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Curvature Inequalities between a Hessian Manifold with Constant Curvature and its Submanifolds

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Abstract
In this paper after a short description of Hessian manifolds, we establish new curvature inequalities between a Hessian manifold and its submanifolds.

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1. Introduction

Among the many Riemannian metrics that may exist on a flat manifold, Hessian metrics are the most compatible with the flat structure. A Riemannian metric on a flat and affine manifold is called a Hessian metric if it is locally expressed by the Hessian of functions with respect to the affine coordinate systems. A pair of a flat structure and a Hessian metric is called Hessian structure and a manifold equipped with a Hessian structure is said to be a Hessian manifold. Typical examples of of these manifolds are regular convex cones and the space of all positive definite real symmetric matrices [1–3] Hirohiko Shima introduced Hessian sectional curvature and its relations with Kaehlerian manifold. He also proved the theorems and gave important remarks on the spaceform of Hessian manifolds. In the light of these studies Yi ldiri m Yilmaz and Bektaş obtained some curvature conditions, results and integral inequalities on this type of manifolds, [4, 5, 8]

In this paper we focus on some new curvature estimates on Hessian manifolds and its submanifolds analogous with [6, 7].

2. Preliminaries

Let M^m be a Hessian manifold with Hessian structure (D, g).We express various geometric concepts for the Hessian structure (D,g)in terms of affine coordinate system $x^1, ..., x^m$ with respect to D , i.e $Ddx^i = 0$

i) The Hessian metric ;

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$$

ii) Let γ be a tensor field of type $(1, 2)$ defined by

$$\gamma(X, Y) = \bigtriangledown_X Y - D_X Y$$

where ∇ is the Riemannian connection for g . Then we have

$$\gamma^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k}$$

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$$\gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}$$
$$\gamma_{ijk} = \gamma_{jik} = \gamma_{kji}$$

where Γ^i_{jk} are the Christoffel ′ s symbols of ∇ .

iii)Let Q be a tensor field of type $(1,3)$ defined by

$$Q = D_\gamma$$

and call it the Hessian curvature tensor for $(D\,,\,g)$ .Then we have

$$Q^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}$$
$$Q_{ijkl} = \frac{1}{2}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^r} - \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s}$$
$$Q_{ijkl} = Q_{ilkj} = Q_{kjil} = Q_{jilk} = Q_{klij}.$$

iv) The Riemannian curvature tensor for ∇ is given by ;

$$R^i_{jkl} = \gamma^i_{rk}\gamma^r_{jl} - \gamma^i_{rl}\gamma^r_{jk}\,,$$

$$R_{ijkl} = \frac{1}{2}\left(Q_{jikl} - Q_{ijkl}\right). \tag{2.1}$$

Definition 2.1. For a non-zero contravariant symmetric tensor ξ_x of degree 2 at x, we set

$$h(\xi_x) = \frac{\langle \varsigma(\xi_x)\,, \xi_x \rangle}{\langle \xi_x\,, \xi_x \rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x, [1].

Theorem 2.1. *Let $(M\,,D\,,\,g)$ be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h(\xi_x)$ depends only x then $(M\,,D\,,\,g)$ is of constant Hessian sectional curvature .$(M\,,D\,,\,g)$ is of constant Hessian sectional curvature c if and only if*

$$Q_{ijkl} = \frac{c}{2}\left(g_{ij}g_{kl} + g_{il}g_{kj}\right)\,. \tag{2.2}$$

Corollary 2.1. *If a Hessian manifold $(M\,,D\,,\,g)$ is a space of constant Hessian sectional curvature c , then the Riemannian manifold $(M\,,\,g)$ is a space of constant sectional curvature $-\frac{c}{4}$, [1].*

3. Curvature equations between a Hessian manifold and Its Submanifolds

Let M^n be an n-dimensional submanifolds of $(n+p)-$ dimensional Hessian manifold M^{n+p}. We choose a local field of orthonormal vectors such that $e_1, ..., e_n$ are tangent to M^n. We use the following convention on the ranges of indices: $1 \leq i, j, k, .. \leq n, n+1 \leq \alpha, \beta\,, \gamma, ... \leq p$.

