Mathematical Sciences and Applications E-Notes
4 (2) 71-78 (2016) ©**MSAEN**

The Application of Kolmogorov's theorem in the one-default model

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(Communicated by Kamile ŞANLI KULA)

Abstract
The problem of modeling a default time is well represented in the literature. There are two main approaches: either the default time τ is a stopping time in the asset's filtration or it is a stopping time in a larger filtration (see Coper and Martin 1996 for a comparison between these approaches). In the first approach, the so-called structural form pioneered by Merton (1974), the default time τ is a stopping time in the filtration of the prices. In the second case, the idea is also to compute the value of the defaultable claim, and it is fruitful to introduce the notion of intensity of the default. The intensity of the default time acts as a change of the spot interest rate in the pricing formula. In this work, we consider the so-called $\natural$-model. It is a one-default model which gives the conditional law of a random time with respect to a reference filtration. In this paper, we work on a stochastic differential equation (called equation ($\natural$) below); this equation plays an essential role in this article, but its application has been submitted to a hypothesis of continuity. Then it is important to know under what conditions the hypothesis of continuity is satisfied. This is the main motivation of our research.

Keywords: *Progressive enlargement of filtration; Semimartingale decomposition; Multiplicative decomposition; Credit risk.*

AMS Subject Classification (2010): *Primary: 60G17 ; Secondary: 60H05.*

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1. Introduction

We consider one-default model, i.e. the data of a random time τ combined with a filtration $\mathbb{F}$ under a probability measure $\mathbb{Q}$. The one-default models are widely applied in modeling financial risk and in price valuation of financial products such as Credit default swap. The usefulness of a one-default model depends upon the way the conditional laws of τ can be computed with respect to the filtration $\mathbb{F}$. The most used examples of random times, therefore, are the independent time, the Cox time, the honest time, the pseudo stopping time, the initial time, etc (for example,[9–11]). In the paper [8] a new class of random times has been introduced. Precisely, on the filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, it is proved that, for any continuous increasing process Λ null at the origin, for any continuous non-negative local martingale N such that $0 < N_t e^{-\Lambda_t} < 1, t > 0$, for any continuous local martingale Y, for any Lipschitz function f on $\mathbb{R}$ null at the origin, there exists a random variable τ such that the family of conditional expectations $X^u_t = \mathbb{Q}[\tau \leq |\mathcal{F}_t]$, $u > 0$, $t < \infty$, satisfy the following stochastic differential equation :

$$(\natural_u) : \begin{cases} dX_t = X_t \left(-\dfrac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(X_t - (1 - Z_t)) dY_t \right), & t \in [u, \infty) \\ X_u = x \end{cases}$$

We call this setting a $\natural$-model, where the initial condition x can be any $\mathcal{F}_u$-mesurable random variable.

Received : *08–April–2015,* ***Accepted :*** *04–October–2016*

There are two remarkable properties about the $\natural$-model. It is the only one in which the conditional laws of τ with respect to $\mathbb{F}$ are defined by a system of dynamic equations. The $\natural$-equation displays the evolution of the defaultable market. The knowledge of market evolution is a valuable property. This evolution form of the $\natural$-model had allowed [8] to establish the so-called enlargement of filtration formula. It is also proved in [8] that, reciprocally, the $\natural$-equation can be recovered from the enlargement of filtration formula in a way similar to that a differentiable function can be deduced from its derivative.

We recall that the formula of enlargement of filtration is essential, when the no-arbitrage price valuation is considered in an one-default model. Much as the enlargement of filtration formula is universally valid before the default time τ, for a long time, the part of the enlargement of filtration formula after τ was merely proved for the honest time model or the initial time model. The $\natural$-models constitute the third family of models where the enlargement of filtration formula is valid on the whole $\mathbb{R}^+$. In addition, the enlargement of filtration formula in the $\natural$-model has a richer structure than that of honest time model, and has a more accurate expression than that of the initial time model.

