.12), if we let

$$
\begin{cases}
\Phi_1(s,v_0)=\frac{k_1(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)-\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0), \\
\Phi_2(s,v_0)=\frac{k_2(s)}{\sqrt{2}}\frac{\partial z}{dv}(s,v_0)+\frac{k_2(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0), \\
\Phi_3(s,v_0)=-\frac{k_2(s)}{\sqrt{2}}\frac{\partial y}{dv}(s,v_0)-\frac{k_1(s)}{\sqrt{2}}\frac{\partial x}{dv}(s,v_0).
\end{cases} \tag{3.20}
$$

we obtain

$$
n(s,v_0)=\Phi_1(s,v_0)T(s)+\Phi_2(s,v_0)N_1(s)+\Phi_3(s,v_0)N_2(s).
$$

From Eqn. (2.2), we get

$$
\begin{aligned}
n(s,v_0) \quad = \quad & \Phi_1(s,v_0)T(s)+(\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\
& +(\cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0))B(s).
\end{aligned}
$$

We know that $\alpha(s)$ is a geodesic curve if and only if
Φ_1(s,v_0)=0,we have

$$
\frac{\partial z}{dv}(s,v_0)=\frac{k_2(s)}{k_1(s)}\frac{\partial y}{dv}(s,v_0) \tag{3.21}
$$

$\cos\theta\,(s)$ Φ_3(s,v_0)-$\sin\theta\,(s)$ Φ_2(s,v_0)=0,we have

$$
\Phi_3(s,v_0)=\tan\theta\,(s)\,\Phi_2(s,v_0) \tag{3.22}
$$

$\cos\theta\,(s)$ Φ_3(s,v_0)-$\sin\theta\,(s)$ Φ_2(s,v_0)$\neq 0$,in using (3.22), we have

$$
k_2\,(s)\neq 0 \text{ and } \frac{\partial x}{dv}(s,v_0)\neq -\frac{\partial z}{dv}(s,v_0) \tag{3.23}
$$

In eqn.(3.23) , using (3.20), (3.21) and $\tan\theta\,(s)=\frac{k_2(s)}{k1(s)}$,we obtain

$$
\frac{\partial x}{dv}(s,v_0)=-\frac{\partial z}{dv}(s,v_0) \tag{3.24}
$$

In equations (3.23) and (3.24) ,the contradiction is obtained. Thus, Smarandache TN_2 curve of unit speed curve $\alpha=\alpha(s)$, is not geodesic on the surface $\varphi(s,v)$.

Let $\alpha(s)$ be a Smarandache TN_1N_2 curve on surface $\varphi(s,v)$.From (3.1) , $\varphi(s,v)$ parametric surface is defined by a given Smarandache TN_1N_2 curve of curve $\alpha(s)$ as follows:

$$
\varphi(s,v)=\frac{1}{\sqrt{2}}\,(T(s)+N_1(s)+N_2(s))+[x(s,v)T(s)+y(s,v)N_1(s)+z(s,v)N_2(s)].
$$

If Smarandache TN_1N_2 curve is an parametric curve on this surface , then there exist a parameter $v=v_0$ such that , $\frac{1}{\sqrt{2}}\,(T(s)+N_1(s)+N_2(s))=\varphi(s,v_0)$, that is,

$$
x(s,v_0)=y(s,v_0)=z(s,v_0)=0 \tag{3.25}
$$

The normal vector can be expressed as

$$\begin{aligned} n(s,v) &= [\frac{\partial z(s,v)}{dv}(\frac{k_1(s)}{\sqrt{3}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v)) \\ &\quad -\frac{\partial y(s,v)}{dv}(\frac{k_2(s)}{\sqrt{3}}+\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v))]T(s) \\ &\quad +[\frac{\partial x(s,v)}{dv}(\frac{\partial z(s,v)}{ds}+k_2(s)x(s,v)+\frac{k_2(s)}{\sqrt{3}}) \\ &\quad -\frac{\partial z(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{3}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v))]N_1(s) \\ &\quad +[\frac{\partial y(s,v)}{dv}(-\frac{k_1(s)+k_2(s)}{\sqrt{3}}+\frac{\partial x(s,v)}{ds}-k_1(s)y(s,v)-k_2(s)z(s,v)) \\ &\quad -\frac{\partial x(s,v)}{dv}(\frac{k_1(s)}{\sqrt{3}}+\frac{\partial y(s,v)}{ds}+k_1(s)x(s,v))]N_2(s) \end{aligned} \tag{3.26}$$