We denote by h^α_{ij} the components of the second fundamental form of M^n with respect to the frame $e_1, ..., e_n, ..., e_{n+p}$ the the mean curvature vector $\overrightarrow{H}$ of M^n, the square H^2 of the mean curvature H of M^n and the square S of the length of the second fundamental form are given , respectively by

$$\overrightarrow{H} = \frac{1}{n}\sum_{\alpha=1}^{n} h^\alpha e_\alpha \text{ where } h^\alpha = \sum_i h^\alpha_{ii} \tag{3.1}$$

$$H^2 = \frac{1}{n^2}\sum_{\alpha=1}^{n}\left(h^\alpha\right)^2 \tag{3.2}$$

$$S = \sum_\alpha \left[\sum_{i,j}^{n}\left(h^\alpha_{ij}\right)^2\right] = \sum_\alpha S_\alpha \quad S_\alpha, = \sum_{i,j}^{n}\left(h^\alpha_{ij}\right)^2 \tag{3.3}$$

S and H^2 are independent of the choice of the orthonormal basis.

And for S and H^2 we have

$$(h_\alpha)^2 \leq nS_\alpha \ \ , nH^2 \leq S \tag{3.4}$$

Let X be an arbitrary unit vector tangent to M^n at a point $x \in M^n$. We denote the the Ricci curvature of M^n in the direction of X by $Ric\,(X, X)$ and the Ricci curvature of M^{n+p} in the some direction X by $\overline{Ric}\,(X, X)$. Supposing that the local frame $e_1, ..., e_n, ...e_{n+p}$ is chosen that $e_n = X$. Then the Ricci curvature $Ric\,(X, X) = Ric\,(e_n, e_n)$ of M^n at a point $x \in M^n$ is equal to

$$Ric\,(e_n, e_n) = \sum_{i=1}^{n-1} K_{in} \tag{3.5}$$

where K_{in} is the sectional curvature of M^n at $x \in M^n$ for the 2-plane section spanned by e_i and e_n.

Let us consider $T_x M^n$ as an n-plane section of M^{n+p} at a point $x \in M^n$ spanned by the $n-$ orthonormal vectors $e_1, ..., e_n$ then the $n-$ Ricci curvature $\overline{Ric_{T_xM^n}}\,(X, X)$ of M^{n+p} for the n-plane section T_xM^n in the direction of $X = e_n$ is equal to

$$\overline{Ric_{T_xM^n}}\,(X, X) = \sum_{i=1}^{n-1} \overline{K_{in}} \tag{3.6}$$

where $\overline{K_{in}}$ is the sectional curvature of M^{n+p} at $x \in M^n$ for the 2-plane section spanned by the vectors e_i and e_n.

Using Gauss equation

$$R_{ijkl} = \overline{K}_{ijkl} + h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk} \tag{3.7}$$

then the $Ric\,(e_n, e_n)$ of M^n at $x \in M^n$ is equal to the following

$$Ric\,(e_n, e_n) = \sum_{i=1}^{n} \overline{K_{in}} + \sum_{\alpha} \left[h^\alpha_{nn} \sum_{i=1}^{n} h^\alpha_{ii} - \sum_{i=1}^{n} (h^\alpha_{in})^2 \right] \tag{3.8}$$

or

$$Ric\,(e_n, e_n) = \overline{Ric_{T_XM^n}}\,(e_n, e_n) + \sum_{\alpha} \left[h^\alpha_{nn} \sum_{i=1}^{n} h^\alpha_{ii} - \sum_{i=1}^{n} (h^\alpha_{in})^2 \right] \tag{3.9}$$

In [6] the author estimates for the Ricci curvature of a submanifold M^n of an arbitrary Riemannian manifold M^{n+p} . After routine calculations we get similar results for Hessian manifolds as follows:

Theorem 3.1. *Let M^n be an $n-$ dimensional submanifold of an $(n + p)$ dimensional Hessian manifold M^{n+p}. For the Ricci curvature $Ric\,(X, X)$ of M^n at a non-totally geodesic point $x \in M^n$ in the direction of a unit vector X tangent to M^n we have*

$$Ric\,(X, X) \leq \overline{Ric_{T_xM^n}}\,(X, X) + \frac{n^2 H^2}{4} \tag{3.10}$$

