An Extension of the $\tau$-Gauss Hypergeometric Functions and its Properties

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Abstract
Recently, an extension of the $\tau$-hypergeometric functions $2R_1^\tau(z)$ was given by Parmar [6]. The main object of this paper is to introduce an extension of the $\tau$-hypergeometric function $3R_2^\tau(z)$ and investigate its various properties such as integral representations, derivative formula, Mellin transform and fractional calculus operators. Some published results are the special cases of our main results.

Keywords: Generalized Gamma functions; Gauss’s hypergeometric function; Generalized Gauss hypergeometric function; Extended $\tau$-hypergeometric function; Mellin transform; Fractional calculus operators.

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1. Introduction and Preliminaries

Chaudhry and Zubair [1, 2] introduced extended gamma function, as follows:

$$\Gamma_p(z) = \left\{ \begin{array}{ll}
\int_0^\infty t^{z-1} \exp \left(-t - \frac{p}{t}\right) \, dt, & (\Re(p) > 0; z \in \mathbb{C}) \\
\Gamma(z), & (\Re(z) > 0; p = 0)
\end{array} \right. .$$

Srivastava et al. [9, p.487, eq.(15)] introduced the following family of generalized hypergeometric functions:

$$\, _rF_s \left[ \begin{array}{c}
(\alpha_1; p), \alpha_2, \cdots, \alpha_r; \\
\beta_1, \beta_2, \cdots, \beta_s;
\end{array} z \right] = \sum_{n=0}^\infty \frac{(\alpha_1; p)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!},$$

where, $(\alpha_1; p)_n$ is the generalized Pochhammer symbol [9, p.485, eq.(8)], defined as

$$\Gamma_p(\lambda + \mu) \Gamma(\lambda) \Gamma(\mu), \quad (\Re(p) > 0; \lambda, \mu \in \mathbb{C})$$

$$\Gamma_p(\lambda + \mu) \Gamma(\lambda) \Gamma(\mu), \quad (\lambda, \mu \in \mathbb{C}; p = 0).$$

The integral representation of (1.1) is given by

$$\Gamma_p(\lambda + \mu) \Gamma(\lambda) \Gamma(\mu), \quad (\Re(p) > 0; \lambda, \mu \in \mathbb{C})$$

$$\Gamma_p(\lambda + \mu) \Gamma(\lambda) \Gamma(\mu), \quad (\lambda, \mu \in \mathbb{C}; p = 0).$$

The $\tau$-hypergeometric function was investigated and studied by Virchenko et al. [11], defined by the following manner:

$$2R_1^\tau(z) = 2R_1(a, b; c; \tau; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{(a)_n \Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!},$$

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with \( \tau > 0; |z| < 1; \Re (c) > \Re (b) > 0. \)

Integral representation of (1.2) given by [11, p.91, eq.(6)]:

\[
2R_1(a, b; c; \tau; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt^\tau)^{-a} dt,
\]

(1.3)

with \( \tau > 0; |\arg (1 - z)| < \pi; \Re (c) > \Re (b) > 0. \)

When \( \tau = 1 \) in (1.2) and (1.3) yields the representations of Gauss’s hypergeometric function [7].

Very recently, Parmar [6, p.423, eq.(2.1)] introduced the extended \( \tau \)-hypergeometric function \( 2R_1^\tau(z) \), as follows:

\[
2R_1^\tau(z) = 2R_1^\tau((a, p), b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a; p)_n}{(c + \tau n)} \frac{\Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{z^n}{n!},
\]

(1.4)

where, \( a, b \in \mathbb{C}, c \in \mathbb{C}\setminus \mathbb{Z}_0^{-}, p \geq 0, \tau > 0, |z| < 1, \) and \( \Re (c) > \Re (b) > 0 \) when \( p = 0, \)

and its integral representation [6, p.423, eq.(3.1)]:

\[
2R_1^\tau((a, p), b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \, _1F_0([a, p); -; zt^\tau] \, dt,
\]

with \( \tau > 0; \Re (p) > 0; \Re (c) > \Re (b) > 0 \) when \( p = 0. \)