Using (3.25), if we let

$$\begin{cases} \Phi_1(s,v_0)=\frac{k_1(s)}{\sqrt{3}}\frac{\partial z}{dv}(s,v_0)-\frac{k_2(s)}{\sqrt{3}}\frac{\partial y}{dv}(s,v_0), \\ \Phi_2(s,v_0)=\frac{k_2(s)}{\sqrt{3}}\frac{\partial x}{dv}(s,v_0)+\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial z}{dv}(s,v_0), \\ \Phi_3(s,v_0)=-\frac{k_1(s)}{\sqrt{3}}\frac{\partial x}{dv}(s,v_0)-\left(\frac{k_1(s)+k_2(s)}{\sqrt{3}}\right)\frac{\partial y}{dv}(s,v_0). \end{cases} \tag{3.27}$$

we obtain

$$n(s,v_0)=\Phi_1(s,v_0)T(s)+\Phi_2(s,v_0)N_1(s)+\Phi_3(s,v_0)N_2(s).$$

From Eqn. (2.2), we get

$$\begin{aligned} n(s,v_0) &= \Phi_1(s,v_0)T(s)+\cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0))N(s) \\ &\quad +(\cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0))B(s). \end{aligned}$$

We know that $\alpha(s)$ is a geodesic curve if and only if

$$\begin{cases} \Phi_1(s,v_0)=0, \\ \cos\theta\,(s)\,\Phi_2(s,v_0)+\sin\theta\,(s)\,\Phi_3(s,v_0)\neq 0 \\ \cos\theta\,(s)\,\Phi_3(s,v_0)-\sin\theta\,(s)\,\Phi_2(s,v_0)=0 \end{cases} \tag{3.28}$$

In (3.28), using (3.27) we obtain

$$\frac{\partial z}{dv}(s,v_0)=\frac{k_2(s)}{k_1(s)}\frac{\partial y}{dv}(s,v_0)\ ,k_1(s)\neq 0 \tag{3.29}$$

$$k_2(s)\frac{\partial x(s,v_0)}{dv}+(k_1(s)+k_2(s))\,\frac{\partial z(s,v_0)}{dv}\neq 0 \tag{3.30}$$

$$-\,\frac{(k_1(s)+k_2(s))}{k_1(s)}\frac{\partial y(s,v_0)}{dv}=\frac{\partial x(s,v_0)}{dv},k_1(s)\neq 0 \tag{3.31}$$

In (3.30) , using (3.29) and (3.31) we have

$$-\frac{k_2(s)\,(k_1(s)+k_2(s))}{k_1(s)}\frac{\partial y(s,v_0)}{dv}+\frac{k_2(s)\,(k_1(s)+k_2(s))}{k_1(s)}\frac{\partial y(s,v_0)}{dv}\neq 0$$

that is not possible.Thus , Smarandache TN_1N_2 curve of unit speed curve $\alpha=\alpha(s)$, is not geodesic on the surface $\varphi(s,v)$.

Thus, we can give the following results:

Conclusion 1. *Smarandache TN_1, TN_2 and TN_1N_2 curves of unit speed curve $\alpha=\alpha(s)$, is not geodesic on the surface .*

Now let us consider other types of the marching-scale functions. In the Eqn. (3.1) marching-scale functions $x(s,v)$, $y(s,v)$ and $z(s,v)$ can be choosen in two different forms:

1) If we choose

$$\begin{cases} x(s,v)=\sum\limits_{k=1}^{p}a_{1k}l(s)^{k}x(v)^{k}, \\ y(s,v)=\sum\limits_{k=1}^{p}a_{2k}m(s)^{k}y(v)^{k}, \\ z(s,v)=\sum\limits_{k=1}^{p}a_{3k}n(s)^{k}z(v)^{k}. \end{cases}$$

then we can simply express the sufficient condition for which the curve $\alpha(s)$ is an Smarandache N_1N_2 isogeodesic curve on the surface $\varphi(s,v)$ as

$$\begin{cases} x(v_0)=y(v_0)=z(v_0)=0, \\ a_{31}\neq 0\ ,n(s)\neq 0 \text{ and } \frac{dz(v_0)}{dv}\neq 0. \\ a_{21}m(s)\frac{dy(v_0)}{dv}=-\tan\theta\left(s\right)a_{31}n(s)\frac{dz(v_0)}{dv} \end{cases} \tag{3.32}$$

where $l(s)$, $m(s)$, $n(s)$, $x(v)$, $y(v)$ and $z(v)$ are C^1 functions, $a_{ij}\epsilon\ IR$, $i=1,2,3,\ j=1,2,...,p$.