We recall also how widely the financial models are defined by stochastic differential equations, because it is one of the best ways to represent the evolution of a financial market. Usually, in a one-default model, there is no such a possibility to design the evolution. Now with the $\natural$-model, this becomes available.

The second remarkable property of the $\natural$-model is its rich and flexible system of parameters Z, Y, f. The parameter Z determines the default intensity. The parameters Y and f describe the evolution of the market after the default time τ. Such a system of parameters sets up a propitious framework for inferring the market behavior and for calibrating the financial data. We believe that the $\natural$-model can be a useful instrument to modeling financial market.

In this paper, we want to show that the continuity of the process $X_t^u(x)$ such as :

$$X_t^u(x) = x + \int_u^t X_s \left(-\frac{e^{-\Lambda_s}}{1-Z_s} \right) dN_s + \int_u^t X_s f(X_s - (1-Z_s)) dY_s \,, u \leq s \leq t$$

is the solution of the equation $(\natural_u)$.

Our aim is to look at the regularity of the process $(u,t,x) \longmapsto X_t^u(x)$ with respect to all the variables u, t, x. Our useful fundamentals are the theorem of Kolmogorov and the lemma of Gronwall.

The continuity of the stochastic flow has been studied by Philip E.Protter (see [1]) for a general system of equations in the form $X_t^x = H_t^x + \int_0^t F(X^x)_{s\_} dZ_s$, where X_t^x and H_t^x are column vectors in $\mathbb{R}^n$, Z is a column vector of m semimartingales, and F is an $n \times m$ matrix. His study is a direct application of Kolmogorov's lemma.

The same study was also done by H.Kunita (see [2]) but with a detailed proof, for a general system of equations of the form $\xi_{st}^m(x) = x + \sum_{k=0}^{m} \int_0^t V_k(r, \xi_{sr}(x)) dB_r^k$, where V_k is a family of vector fields on $\mathbb{R}^d$ and B^k is a family of standard brownian motions, such that this result has been based on Kolmogorov's lemma, Itô's formula and the inequality of Burkholder-Davis-Gundy.

A technical proof based on the use of the Kolmogorov's lemma has been also done by G.Barles and Bernt Oksendal (see [3, 4]), for a general system $X_t = Z + \int_0^t \alpha(r, X_r) dr + \int_0^t \sigma(r, X_r) dW_r$, where α and σ are measurable functions, Z is a square integrable random variable and W is a d-dimensional brownian motion.

The paper is organized as follows. In the next section, we prove the theorem of Kolmogorov, Gronwall's lemma, and an other lemma to finish our calculus. Section 3 presents the main result of this paper.

2. The Kolmogorov's theorem and basic lemmas

There are several versions of Kolmogorov's theorem; we give here a quite general one.
Theorem 2.1. [8]. Let (E, d) be a complete metric space, and let U^x be an E-valued random variable for all x dyadic rationals in $\mathbb{R}^n$. Suppose that for all x and y, we have $d(U^x, U^y)$ which is a random variable and that there exist strictly positive constants ε, C, β such that

$$\mathbb{E}\{d(U^x, U^y)^\varepsilon\} \leq C\|x-y\|^{n+\beta}$$

Then for almost all ω the function $x \longmapsto U^x$ can be extended uniquely to a continuous function from $\mathbb{R}^n$ to E.

Proof. We prove the theorem for the unit cube $[0,1]^n$. Before the statement of the theorem we establish some notations. Let Δ denote the dyadic rational points of the unit cube $[0,1]^n$ in $\mathbb{R}^n$, and let Δ_m denote all $x \in \Delta$ whose coordinates are of the form $k2^{-m}, 0 \leq k \leq 2^m$. Two points x and y in Δ_m are neighbors if $\sup_i |x^i y^i| = 2^{-m}$. We use Chebyshev's inequality on the inequality hypothesized to get

$$\mathbb{P}\{d(U^x, U^y) \geq 2^{-\alpha m}\} \leq C 2^{\alpha\varepsilon m} 2^{-m(n+\beta)}$$