$$\begin{aligned} Ric\,(X, X) \ &\geq \ \overline{Ric_{T_xM^n}}\,(X, X) + (n-1) \sum_{\alpha} \lambda^\alpha_1 \lambda^\alpha_n \\ &= \ \overline{Ric_{T_xM^n}}\,(X, X) + \\ &\quad \frac{(n-1)}{n} \left[2nH^2 - S - (n-2) \sqrt{\frac{nH^2\,(S - nH^2)}{n-1}} \right] \end{aligned} \tag{3.11}$$

where λ^α_1 and λ^α_n are two different eigenvalues of each one of the matrices $\left(h^\alpha_{ij}\right)$ of the second fundamental form. The equality in (3.10) is held only when all the matrices $\left(h^\alpha_{ij}\right)$ of the second fundamental form with respect to an orthonormal basis $e_1, ..., e_n = X, ..., e_{n+p}$ are of the form

$$\begin{pmatrix} \widetilde{h}^\alpha_{11} & \widetilde{h}^\alpha_{12} & ... & \widetilde{h}^\alpha_{1n-1} & 0 \\ \widetilde{h}^\alpha_{12} & \widetilde{h}^\alpha_{22} & ... & \widetilde{h}^\alpha_{2,n-1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{h}^\alpha_{1,n-1} & \widetilde{h}^\alpha_{2,n-1} & ... & \widetilde{h}^\alpha_{n-1,n-1} & 0 \\ 0 & 0 & ... & 0 & \frac{\widetilde{h^\alpha}}{2} \end{pmatrix} \tag{3.12}$$

where $\widetilde{h}_{11}^{\alpha}+\widetilde{h}_{22}^{\alpha}+...+\widetilde{h}_{n-1,n-1}^{\alpha}=\frac{\widetilde{h}_{\alpha}}{2}, \widetilde{h}_{ij}^{\alpha}-$ *arbitrary* $1\leq i\leq j\leq n-1$.

The equality in (2.11) *is fulfilled if and only if the matrices* $\left(\widetilde{h}_{ij}^{\alpha}\right)$ *are the following*

$$\begin{pmatrix} \lambda_1^{\alpha} & 0 & ... & 0 \\ 0 & \lambda_1^{\alpha} & ... & 0 \\ \vdots & \vdots & \ddots & \\ ... & ... & ... & \lambda_n^{\alpha} \end{pmatrix} \tag{3.13}$$

where

$$\lambda_1^{\alpha}=\frac{1}{n}\left(h^{\alpha}\mp\sqrt{\frac{nS_{\alpha}-(h_{\alpha})^2}{n-1}}\right),...,\lambda_n^{\alpha}=\frac{1}{n}\left(h^{\alpha}\mp(n-1)\sqrt{\frac{nS_{\alpha}-(h_{\alpha})^2}{n-1}}\right)$$

as $(n-1)\lambda_1^{\alpha}+\lambda_n^{\alpha}=h_{\alpha}$ *and* $(n-1)(\lambda_1^{\alpha})^2+(\lambda_n^{\alpha})^2=S_{\alpha}$.

For the proof of the Riemannian version of the theorem we refer to [6].

Corollary 3.1. *If*

$$\frac{n^2H^2}{4}\leq-\overline{Ric_{T_xM^n}}(X,X) \tag{3.14}$$

then $Ric(X,X)\leq 0$ *at* $x\in M^n$ *for the unit vector* $x\in T_xM^n$. *Particularly when the ambient space* M^{n+p} *is a space form on positive curvature then*

$$Ric(X,X)\leq 0 \;\; if \;\; H^2\leq\frac{(n-1)c}{n^2}.$$

Corollary 3.2. *If*

$$2(n-1)H^2-\frac{n-1}{n}S-\frac{(n-2)}{n}\sqrt{n(n-1)H^2(S-nH^2)}\geq 0 \tag{3.15}$$

then $Ric(X,X)\geq\overline{Ric_{T_xM^n}}(X,X)$ *at* $x\in M^n$ *in the direction of* $x\in T_xM^n$.