2) If we choose

$$\begin{cases} x(s,v)=f\left(\sum\limits_{k=1}^{p}a_{1k}l(s)^{k}x(v)^{k}\right), \\ y(s,v)=g\left(\sum\limits_{k=1}^{p}a_{2k}m(s)^{k}y(v)^{k}\right), \\ z(s,v)=h\left(\sum\limits_{k=1}^{p}a_{3k}n(s)^{k}z(v)^{k}\right). \end{cases}$$

then we can write the sufficient condition for which the curve $\alpha(s)$ is an Smarandache N_1N_2 isogeodesic curve on the surface $\varphi(s,v)$ as

$$\begin{cases} x(v_0)=y(v_0)=z(v_0)=f(0)=g(0)=h(0)=0, \\ a_{31}\neq 0\ ,n(s)\neq 0\ ,\ \frac{dz(v_0)}{dv}\neq 0 \text{ and } h^{'}(0)\neq 0 \\ a_{21}m(s)\frac{dy(v_0)}{dv}g'(0)=-\tan\theta\left(s\right)a_{31}n(s)\frac{dz(v_0)}{dv}h^{'}(0) \end{cases} \tag{3.33}$$

where $l(s)$, $m(s)$, $n(s)$, $x(v)$, $y(v)$, $z(v)$, f, g and h are C^1 functions.

Also conditions for different types of marching-scale functions can be obtained by using the Eqn. (3.4) and (3.9).

Example 3.1. Let $\alpha(s)=\left(\frac{3}{5}\cos(s),\frac{3}{5}\sin(s),\frac{4}{5}\right)$ be a unit speed curve. Then it is easy to show that
$T(s)=\left(-\frac{3}{5}\sin(s),\frac{3}{5}\cos(s),0\right)$, $\kappa=\frac{3}{5}$, $\tau=\frac{4}{5}$.

From Eqn.(2.3), $\tau\left(s\right)=-\frac{d\theta(s)}{ds}\Rightarrow\theta\left(s\right)=\frac{4s}{5}, c=cons\tan t$. Here $c=0$ can be taken.
From Eqn. (2.4),$k_1=\frac{3}{5}\cos(\frac{4s}{5}), k_2=\frac{3}{5}\sin(\frac{4s}{5})$.
From Eqn. (2.1),$N_1=-\int k_1T$, $N_2=-\int k_2T$.
$N_1(s)=\left(-\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5}),-\frac{1}{10}\sin(\frac{9s}{5})-\frac{9}{10}\sin(\frac{s}{5}),-\frac{3}{5}\sin(\frac{4s}{5})\right)$,
$N_2(s)=\left(\frac{9}{10}\sin(\frac{s}{5})-\frac{1}{10}\sin(\frac{s}{5}),\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5}),\frac{3}{5}\cos(\frac{4s}{5})\right)$.

If we take $x(s,v)=0,\ y(s,v)=-\tan(\frac{4s}{5})\sin(v),\ z(s,v)=\sin(v)$,we obtain a member of the surface with common curve $\alpha(s)$ as

$$\varphi_1(s,v)=\begin{pmatrix} \frac{3}{5}\cos(s)-\left(\tan(\frac{4s}{5})\sin(v)\right)\left(-\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right)+\sin(v)\left(\frac{9}{10}\sin(\frac{s}{5})-\frac{1}{10}\sin(\frac{s}{5})\right), \\ \frac{3}{5}\sin(s)-\left(\tan(\frac{4s}{5})\sin(v)\right)\left(-\frac{1}{10}\sin(\frac{9s}{5})-\frac{9}{10}\sin(\frac{s}{5})\right)+\sin(v)\left(\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right), \\ \frac{4s}{5}+\frac{3}{5}\tan(\frac{4s}{5})\sin(v)\sin(\frac{4s}{5})+\frac{3}{5}\sin(v)\cos(\frac{4s}{5}) \end{pmatrix}$$

where $0\leq s\leq 2\pi$, $-4\leq v\leq 4$ (Fig. 1).