Let

$$\Lambda_m = \{\omega : \exists\, neighbors\ x, y \in \Delta_m\ with\ d(U^x(\omega), U^y(\omega)) \geq 2^{-\alpha m}\}$$

Since each $x \in \Delta_m$ has at most 3^n neighbors, and the cardinality of Δ_m is 2^{mn}, we have

$$\mathbb{P}(\Lambda_m) \leq c 2^{m(\alpha\varepsilon - \beta)}$$

Where the constant $c = 3^n C$. Take α a sufficiently small so that $\alpha\varepsilon < \beta$. Then

$$\mathbb{P}(\Lambda_m) \leq c 2^{-m\delta}$$

Where $\delta = \beta - \alpha\varepsilon > 0$. The Borel-Cantelli lemma then implies $\mathbb{P}(\Lambda_m\ infinitely\ often) = 0$. In other words, there exists an m_0 such that for $m \geq m_0$ and every pair (u, v) of points of Δ_m that are neighbors,

$$d(U^u, U^v) \leq 2^{-\alpha m}$$

We now use the preceding to show that $x \longmapsto U^x$ is uniformly continuous on Δ and hence extendable uniquely to a continuous function on $[0,1]^n$. To this end, let $x, y \in \Delta$ be such that $\|x-y\| \leq 2^{-k-1}$. We will show that $d(U^x, U^y) \leq c 2^{-\alpha k}$ for a constant c, and this will complete the proof.

Without loss of generality assume $k \geq m_0$. Then $x = (x^1, ..., x^n)$ and $y = (y^1, ..., y^n)$ in Δ with $\|x-y\| \leq 2^{-k-1}$ have dyadic expansions of the form

$$x^i = u^i + \sum_{j>k} a^i_j 2^{-j}$$

$$y^i = v^i + \sum_{j>k} b^i_j 2^{-j}$$

where a^i_j, b^i_j are each 0 or 1 and u,v are points of Δ_k which are either equal or neighbors. Next set $u_0 = u, u_1 = u_0 + a_{k+1} 2^{-k-1}, u_2 = u_1 + a_{k+2} 2^{-k-2}$,...
We also make analogous definitions for $v_0, v_1, v_2, ...$ then u_{i-1} and u_i are equal or neighbors in Δ_{k+i} each i, and analogously for v_{i-1} and v_i. Hence

$$d(U^x(\omega), U^u(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j}$$

$$d(U^y(\omega), U^v(\omega)) \leq \sum_{j=k}^{\infty} 2^{-\alpha j}$$

and moreover

$$d(U^u(\omega), U^v(\omega)) \leq 2^{-\alpha k}$$

The result now follows by the triangle inequality. □

The following section is the heart of our article. To show our main result, we need the following lemmas :

Lemma 2.1. *[6]. Let $a(t)$ be a non-negative right-continuous increasing (extented real-valued) function on $\mathbb{R}_+$.Set*

$$C(t) = \inf\{s : a(s) > t\},\ t \in \mathbb{R}_+$$

Then $C(t)$ is a non-negative right-continuous increasing function on $\mathbb{R}_+$, and is called the right-inverse function of $a(t)$. For $t \in \mathbb{R}_+$,$C(t) < +\infty$ if and only if $t < a(\infty) = \lim_{t \longrightarrow \infty} a(t)$. Set

$$a_-(t) = a(t-) = \lim_{s \uparrow\uparrow t} a(s),\ t > 0 \text{ (such that } s \uparrow\uparrow t \text{ means } s \longrightarrow t, s < t\text{)},$$
$$C_-(t) = C(t-) = \lim_{s\uparrow\uparrow t} C(s) = \inf\{s : a(s) \geq t\} = \sup\{s : a(s) < t\}, t > 0,$$
$$a(0-) = a(0),\ C(0-) = C(0).$$