Corollary 3.3. *If*

$$\overline{Ric_{T_xM^n}}(X,X)\geq 2(1-n)H^2+\frac{n-1}{n}S+\frac{n-2}{n}\sqrt{n(n-1)H^2(S-nH^2)}\geq 0 \tag{3.16}$$

then $Ric(X,X)\geq 0$ *for* $x\in T_xM^n$. *Next we may expess the following theorem*

Theorem 3.2. *Let* M^n *be an n-dimensional submanifold isometrically immersed in an* $(n+p)-$ *dimensional arbitrary Hessian manifold* M^{n+p}. *The equality*

$$Ric(X,X)=\overline{Ric_{T_xM^n}}(X,X)+\frac{n-1}{n}\left[2nH^2-S-(n-2)\sqrt{\frac{nH^2(S-nH^2)}{n-1}}\right] \tag{3.17}$$

is fulfilled at a non-totally geodesic point $x\in M^n$ *in the direction of a tangent unit vector* $X\in T_xM^n$ *if and only if the following conditions are satisfied:*

$$i)R_{\beta kl}^{\alpha}=\overline{K}_{\beta kl}^{\alpha} \; at \; x\in M^n.$$

$ii)$ *Each one of the matrices* $\left(h_{ij}^{\alpha}\right)$ *of the second fundamental form of* M^n *has exactly* $(n-1)$ *eigenvalues equal to* λ_1^{α} *and are equal to the corresponding* λ_n^{α} *from* (3.13).

$iii)$ *The vector* X *is their common eigenvector corresponding to their simple eigenvalue* λ_n^{α}.

4. Submanifold of a Hessian space form

In the case when the ambient space M^{n+p} is a space form with curvature $-\frac{c}{4}$, then we may compute the equation (3.10) and (3.11) as follows by using (3.17) and $i)$

$$Ric\,(X,X) \leq (1-n)\,\frac{c}{4} + \frac{n^2H^2}{4} \tag{4.1}$$

$$Ric\,(X,X) \geq (1-n)\,\frac{c}{4} + \frac{n-1}{n}\left[2nH^2 - S - (n-2)\sqrt{\frac{nH^2\,(S-nH^2)}{n-1}}\right] \tag{4.2}$$

$$Ric\,(X,X) = (n-1)\left[2H^2 - \frac{1}{n}S - \frac{(n-2)}{n}\sqrt{\frac{nH^2\,(S-nH^2)}{n-1}} - \frac{c}{4}\right] \tag{4.3}$$

Applying Theorem 2.1 to an arbitrary submanifold M^n of a space form $M^{n+p}\left(-\frac{c}{4}\right)$ gives the following

Theorem 4.1. *Let M^n be a non-totally geodesic submanifold isometrically immersed in a Hessian space form $M^{n+p}\left(-\frac{c}{4}\right)$. The equality*

$$\min_X Ric\,(X,X) = (n-1)\left[2H^2 - \frac{1}{n}S - \frac{(n-2)}{n}\sqrt{\frac{nH^2\,(S-nH^2)}{n-1}} - \frac{c}{4}\right] \tag{4.4}$$

when X runs over all unit tangent vectors of M^n at a point $x \in M^n$, holds for all points $x \in M^n$ if and only if the normal bundle of M^n is flat, each one of the matrices $\left(h_{ij}^{\alpha}\right)$ has exactly $(n-1)$ eigenvalues equal to the corresponding λ_1^{α} and one equal to λ_n^{α} from (3.13) and the vector X_0, for which the minimum of $Ric\,(X,X)$ is achieved, is their common eigenvector corresponding to their simple eigenvalue λ_n^{α}.

Now let us consider a Hessian domain $(\Omega, D, g = Dd\varphi)$ in $\mathbb{R}^n$ of constant Hessian sectional curvature c as indicated [1].

Proposition 4.1. *The following Hessian domains are examples of spaces of constant Hessian sectional curvature 0.*

(1) Euclidean space $\left(\mathbb{R}^n, D, g = Dd\left(1/2\sum_{i=1}^{n}\left(x^i\right)^2\right)\right)$.

(2) $\left(\mathbb{R}^n, D, g = Dd\left(\sum_{i=1}^{n} e^{x^i}\right)\right)$.

Proposition 4.2. *Let c be positive real number and let Ω be given by*

$$\Omega = \left\{\left(x^1,...,x^n\right) \in \mathbb{R}^n \,\middle|\, x^n > \frac{c}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2\right\},$$

and let φ be a smooth function on Ω defined by

$$\varphi = -\frac{1}{c}\log\left\{x^n - \frac{1}{2}\sum_{i=1}^{n-1}\left(x^i\right)^2\right\}.$$

Then $\left(\Omega, D,\ g = D^2\varphi\right)$ is a simply connected Hessian manifold of positive constant Hessian sectional curvature c.

