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Some Special Curves and Mannheim Curves in Three Dimensional Euclidean Space

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Abstract
In this study we investigate some special curves which have Mannheim curves and Mannheim partner curves and obtain some characterizations about them.

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1. Introduction

In the differential geometry of a regular curve in the Euclidean 3-Space E^3, it is well known that one of the important problem is the characterization of a regular curve. The curvature functions κ and τ of a regular curve play an important role to determine the shape and size of the curve. For example; if $\kappa = \tau = 0$ then the curve is geodesic. If $\kappa \neq 0$(constant) and $\tau = 0$, then the curve is a circle with radius $\frac{1}{\kappa}$. If $\kappa \neq 0$(constant) and $\tau \neq 0$(constant), then the curve is a helix [2]. If a curve in E^3 with $\kappa > 0$ is congruent to a rectifying curve if and only if the ratio $\frac{\tau}{\kappa} = as + b$ of the curve is a nonconstant linear function in arclength function s , then the curve rectifying curve [2]. If κ is constant and τ is non-constant, then the curve is Salkowski curve [6]. If κ is non-constant and τ is constant, then the curve is anti-Salkowski curve [5].

Another way to classification and characterization of curves is the relationship between the Frenet vectors of the curves. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. For the Bertrand curve Γ, it shares the normal lines with another curve Γ_1, called Bertrand partner curve Γ. In this study we are concerned with another kind of associated curves, called Mannheim curve and Mannheim partner curve. If there is a corresponding relationship between the space curves Γ and Γ_1 such that , at the corresponding points of the curves, the principal normal lines of Γ coincides with the binormal lines of Γ_1, then Γ is called a Mannheim curves and Γ_1 is called Mannheim partner curve of Γ. Mannheim partner curves was studied by Liu and Wang in 3-Euclidean space [3].

2. Basic concepts

Let $\Gamma : x(s)$ be a Mannheim curve in E^3 parameterized by its arc-length s and $\Gamma_1 : x_1(s_1)$ the Mannheim partner curve of Γ with an arc-length parameter s_1.

Let $\{\kappa, \tau\}$ and $\{T(s), N(s), B(s)\}$ be curvatures and Frenet frame of the Mannheim curve Γ, respectively. Then Frenet formulas are given by

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

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where we use $(')$ to denote the derivate with respect to the arc-length parameter of the curve Γ.

In a similar way, let $\{\kappa_1, \tau_1\}$ and $\{T_1(s_1), N_1(s_1), B_1(s_1)\}$ be curvatures and the Frenet frame of the Mannheim partner curve of Γ, respectively. Then the Frenet formulas of Γ_1 are given by

$$\begin{bmatrix} \dot{T_1}(s_1) \\ \dot{N_1}(s_1) \\ \dot{B_1}(s_1) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \tau_1 \\ 0 & -\tau_1 & 0 \end{bmatrix} \begin{bmatrix} T_1(s_1) \\ N_1(s_1) \\ B_1(s_1) \end{bmatrix}$$

where we use $(\cdot)$ to denote the derivate with respect to the arc-length parameter of the curve Γ_1.

3. Some special curves and Mannheim curves

In this section, we define Mannheim curves in E^3, and we give some characterization for some special curves which have Mannheim curves in the same space.

Theorem 3.1. *The space curve in E^3 is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the formula*

$$\kappa = \lambda(\kappa^2 + \tau^2) \tag{3.1}$$

where λ is a nonzero constant [4].