2. Extended \( \tau \)-Hypergeometric Function \( 3R_2^\tau(z) \)

Motivated mainly by investigations of the extended \( \tau \)-hypergeometric function \( 2R_1^\tau(z) \) defined by (1.2), we introduced the extended \( \tau \)-hypergeometric function \( 3R_2^\tau(z) \) as follows:

For \( \lambda, a, b \in \mathbb{C} \) and \( c, d \in \mathbb{C}\setminus \mathbb{Z}_0^{-}, \) we have

\[
3R_2^\tau(z) = 3R_2^\tau((\lambda, p), a, b; c; d; z) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(a + \tau n)} \frac{\Gamma(\lambda + \tau n)}{\Gamma(a + \tau n)} \frac{\Gamma(b + \tau n)}{\Gamma(c + \tau n)} \frac{\Gamma(d + \tau n)}{\Gamma(d + \tau n)} \frac{z^n}{n!},
\]

(2.1)

where, \( p \geq 0, \tau > 0, |z| < 1, \) and \( \Re (d) > \Re (a) > 0, \Re (c) > \Re (b) > 0 \) when \( p = 0. \)

Special Cases:

1. If we take \( b = d, \) then (2.1) reduces to the extended \( \tau \)-hypergeometric function \( 2R_1^\tau(z) \) given by Parmar [6, p.422, eq.(2.1)] as defined in (1.4).

2. If we put \( b = d \) and set \( \tau = 1, \) then (2.1) reduces to the extended Gauss hypergeometric function [9, p.487, eq.(17)] given by

\[
_2F_1((\lambda, p); a; c; z) = \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(a)_n} \frac{z^n}{n!}.
\]

3. If we take \( \tau = 1 \) and \( p = 0 \) in (2.1), then it reduces to the classical Gauss’s hypergeometric function as

\[
_3F_2(\lambda, a, b; c; d; z) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} \frac{(a)_n}{(b)_n} \frac{(b)_n}{(d)_n} \frac{z^n}{n!}.
\]

4. If we put \( b = d \) and set \( \tau = 1, p = 0 \) in (2.1), then it reduces to the classical Gauss’s hypergeometric function as

\[
_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{(b)_n}{(c)_n} \frac{z^n}{n!}.
\]
3. Integral Representation and Derivative Formula

In this section, we obtain integral representation and differential formula for $3R_2^\tau(z)$ as given in (2.1).

**Theorem 1.** The following integral representation for $3R_2^\tau(z)$ in (2.1) holds true:

$$3R_2^\tau((\lambda, p), a, b; c; d; z) = \frac{1}{B(a, c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} 2R_1^\tau[(\lambda, p), b, d; zt^\tau] \, dt, \quad (3.1)$$

where $2R_1^\tau(z)$ is given in (1.4), and $\tau > 0$; $\Re(p) > 0$; $\Re(d) > \Re(a) > 0$, $\Re(c) > \Re(b) > 0$ when $p = 0$.

**Proof.** Using (2.1) and considering the following elementary identity for the Beta function:

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^1 t^{m-1} (1-t)^{n-1} \, dt, \quad (3.2)$$

then we arrive at

$$3R_2^\tau((\lambda, p), a, b; c; d; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \sum_{n=0}^\infty \frac{(\lambda;p)_n}{\Gamma(d+\tau n)} \frac{\Gamma(b+\tau n)}{B(a+\tau n, c-a)} \frac{z^n}{n!},$$

now using (3.2), then we have

$$= \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \sum_{n=0}^\infty \frac{(\lambda;p)_n}{\Gamma(d+\tau n)} \frac{\Gamma(b+\tau n)}{B(a+\tau n, c-a)} \int_0^1 t^{a+\tau n-1} (1-t)^{c-a-1} \frac{z^n}{n!} \, dt.$$

Next, interchanging the order of integration and summation which is permissible

$$= \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \left\{ \frac{\Gamma(d)}{\Gamma(b)} \sum_{n=0}^\infty \frac{(\lambda;p)_n}{\Gamma(c+\tau n)} \frac{\Gamma(b+\tau n)}{(c+\tau n)} \frac{(zt^\tau)^n}{n!} \right\} \, dt,$$

by using (1.4), then we easily get the desired result in (3.1). This complete the proof of the Theorem 1. \hfill \Box