Hence the following theorem can be proved as a consequence of the properties above.

It is really surprising that (Ω, g) is isometric to hyperbolic space form $\left(H\left(-\frac{c}{4}\right), g\right)$ of constant sectional curvature $-c/4$;

$$\begin{aligned} H &= \left\{\left(\xi^1,...,\xi^{n-1},\xi^n\right) \in \mathbb{R}^n \left|\xi^n > 0\right.\right\}, \\ g &= \frac{1}{\left(\xi^n\right)^2}\left\{\sum_{i=1}^{n}\left(d\xi^i\right)^2 + \frac{4}{c}\left(d\xi^n\right)^2\right\}. \end{aligned}$$

Proposition 4.3. *Let φ be a smooth function on $\mathbb{R}^n$ defined by*

$$\varphi = -\frac{1}{c}\log\left(\sum_{A=1}^{n} e^{-cx^A} + 1\right),$$

where c is a negative constant. Then $\left(\mathbb{R}^n, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ is a simply connected Hessian manifold of negative constant Hessian sectional curvature c. The Riemannian manifold $(\mathbb{R}^n, g)$ is isometric a domain of the sphere $\sum_{i=1}^{n+1} \xi_A^2 = -\frac{4}{c}$ defined by $\xi_A > 0$ for all A, [1].

Then we re-formulate the (4.1)-(4.4) as follows:

Corollary 4.1. *Let the ambient space be one of the following space indicated below*

(1) *Euclidean space* $\left(\mathbb{R}^n, D, g = Dd\left(1/2\sum_{i=1}^{n}\left(x^i\right)^2\right)\right)$.

(2) $\left(\mathbb{R}^n, D, g = Dd\left(\sum_{i=1}^{n} e^{x^i}\right)\right)$

then we get

$$Ric\,(X,X) \leq \frac{n^2H^2}{4}.$$

Corollary 4.2. *If (Ω, g) is isometric to hyperbolic space form $\left(H\left(-\frac{c}{4}\right), g\right)$ of constant sectional curvature $-c/4$;*

$$\begin{aligned} H &= \left\{\left(\xi^1, ..., \xi^{n-1}, \xi^{n+p}\right) \in \mathbb{R}^{n+p} \left| \xi^{n+p} > 0\right.\right\}, \\ g &= \frac{1}{\left(\xi^{n+p}\right)^2}\left\{\sum_{i=1}^{n+p}\left(d\xi^i\right)^2 + \frac{4}{c}\left(d\xi^n\right)^2\right\}. \end{aligned}$$

.Let us take ambient space as $M^{n+p} = (\Omega, g)$ then

$$Ric\,(X,X) \geq (1-n)\frac{c}{4} + \frac{n-1}{n}\left[2nH^2 - S - (n-2)\sqrt{\frac{nH^2\left(S - nH^2\right)}{n-1}}\right]$$

and

$$Ric\,(X,X) = (n-1)\left[2H^2 - \frac{1}{n}S - \frac{(n-2)}{n}\sqrt{\frac{nH^2\left(S-nH^2\right)}{n-1}} - \frac{c}{4}\right].$$

Corollary 4.3. *Let φ be a smooth function on $\mathbb{R}^{n+p}$ defined by*

$$\varphi = -\frac{1}{c}\log\left(\sum_{A=1}^{n+p} e^{-cx^A} + 1\right),$$

where c is a negative constant. Then $\left(\mathbb{R}^{n+p}, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ is isometric a domain of the sphere $\sum_{i=1}^{n+1} \xi_A^2 = -\frac{4}{c}$ defined by $\xi_A > 0$ for all A. If the ambient space is $\left(\mathbb{R}^{n+p}, \widetilde{D},\ g = \widetilde{D}^2\varphi\right)$ we get

$$Ric\,(X,X) \geq (n-1)\frac{c}{4} + \frac{n-1}{n}\left[2nH^2 - S - (n-2)\sqrt{\frac{nH^2\left(S-nH^2\right)}{n-1}}\right]$$

$$Ric\,(X,X) = (n-1)\left[2H^2 - \frac{1}{n}S - \frac{(n-2)}{n}\sqrt{\frac{nH^2\left(S-nH^2\right)}{n-1}} + \frac{c}{4}\right].$$

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