If we take $x(s,v)=0,\ y(s,v)=-\tan(\frac{4s}{5})\sin(v),\ z(s,v)=\sin(v)$ and $v_0=0$ then the Eqns.(3.4) and (3.7) are satisfied. Thus, we obtain a member of the surface with common Smarandache N_1N_2 geodesic curve as

$$\varphi_2(s,v)=\begin{pmatrix} -\frac{1}{10\sqrt{2}}\left(\cos(\frac{9s}{5})+\sin(\frac{9s}{5})\right)+9\left(\cos(\frac{s}{5})-\sin(\frac{s}{5})\right)- \\ \left(\tan(\frac{4s}{5})\sin(v)\right)\left(-\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right) \\ +\sin(v)\left(\frac{9}{10}\sin(\frac{s}{5})-\frac{1}{10}\sin(\frac{9s}{5})\right), \\ -\frac{1}{10\sqrt{2}}\left(\sin(\frac{9s}{5})-\cos(\frac{9s}{5})\right)+9\left(\cos(\frac{s}{5})+\sin(\frac{s}{5})\right) \\ -\left(\tan(\frac{4s}{5})\sin(v)\right)\left(\frac{-1}{10}\sin(\frac{9s}{5})-\frac{9}{10}\sin(\frac{s}{5})\right)+\sin(v)\left(\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right), \\ -\frac{3}{5\sqrt{2}}\left(\sin(\frac{4s}{5})+\cos(\frac{4s}{5})\right)+\frac{3}{5}\sin(v)\left(\tan(\frac{4s}{5})\sin(\frac{4s}{5})+\cos(\frac{4s}{5})\right) \end{pmatrix}$$

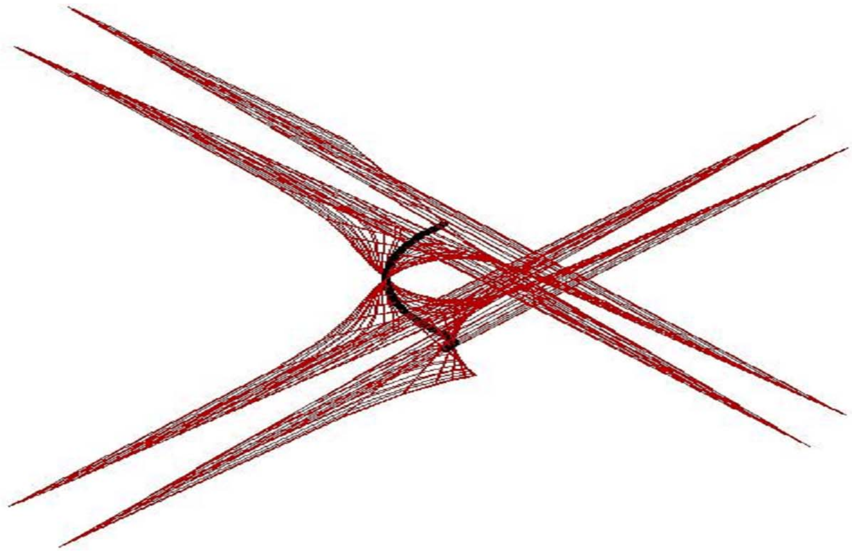

Figure 1. $\varphi_1(s,v)$ as a member of surfaces and curve $\alpha(s)$.

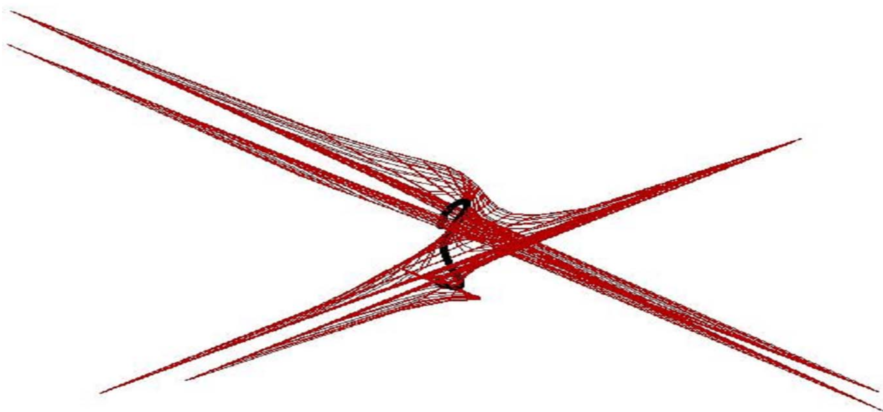

Figure 2. $\varphi_2(s,v)$ as a member of surfaces and its Smarandache N_1N_2 geodesic curve of $\alpha(s)$.

where $0 \leq s \leq 2\pi$, $-4 \leq v \leq 4$ (Fig. 2).