Theorem 3.2. *If x Mannheim curve is a general helix then x is a circular helix. The curvature and torsion of x are obtained as follows:*

$$\kappa = \frac{1}{\lambda(1+c^2)},$$

$$\tau = \frac{c}{\lambda(1+c^2)}.$$

Proof. Let x be a general helix. From the definition of helix, the curvature and torsion of x satisfy the following equation

$$\frac{\tau}{\kappa} = c \tag{3.2}$$

(3.2) equation can be written as (3.3)

$$\tau = c\kappa \tag{3.3}$$

Also x curve is a Mannheim curve, then it provides (3.1) equation. If (3.3) equation is written instead of (3.1) equation, we get

$$\kappa = \frac{1}{\lambda(1+c^2)},$$

$$\tau = \frac{c}{\lambda(1+c^2)}.$$

The proof is completed. □

Theorem 3.3. *If x Mannheim curve is a rectifying curve, then the curvature and torsion of x are obtained as follows:*

$$\kappa = \frac{1}{\lambda(1+(as+b)^2)},$$

$$\tau = \frac{as+b}{\lambda(1+(as+b)^2)}.$$

Proof. Let x be a rectifying curve. From the definition of rectifying curve, the curvature and torsion of x satisfy the following equation

$$\frac{\tau}{\kappa} = as + b \tag{3.4}$$

(3.4) equation can be written as (3.5)

$$\tau = \kappa(as + b) \tag{3.5}$$

Also x curve is a Mannheim curve, then it provides (3.1) equation. If (3.5) equation is written instead of (3.1) equation, we get

$$\kappa = \frac{1}{\lambda(1 + (as + b)^2)},$$

$$\tau = \frac{as + b}{\lambda(1 + (as + b)^2)}.$$

The proof is completed. □

Theorem 3.4. *There is no Mannheim Salkowski curve.*

Proof. Let x be a Salkowski curve. From the definition of Salkowski curve, the curvature and torsion of x satisfy the following equation

$$\kappa = \text{constant} = c, \qquad \tau = \tau(s) \tag{3.6}$$

If (3.6) equation is written instead of (3.1) equation, torsion function of x is founded constant. This is a contradiction. The proof is completed. □

Theorem 3.5. *There is no Mannheim anti-Salkowski curve.*

Proof. Let x be a anti-Salkowski curve. From the definition of anti-Salkowski curve, the curvature and torsion of x satisfy the following equation

$$\kappa = \kappa(s), \qquad \tau = \text{constant} = c \tag{3.7}$$

If (3.7) equation is written instead of (3.1) equation , curvature function of x is founded constant. This is a contradiction.
The proof is completed. □

Remark 3.1. If any curve is Mannheim curve in 3-dimensional Euclidean space, then curvature functions of x are either both fixed or are variable.

4. Some special curves and Mannheim partner curves

In this section, we define Mannheim partner curves in E^3, and we give some characterization for some special curves which have Mannheim partner curves in the same space.

Theorem 4.1. *The space curve in E^3 is a Mannheim partner curve if and only if its curvature κ_1 and torsion τ_1 satisfy the following equation*

$$\dot{\tau}_1 = \frac{d\tau_1}{ds_1} = \frac{\kappa_1}{\lambda}(1 + \lambda^2\tau_1^2) \tag{4.1}$$

[3].

Theorem 4.2. *If x_1 Mannheim partner curve is a general helix then the curvature and torsion of x_1 are obtained as follows:*

$$\kappa_1 = \frac{e^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}},$$

$$\tau_1 = \frac{ce^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}}.$$

where c, d are real constants.

Proof. Let x_1 be a general helix. From the definition of helix, the curvature and torsion of x_1 satisfy the following equation

$$\frac{\tau_1}{\kappa_1} = c \tag{4.2}$$

(4.2) equation can be written as (4.3)

$$\tau_1 = c\kappa_1 \tag{4.3}$$

Also x_1 curve is a Mannheim partner curve, then it provides (4.1) equation. If (4.3) equation is written instead of (4.1) equation, we get differential equation as follows:

$$\dot{\kappa_1} = \frac{\kappa_1}{c\lambda} + \lambda c \kappa_1^3$$

This differential equation is a Bernoulli differential equation.If the $z = \kappa_1^{-2}$ transformation is made, we get

