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A Modified Expansion Approach for Capturing Nonlinearity and Dispersion in the Benney–Luke Equation

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Research Article

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Abstract

This study investigates traveling wave solutions of the Benney–Luke equation, a higher-order dispersive model for shallow-water wave propagation. A modified expansion method combining an auxiliary function approach with Jacobi elliptic functions is proposed to obtain explicit analytical solutions. The derived solutions exhibit various wave forms depending on key parameters, and their accuracy is supported by numerical simulations. The results confirm the method's effectiveness in capturing nonlinear and dispersive effects in shallow-water wave dynamics.

Keywords: Benney–Luke Equation, Traveling Wave Solutions, Modified Expansion Technique, Jacobi Elliptic Functions

Benney–Luke Denklemi'nde Doğrusal Olmayanlık ve Dağılımı Yakalamak için Modifiye Edilmiş bir Genişleme Yaklaşımı

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Öz

Bu çalışma, sığ su dalga yayılımını modelleyen daha yüksek mertebeden bir dispersif model olan Benney–Luke denkleminin hareketli dalga çözümlerini incelemektedir. Açık biçimli analitik çözümler elde etmek için yardımcı fonksiyon yaklaşımı ile Jacobi eliptik fonksiyonlarını birleştiren değiştirilmiş bir genişletme yöntemi önerilmiştir. Elde edilen çözümler, parametrelere bağlı olarak çeşitli dalga formları göstermekte olup, doğrulukları sayısal simülasyonlarla desteklenmiştir. Sonuçlar, yöntemin doğrusal olmayanlık ve dispersiyon etkilerini başarıyla yakaladığını göstermektedir.

Anahtar Kelimeler: Benney-Luke Denklemi, Seyreden Dalga Çözümleri, Modifiye Genişleme Tekniği, Jacobi Eliptik Fonksiyonlar