If we put $b = d$, then (3.1) reduces to the known integral representation of the extended $\tau$-hypergeometric function $2R_1^\tau(z)$ given by Parmar [6, p.423, eq.(3.1)], as given in the following corollary:

**Corollary 1.1.**

$$2R_1^\tau((\lambda, p), b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} 1_F_0[(\lambda, p); -; zt^\tau] \, dt, \quad (3.3)$$

where $\Re(p) > 0$; $\tau > 0$; $\Re(c) > \Re(b) > 0$ when $p = 0$.

**Remark 3.1.** If we set $\tau = 1$ in above corollary then we obtain the following result in terms of the extended hypergeometric function [9, p.488, eq.(24)]:

$$2F_1((\lambda, p), b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} 1_F_0[(\lambda, p); -; z] \, dt. \quad (3.3)$$

Further, if we take $p = 0$ in (3.3), then we arrive at the known integral representation of the classical Gauss’s hypergeometric function [3, p.19, eq.(1.11.10)], as follows.

$$2F_1(\lambda, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-\lambda} \, dt.$$

**Theorem 2.** The following derivative formula for $3R_2^\tau(z)$ in (2.1) holds true:

$$\left( \frac{d}{dz} \right)^n [z^{c-1} 3R_2^\tau((\lambda, p), a, b; c; d; uz^\tau)] = z^{c-n-1} \frac{\Gamma(c)}{\Gamma(c-n)} 3R_2^\tau[(\lambda, p), a, b; c-n; d; uz^\tau]. \quad (3.4)$$
Proof. By using the series representation of $3R_2^c(z)$ as given in (2.1), and interchanging the order of differentiation and summation, then we arrive at the following:

$$\left( \frac{d}{dz} \right)^n [z^{c-1} 3R_2^c ((\lambda, p), a, b; c; d; \nu z^\tau)]$$

$$= \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} \sum_{m=0}^{\infty} \frac{(\lambda; p)_m}{(c + \tau m) \Gamma(d + \tau m)} \frac{\Gamma(a+\tau m) \Gamma(b+\tau m) \nu^m}{m!} \left( \frac{d}{dz} \right)^n (z^{c+\tau m-1}),$$

now, differentiating term by term under the sign of summation, we have

$$= \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} \sum_{m=0}^{\infty} \frac{(\lambda; p)_m}{(c - n + \tau m) \Gamma(d + \tau m)} \frac{\Gamma(a+\tau m) \Gamma(b+\tau m) \nu^m}{m!} z^{c+\tau m-n-1}$$

$$= z^{c-n-1} \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} \sum_{m=0}^{\infty} \frac{(\lambda; p)_m}{(c - n + \tau m) \Gamma(d + \tau m)} \frac{\Gamma(a+\tau m) \Gamma(b+\tau m) (\nu z^\tau)^m}{m!}.$$  

By using (2.1), we obtain the R.H.S. of (3.4) after little simplifications. This complete the proof.

4. Mellin Transform of the Function $3R_2^c(z)$

The Mellin transform of a function $f(x)$ is defined by

$$\mathcal{M} [f(x); s] = F(s) = \int_0^\infty x^{s-1} f(x) \, dx, \quad (s \in \mathbb{C}),$$

provided that the improper integral in (4.1) exists.

**Theorem 3.** The Mellin transform of the extended $\tau$-hypergeometric function $3R_2^c(z)$ defined in (2.1) is given by

$$\mathcal{M} [3R_2^c ((\lambda, p), a, b; c; d; z); s] = \Gamma(s) \Gamma(\lambda) 3R_2^c (\lambda + s, a, b; c; d; z),$$

where $\Re(s) > 0$ and $\Re(\lambda + s) > 0$ when $p = 0$.