$$\dot{z} + \frac{2z}{\lambda c} = -2\lambda c$$

linear differential equation. If this differential equation is solved, we get

$$\kappa_1 = \frac{e^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}},$$

From (4.3) equation,

$$\tau_1 = \frac{ce^{\frac{s_1}{\lambda c}}}{\sqrt{-e^{\frac{2s_1}{\lambda c}}\lambda^2 c^2 + d}}$$

is founded. The proof is completed. □

Theorem 4.3. *If x_1 Mannheim partner curve is a rectifying curve, then the curvature and torsion of x_1 are obtained as follows:*

$$\kappa_1 = \frac{1}{\sqrt{-\lambda^2(as_1+b)^2 + \frac{d}{(as_1+b)^{(\frac{2}{\lambda a}-2)}}}},$$

$$\tau_1 = \frac{(as_1+b)}{\sqrt{-\lambda^2(as_1+b)^2 + \frac{d}{(as_1+b)^{(\frac{2}{\lambda a}-2)}}}}.$$

where d is real constant.

Proof. Let x_1 be a rectifying curve. From the definition of rectifying curve, the curvature and torsion of x_1 satisfy the following equation

$$\frac{\tau_1}{\kappa_1} = as_1 + b \tag{4.4}$$

(4.4) equation can be written as (4.5)

$$\tau_1 = \kappa_1(as_1 + b) \tag{4.5}$$

Also x_1 curve is a Mannheim partner curve, then it provides (4.1) equation. If (4.5) equation is written instead of (4.1) equation, we get differential equation as follows:

$$\dot{\kappa_1} = \kappa_1\left(\frac{1}{\lambda(as_1 + b)} - \frac{a}{(as_1 + b)}\right) + \lambda(as_1 + b)\kappa_1^3$$

This differential equation is a Bernoulli differential equation.If the $z = \kappa_1^{-2}$ transformation is made, we get

$$\dot{z} + \left(\frac{2}{\lambda(as_1 + b)} - \frac{2a}{(as_1 + b)}\right)z = -2\lambda(as_1 + b)$$

linear differential equation. If this differential equation is solved, we get

$$\kappa_1 = \frac{1}{\sqrt{-\lambda^2(as_1 + b)^2 + \frac{d}{(as_1 + b)^{(\frac{2}{\lambda a} - 2)}}}}$$

From (4.5) equation,

$$\tau_1 = \frac{(as_1 + b)}{\sqrt{-\lambda^2(as_1 + b)^2 + \frac{d}{(as_1 + b)^{(\frac{2}{\lambda a} - 2)}}}}$$

is founded. The proof is completed. □

Theorem 4.4. *If x_1 Mannheim partner curve is a Salkowski curve, then the curvature and torsion of x_1 are obtained as follows:*

$$\tau_1 = \frac{\tan(cs_1 + \frac{d}{\lambda})}{\lambda}$$

where c, d are real constants.

Proof. Let x_1 be a Salkowski curve. From the definition of Salkowski curve, the curvature and torsion of x_1 satisfy the following equation

$$\kappa_1 = c \text{ and } \tau_1 = \tau_1(s_1) \tag{4.6}$$

Also x_1 curve is a Mannheim partner curve, then it provides (4.1) equation. If (4.6) equation is written instead of (4.1) equation, we get differential equation as follows:

$$\dot{\tau_1} = \frac{c}{\lambda} + c\lambda\tau_1^2$$

If this differential equation is solved with method separation of variables, we get

$$\tau_1 = \frac{\tan(cs_1 + \frac{d}{\lambda})}{\lambda}.$$

The proof is completed. □

Theorem 4.5. *There is no Mannheim partner anti-Salkowski curve.*

Proof. Let x_1 be a anti-Salkowski curve. From the definition of anti-Salkowski curve, the curvature and torsion of x satisfy the following equation

$$\kappa = \kappa(s), \qquad \tau = constant = c \tag{4.7}$$

If (4.7) equation is written instead of (4.1) equation, curvature function of x is founded constant. This is a contradiction.
The proof is completed. □

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