If we take $x(s,v)=0,\ y(s,v)=-\sum_{k=1}^{3}\left(\tan(\frac{4s}{5})\cos(s)\right)^k\sin^k(v),\ z(s,v)=\sum_{k=1}^{3}\cos^k(s)\sin^k(v)$ and $v_0=0$ then the Eqn.(3.32) is satisfied. Thus, we obtain a member of the surface with common Smarandache N_1N_2 geodesic curve as

$$\varphi_3(s,v)=\begin{pmatrix} -\frac{1}{10\sqrt{2}}\left(\cos(\frac{9s}{5})+\sin(\frac{9s}{5})\right)+9\left(\cos(\frac{s}{5})-\sin(\frac{s}{5})\right)-\sum_{k=1}^{3}\left(\tan(\frac{4s}{5})\cos(s)\right)^k\sin^k(v) \\ \left(-\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right)+\sum_{k=1}^{3}\cos^k(s)\sin^k(v)\ \left(\frac{9}{10}\sin(\frac{s}{5})-\frac{1}{10}\sin(\frac{9s}{5})\right), \\ -\frac{1}{10\sqrt{2}}\left(\sin(\frac{9s}{5})-\cos(\frac{9s}{5})\right)+9\left(\cos(\frac{s}{5})+\sin(\frac{s}{5})\right) \\ -\sum_{k=1}^{3}\left(\tan(\frac{4s}{5})\cos(s)\right)^k\sin^k(v)\left(\frac{-1}{10}\sin(\frac{9s}{5})-\frac{9}{10}\sin(\frac{s}{5})\right) \\ +\sum_{k=1}^{3}\cos^k(s)\sin^k(v)\ \left(\frac{1}{10}\cos(\frac{9s}{5})-\frac{9}{10}\cos(\frac{s}{5})\right), -\frac{3}{5\sqrt{2}}\left(\sin(\frac{4s}{5})-\cos(\frac{4s}{5})\right) \\ -\frac{3}{5}\sin(\frac{4s}{5})\sum_{k=1}^{3}\left(\tan(\frac{4s}{5})\cos(s)\right)^k\sin^k(v)+\frac{3}{5}\cos(\frac{4s}{5})\sum_{k=1}^{3}\cos^k(s)\sin^k(v) \end{pmatrix}$$

where $\frac{-\pi}{9} \leq s \leq \frac{5\pi}{9}$, $-1 \leq v \leq 1$ (Fig. 3).

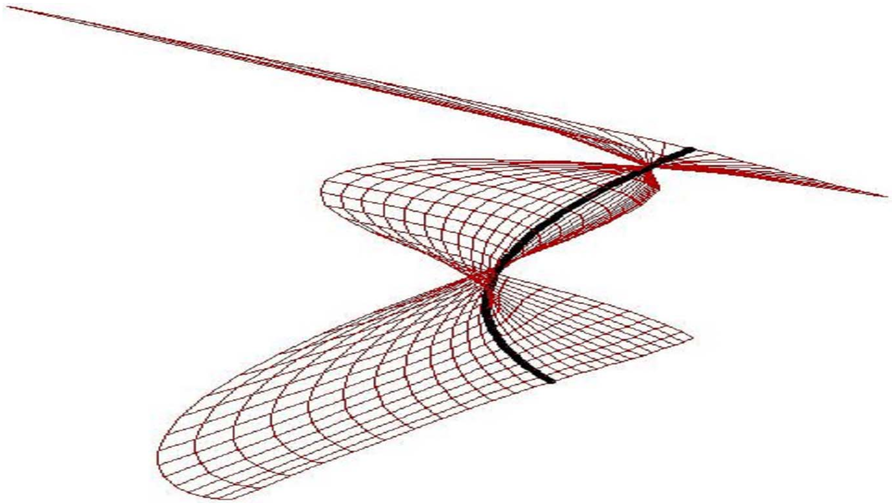

Figure 3. $\varphi_3(s,v)$ as a member of surfaces and its Smarandache N_1N_2 geodesic curve of $\alpha(s)$.

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