Then we have

$$a_-(C_-(t)) \leq a_-(C(t)) \leq t\ ,\ t \in \mathbb{R}_+$$

and

$$a(C(t)) \geq a(C_-(t)) \geq t\ , t < a(\infty)$$

Lemma 2.2. *[7]. Let $(a, b) \in \mathbb{R}^2$ with $a < b$, φ and $\psi : [a, b] \longrightarrow \mathbb{R}$ non-negative continuous functions, such that $\exists \rho \in \mathbb{R}^+, \forall t \in [a, b], \varphi(t) \leq \rho + \int_a^t \varphi(s)\psi(s)ds$ then:*

$$\forall t \in [a, b], \varphi(t) \leq \rho \exp(\int_a^t \psi(s)ds)$$

Proof. We assume $G : [a, b] \longrightarrow \mathbb{R}$

$$u \longmapsto \left(\int_a^u \varphi(s)\psi(s)ds\right) \exp\left(-\int_a^u \psi(s)ds\right)$$

Because φ and ψ are continuous functions, then G is the is continuously derivable on $[a, b]$ and:

$$\forall u \in [a, b],\ \grave{G}(u) = \varphi(u)\psi(u) \exp\left(-\int_a^u \psi(s)ds\right) - \psi(u)\left(\int_a^u \varphi(s)\psi(s)ds\right) \exp\left(-\int_a^u \psi(s)ds\right)$$

$$\forall u \in [a, b],\ \grave{G}(u) = \psi(u) \exp\left(-\int_a^u \psi(s)ds\right)\left(\varphi(u) - \int_a^u \varphi(s)\psi(s)ds\right)$$

But, by hypothesis

$$\forall u \in [a, b], \varphi(u) \leq \rho + \int_a^u \varphi(s)\psi(s)ds$$

So

$$\forall u \in [a, b], \grave{G}(u) \leq \rho\ \psi(u) \exp\left(-\int_a^u \psi(s)ds\right)$$

Let $t \in [a, b]$, integrating this inequality for i from a and t:

$$G(t) - G(a) \leq \rho \int_a^t \psi(u) \exp\left(-\int_a^u \psi(s)ds\right) du$$

By definition of G and as $G(a) = 0$:

$$\left(\int_a^t \varphi(s)\psi(s)ds\right)\exp\left(-\int_a^t \psi(s)ds\right) \leq \rho\left[-\exp\left(-\int_a^u \psi(s)ds\right)\right]_a^t$$
$$\leq -\rho\exp\left(-\int_a^t \psi(s)ds\right) + \rho\exp(0)$$

From where

$$\left(\int_a^t \varphi(s)\psi(s)ds\right) \leq -\rho + \rho\exp\left(\int_a^t \psi(s)ds\right)$$

and finally

$$\varphi(t) \leq \rho\exp\left(\int_a^t \psi(s)ds\right)$$

□

3. Our approach to the ♮-model

In our model, we show the continuity of the solution of the ♮-equation by applying the theorem of Kolmogorov presented in the previous section and the lemma of Gronwall (lemma 2.2) such that we take $\varepsilon = p$ and $\beta = p - n$ with $p > 0$. We have for $u \leq s \leq t$:

$$X_t^u(x) = x + \int_u^t X_s\left(-\frac{e^{-\Lambda_s}}{1 - Z_s}\right)dN_s + \int_u^t X_s f(X_s - (1 - Z_s))dY_s$$

We know that the quantity $f(X_s - (1 - Z_s))$ is bounded because f is a Lipschitz function, but as we do not know a priori if the quantity $\left(-\frac{e^{-\Lambda_s}}{1 - Z_s}\right)$ is finite or not, we introduce the stopping time $\tau_n = \inf\{t, 1 - Z_t < \frac{1}{n}\}$. Therefore, we assume the process $\tilde{X}$ instead of X:

$$d\tilde{X}_t = \tilde{X}_t\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}dN_t + f(\tilde{X}_t - (1 - Z_t))dY_t\right), \text{Such as } \tilde{X}_t = X_t, \quad \forall\, t \leq \tau_n\,,\, n \in \mathbb{N}$$