**Proof.** Using (2.1) and applying Mellin transform, then we get

$$\mathcal{M} [3R_2^c ((\lambda, p), a, b; c; d; z); s]$$

$$= \int_0^\infty p^{s-1} \left( \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(c + \tau n) \Gamma(d + \tau n)} \frac{\Gamma(a+\tau n) \Gamma(b+\tau n) z^n}{n!} \right) dp.$$  

Interchanging the order of integration and summation which is permissible, we have

$$= \frac{\Gamma(c) \Gamma(d)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(c + \tau n) \Gamma(d + \tau n)} \frac{\Gamma(a+\tau n) \Gamma(b+\tau n) z^n}{n!} \frac{1}{\Gamma(\lambda)} \int_0^\infty p^{s-1} \Gamma_p (\lambda + n) \, dp.$$  

Now, using the result [2, p.16, eq.(1.110)] given by

$$\int_0^\infty p^{s-1} \Gamma_p (\lambda + n) \, dp = \Gamma(\lambda + s + n) \Gamma(s), \quad (\Re(s) > 0),$$

then we arrive at the following:

$$\mathcal{M} [3R_2^c ((\lambda, p), a, b; c; d; z); s]$$

$$= \frac{\Gamma(s) \Gamma(c) \Gamma(d)}{\Gamma(\lambda) \Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(c + \tau n) \Gamma(d + \tau n)} \frac{\Gamma(a+\tau n) \Gamma(b+\tau n) z^n}{n!} \Gamma(s) \Gamma(c + \tau n) \Gamma(d + \tau n),$$

$$= \frac{\Gamma(s) \Gamma(c) \Gamma(d)}{\Gamma(\lambda) \Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda; p)_n}{(c + \tau n) \Gamma(d + \tau n)} \frac{\Gamma(a+\tau n) \Gamma(b+\tau n) z^n}{n!}.$$

By using the definition of extended $\tau$-hypergeometric function as given by (2.1), we obtain the desired result in (4.2). This complete the proof of the Theorem 3.
5. Fractional Calculus Approach

The Riemann-Liouville left-sided fractional calculus operators $I^\alpha_{a+}$ and $D^\alpha_{a+}$ of order $\alpha$ are defined by [8]

$$I^\alpha_{a+} f(x) = a I^\alpha_{x} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) \, dt, \quad (x > a),$$

is called the Riemann-Liouville left-sided fractional integral of order $\alpha$.

$$D^\alpha_{a+} f(x) = a D^\alpha_{x} f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^{n} \int_{a}^{x} \frac{f(t) \, dt}{(t-x)^{n-\alpha+1}}, \quad (n = [\alpha] + 1),$$

is called the left-sided Riemann-Liouville derivative of order $\alpha$.

In this section we consider the fractional differintegral operators containing extended $\tau$-hypergeometric function $^3R^\tau_2(z)$ as in kernel.

**Theorem 4.** Let $\mu \in \mathbb{R}_+$, $\lambda, \alpha, a, b, c, d, \nu \in \mathbb{C}$ and $\Re(\alpha) > 0$, $\Re(c) > 0$, $\Re(\tau) > 0$, then for $x > \mu$ the following result holds true:

$$\left( I^\alpha_{\mu+} \left[ (t-\mu)^{c-1} \, ^3R^\tau_2 \left( (\lambda, p), a, b; d; \nu (t-\mu)^\tau \right) \right] \right)(x) = \frac{(x-\mu)^{c+\alpha-1}}{\Gamma(c+\alpha)} \, I^\alpha_{\mu+} \left( (t-\mu)^{c+\tau n-1} \right)(x).$$

**Proof.** By using the series representation of extended $\tau$-hypergeometric function $^3R^\tau_2(z)$ as given by (2.1) and interchanging the order of integration and summation, we have

$$\left( I^\alpha_{\mu+} \left[ (t-\mu)^{c-1} \, ^3R^\tau_2 \left( (\lambda, p), a, b; d; \nu (t-\mu)^\tau \right) \right] \right)(x) = \frac{\Gamma(c) \Gamma(d)}{\Gamma(\alpha) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda/p)_n \, \Gamma(a+\tau n) \, \Gamma(b+\tau n) \, \nu^n}{\Gamma(c+\tau n) \, \Gamma(d+\tau n)} \, \frac{1}{n!} \, \left( (t-\mu)^{c+\tau n-1} \right)(x),$$

now for $x > \mu$, taking the following power function formula into account:

$$I^\alpha_{\mu+} (x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \, (x-a)^{\alpha+\beta-1}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0),$$

then we have

$$\left( I^\alpha_{\mu+} \left[ (t-\mu)^{c-1} \, ^3R^\tau_2 \left( (\lambda, p), a, b; d; \nu (t-\mu)^\tau \right) \right] \right)(x) = \frac{(x-\mu)^{c+\alpha-1}}{\Gamma(c+\alpha)} \, \frac{\Gamma(c) \Gamma(d)}{\Gamma(\alpha) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(\lambda/p)_n \, \Gamma(a+\tau n) \, \Gamma(b+\tau n) \, \nu^n}{\Gamma(c+\alpha+\tau n) \, \Gamma(d+\tau n)} \, \frac{1}{n!} \, \left[ (x-\mu)^\tau \right]^n,$$

next, by using the definition (2.1), we obtain the desired result in (5.1).

**Theorem 5.** Let $\mu \in \mathbb{R}_+$, $\lambda, \alpha, a, b, c, d, \nu \in \mathbb{C}$ and $\alpha > 0$, $\Re(c) > 0$, $\Re(\tau) > 0$, then for $x > \mu$ the following relation holds true:

$$\left( D^\alpha_{\mu+} \left[ (t-\mu)^{c-1} \, ^3R^\tau_2 \left( (\lambda, p), a, b; d; \nu (t-\mu)^\tau \right) \right] \right)(x) = \frac{(x-\mu)^{c+\alpha-1}}{\Gamma(c-\alpha)} \, \frac{\Gamma(c)}{\Gamma(c-a)} \, ^3R^\tau_2 \left( (\lambda, p), a, b-\alpha; d; \nu (x-\mu)^\tau \right).$$

**Proof.** By using the series representation given by (2.1) and using the following relation

$$(D^\alpha_{\mu+} f)(x) = \left( \frac{d}{dx} \right)^n (I^{\alpha-n}_{\mu+} f)(x), \quad (\alpha \in \mathbb{C}, \alpha > 0; \, n = [\alpha] + 1),$$
then we have
\[
\left( D_{\mu+}^\alpha \left[ (t - \mu)^{c-1} \, 3R_2^\tau (\lambda, p), a, b; c; d, \nu (t - \mu) \right] \right) (x)
\]
\[
= \left( \frac{d}{dx} \right)^n \left( I_{\mu+}^{n-\alpha} \left[ (t - \mu)^{c-1} \, 3R_2^\tau (\lambda, p), a, b; c; d, \nu (t - \mu) \right] \right) (x),
\]
by using (5.1) and setting \( \alpha = n - \alpha \), then we arrive at the following:
\[
\left( D_{\mu+}^\alpha \left[ (t - \mu)^{c-1} \, 3R_2^\tau (\lambda, p), a, b; c; d, \nu (t - \mu) \right] \right) (x)
\]
\[
= \left( \frac{d}{dx} \right)^n \left( \frac{(x - \mu)^{c+n-\alpha-1} \, \Gamma (c)}{\Gamma (c + n - \alpha)} \, 3R_2^\tau (\lambda, p), a, b; c + n - \alpha; d, \nu (x - \mu) \right). \]

next, by using the result (3.4), we obtain the desired result in (5.2) after little simplification. This complete the proof of the Theorem 5.

**Remark 5.1.** For the detail of fractional calculus operators the reader can refer the work (see, [4, 5]).

### 6. Concluding Remarks

In the present paper we derive a new extended \( \tau \)-hypergeometric function \( 3R_2^\tau (z) \). Our results motivated mainly by investigations of the \( \tau \)-hypergeometric function \( 2R_1^\tau (z) \) [11] and its extension [6]. We obtained certain integral representation, a derivative formula, Mellin transform and fractional calculus approach of this new extended \( \tau \)-hypergeometric function \( 3R_2^\tau (z) \). The provided results are new and have uniqueness identity in the literature. On account of the general nature of the extended \( \tau \)-hypergeometric function, a number of known results can easily be found as special cases of our main results.

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### References


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