We denote $A_t = \tilde{X}_t^u(x) - \tilde{X}_t^u(y)$ and we apply Itô's formula to the process $|A_t|^p$, we find:
$A = \tilde{X}^x - \tilde{X}^y \Longrightarrow dA_t = d(\tilde{X}_t^x - \tilde{X}_t^y)$

$$\Longrightarrow d|A_t|^p = p|A_t|^{p-1}dA_t + \tfrac{|A_t|^{p-2}}{2}p(p-1)[d < A_t, A_t >]$$

Such as
$dA_t = d(\tilde{X}_t^x - \tilde{X}_t^y)$

$$= (\tilde{X}_t^x - \tilde{X}_t^y)\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)dN_t + [\tilde{X}_t^x f(\tilde{X}_t^x - (1 - Z_t)) - \tilde{X}_t^y f(\tilde{X}_t^y - (1 - Z_t))]dY_t$$

Noting

$$\mathcal{V}_t(\tilde{X}_t^x) = \tilde{X}_t^x f(\tilde{X}_t^x - (1 - Z_t))$$
$$\mathcal{V}_t(\tilde{X}_t^y) = \tilde{X}_t^y f(\tilde{X}_t^y - (1 - Z_t))$$

So

$$d|A_t|^p = p|A_t|^{p-1}dA_t + \tfrac{|A_t|^{p-2}}{2}p(p-1)d < A_t, A_t >$$

$$d|A_t|^p = p|A_t|^{p-1}dA_t + \tfrac{|A_t|^{p-2}}{2}p(p-1)[(\tilde{X}_t^x - \tilde{X}_t^y)^2\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)^2 d < N, N >_t + (\mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y))^2 d < Y, Y >_t + 2(\tilde{X}_t^x - \tilde{X}_t^y)\left(-\frac{e^{-\Lambda_t}}{1 - Z_{t\wedge\tau_n}}\right)(\mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y))d < N, Y >_t]$$

By lemma of Jacod (see [5], page 128, 129), there always exists an increasing matrix G, such that: $C_{11}dG = d<N,N>$, $C_{22}dG = d<Y,Y>$ and $C_{12}dG = d<N,Y>$ with $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ is a symmetric nonnegative matrix, and the choice of the latter is arbitrary, then:

$d|A_t|^p = p|A_t|^{p-1}dA_t + \frac{A_t^{p-2}}{2}p(p-1)[\Big((\tilde{X}^x - \tilde{X}^y), \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y)\Big) \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$
$\begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{X}_t^x - \tilde{X}_t^y \\ \mathcal{V}_t(\tilde{X}_t^x) - V_t(\tilde{X}_t^y) \end{pmatrix}]dG_t$

We denote

$$W_t^T = \Big((\tilde{X}^x - \tilde{X}^y), \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y)\Big)$$

$$M = \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} -\dfrac{e^{-\Lambda_t}}{1-Z_{t\wedge\tau_n}} & 0 \\ 0 & 1 \end{pmatrix}$$

$$W_t = \begin{pmatrix} \tilde{X}_t^x - \tilde{X}_t^y \\ \mathcal{V}_t(\tilde{X}_t^x) - \mathcal{V}_t(\tilde{X}_t^y) \end{pmatrix}$$

So

$d|A_t|^p = p|A_t|^{p-1}dA_t + \frac{A_t^{p-2}}{2}p(p-1)[W_t^T M W_t]dG_t$
$\Longrightarrow |A_t|^p = |x-y|^p + \left[\int_u^t p|A_s|^{p-1}dA_s + \int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s\right]$
$\Longrightarrow \mathbb{E}[|A_t|^p] = |x-y|^p + \mathbb{E}[\int_u^t p|A_s|^{p-1}dA_s] + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s]$
$\Longrightarrow \mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}W_s^T M W_s dG_s]$
$\leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s|W_s|^2 dG_s]$

such that

$$M = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \text{ and } m = |b_{11}| + |b_{12}| + |b_{21}| + |b_{22}|$$

So

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s((\tilde{X}_s^x - \tilde{X}_s^y)^2 + (V_s(\tilde{X}_s^x) - V_s(\tilde{X}_s^y))^2)dG_s]$$

But f is a Lipschitz function, then there exists a real positive constant K, so:

$$|V_s(\tilde{X}_s^x) - V_s(\tilde{X}_s^y)|^2 \leq K|\tilde{X}_s^x - \tilde{X}_s^y|^2$$

Therefore

$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}A_s^{p-2}m_s(A_s^2 + K|A_s|^2)dG_s]$$
$$\mathbb{E}[|A_t|^p] \leq |x-y|^p + \mathbb{E}[\int_u^t \frac{p(p-1)}{2}|A_s|^p m_s(1+K)dG_s]$$
$$\leq |x-y|^p + \frac{p(p-1)}{2}(1+K)\mathbb{E}[\int_u^t |A_s|^p m_s dG_s]$$

We denote

$$a = |x-y|^p$$

$$b = \frac{p(p-1)}{2}(1+K)$$

Then

$$\mathbb{E}[|A_t|^p] \leq a + b\mathbb{E}[\int_u^t |A_s|^p m_s dG_s]$$

To apply Gronwall's lemma (lemma 2.2) we must use the technique of change of time to eliminate the matrix G, so for this we will use the lemma (2.1). In our case, putting $G(s) = a(s)$, we consider the stopping time:

$$C(t) = \inf\{s, G(s) > t\}$$

Such that, for $t \in \mathbb{R}_+$, $C(t) < \infty$ if and only if $t < G(\infty) = \lim_{t \longrightarrow \infty} G(t)$ and

$$G(C(t)) \geq G(C_-(t)) \geq t \text{ , } t \in \mathbb{R}_+.$$

In fact

$$\mathbb{E}[|A_t|^p] \leq a + b\mathbb{E}[\int_u^t |A_s|^p m_s dG_s]$$

For $s, t, \alpha \in \mathbb{R}^+$ such that $s < \alpha$:

$$\mathbb{E}[\sup_{t \leq C(\alpha)} |A_t|^p] \leq a + b\mathbb{E}[\sup_{t \leq C(\alpha)} \int_u^{C(\alpha)} |A_s|^p m_s dG_s]$$

We denote $B_t = \sup_{t \leq C(\alpha)} |A_t|$, then

$$\begin{aligned}
\mathbb{E}[B_t^p] &\leq a + b\mathbb{E}[\int_u^\alpha B_{C(s)}^p m_{C(s)} dG_{C(s)}] \\
&\leq a + b\mathbb{E}[\int_u^\alpha B_{C(s)}^p m_{C(s)} ds] \\
&\leq a + b\mathbb{E}[\int_u^{C(\alpha)} B_s^p m_s ds]
\end{aligned}$$

Now, we can apply the lemma of Gronwall (lemma 2.2) to this last expression, we have:

$$\mathbb{E}[B_t^p] \leq a + b\mathbb{E}\left[\int_u^{C(\alpha)} B_s^p m_s ds\right]$$

We take

$$\varphi(t) = \mathbb{E}[B_t^p]$$

$$\psi(s) = m_s$$

$$a = \rho$$

So, we find

$$\mathbb{E}[B_t^p] \leq a \exp\left(b \int_u^{C(\alpha)} m_s ds\right)$$

Eventually, if the quantity $\left(\int_u^{C(\alpha)} m_s ds\right)$ is finite , there exists the constant C which satisfies the condition of Kolmogorov's lemma that is to say that $\mathrm{C} = \exp\left(b \int_u^{C(\alpha)} m_s ds\right)$.

4. Conclusion

This document contains a new and original methodological approach to the subject in question and could therefore be a good contribution to the theory of stochastic processes, based on a very interesting lemma of Kolomogorov. Some difficulties have been encountered because the subject deals with a difficult area "the stochastic differential equations". As prospects, we try to prove the same result of the paper, but in a vectorial case; moreover, we also think of demonstrating that the stochastic flow associated with our model will be a diffeomorphism with multidimensional parameters on the same space, and we will investigate whether it is possible to have the same work on manifolds.

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