Introduction

The exploration of nonlinear wave equations occupies a central place in mathematical physics, bridging theoretical frameworks with real-world phenomena observed in disciplines such as fluid dynamics, optics, and plasma physics. Within this rich landscape, the Benney–Luke equation emerges as a sophisticated and versatile model tailored to the dynamics of water waves in shallow-water regimes. First introduced in 1964 [1], this equation represents a significant advancement over earlier models by providing a higher-order approximation of the Euler equations for inviscid, irrotational fluid flow. Unlike the widely studied Korteweg-de Vries (KdV) equation, which excels at describing unidirectional wave propagation [2, 3], the Benney–Luke equation captures bidirectional wave motion while accounting for both gravitational forces and surface tension effects. This dual capability positions it as a critical tool for studying complex wave interactions where capillary influences often overlooked in simpler shallow-water theories cannot be ignored, such as in the propagation of small-amplitude waves or in fluids with pronounced surface tension characteristics [4]. Central to this research is the exploration of traveling wave solutions for the Benney–Luke equation. These solutions characterize waves that advance at a constant velocity while maintaining their form, a feature that makes them particularly valuable for modeling coherent wave patterns such as solitons, cnoidal waves, or periodic wave trains [5, 6]. Within the framework of the Benney–Luke equation, such solutions provide a means to investigate the interplay between nonlinear and dispersive processes that sustain stable, propagating waveforms in shallow-water systems where bidirectional wave motion and capillary effects are significant [7]. The importance of traveling wave solutions extends beyond their theoretical appeal [8]. Physically, they correspond to observable phenomena such as solitary waves breaking on a shore or periodic wave patterns in controlled experiments [9]. Mathematically, they simplify the analysis of complex partial differential equations (PDEs) by reducing them to ordinary differential equations (ODEs) through a traveling wave transformation [10]. For the Benney–Luke equation, this transformation involves adopting a moving reference frame defined by a wave variable. This approach renders the wave profile stationary in the transformed coordinate, allowing us to focus on its spatial structure while implicitly accounting for its temporal evolution. To uncover these traveling wave solutions, we introduce a modified expansion technique that represents a novel synthesis of traditional expansion methods and the properties of Jacobi elliptic functions [11]. This hybrid approach draws inspiration from recent innovations in the field while extending the methodology to address the unique challenges posed by the Benney–Luke equation [12]. Traditional expansion techniques, such as perturbation methods or polynomial trivial solutions, often excel at handling specific classes of solutions [13] e.g., exponential decays or power-series approximations but may struggle to capture the full spectrum of wave behaviors exhibited by equations with mixed dispersive [14] and nonlinear terms [15]. In contrast, our modified technique integrates an auxiliary function-based expansion with the periodic and quasi-periodic properties of Jacobi elliptic functions [16], specifically the elliptic sine function (sn function), to achieve a more versatile and comprehensive solution framework [17]. The present method constructs a trial solution comprising two synergistic components: an auxiliary function that governs the wave's decay or growth characteristics and a Jacobi elliptic function that introduces periodic oscillations. The modified expansion technique offers several advantages over conventional methods. First, its hybrid nature [18]allows it to capture a broader range of wave behaviors than standalone approaches [19], our method bridges these regimes by leveraging the elliptic modulus as a tuning parameter [20]. Lower values of m produce cnoidal waves with gentle, periodic oscillations, while higher values yield solitary like profiles with localized peaks and extended tails a flexibility that mirrors the physical transitions observed in shallow-water wave phenomena as environmental conditions (e.g., depth or surface tension) vary. Second, the inclusion of auxiliary parameters enhances the strongness of the method, enabling fine adjustments to ensure mathematical consistency and physical relevance [21]. These parameters, determined through the algebraic constraints, act as degrees of freedom that stabilize the solution across the spatial domain, preventing singularities or unphysical artifacts. This adaptability is particularly valuable for the Benney–Luke equation [22], whose mixed derivatives and nonlinear terms introduce complexities [23] that demand a nuanced analytical approach [24]. Third, the technique's reliance on Jacobi elliptic functions provides a natural framework for exploring wave periodicity and amplitude, key features in shallow-water dynamics. By systematically varying the elliptic modulus and auxiliary parameters, we can map out a continuum of solutions, revealing how wave profiles evolve from one regime to another [25]. For instance, a modest increase in m might transform a wave train with uniform crests into a solitary wave with a pronounced peak, offering insights into the conditions under which different wave types emerge a capability with direct implications for interpreting experimental or field data [26]. To substantiate our analytical findings, we complement the derivation with computational experiments that include two- and three-dimensional graphical visualizations. These tools serve multiple purposes: they validate the stability of the derived solutions by demonstrating that the wave profiles maintain their coherence as they propagate over time; they illustrate the dynamic interplay between exponential decay (from the auxiliary function) and elliptic oscillations (from the sn function); and they provide an intuitive understanding of how parameter variations influence the wave's shape and behavior. This research contributes to the field of nonlinear wave equations by providing an in depth examination of traveling wave solutions for the Benney–Luke equation, utilizing a novel methodological approach. The modified expansion technique not only enhances our comprehension of the equation's mathematical framework but also highlights its significance in physical systems where shallow water waves are of critical importance. The applications are extensive in coastal engineering, where bidirectional wave models are essential for designing breakwaters and harbors; in oceanography, where capillary effects impact wave dissipation; and in experimental fluid dynamics, where accurate analytical solutions aid in interpreting laboratory observations.

This paper is structured as follows: Methodology section describes the modified expansion technique used to obtain the traveling wave solutions, highlighting the integration of the auxiliary function method with Jacobi elliptic functions for the Benney-Luke equation. In section Results and Discussion, the analytical findings are presented and discussed, with particular emphasis on how parameter variations influence the solution structure. Moreover, it complements the analytical results with computational visualizations, demonstrating the stability and powerful of the derived wave profiles. Finally, the paper is concluded with a summary of the main contributions and suggestions for future research directions.

Methodology

The Benney–Luke equation, derived as a higher-order asymptotic model for water-wave propagation in shallow-water regimes, is mathematically expressed as

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} + u_t\, u_{xx} + 2\, u_x\, u_{xt} = 0, \tag{1}$$

where $u(x,t)$ denotes the deviation of the free surface from its undisturbed state, $x \in \mathbb{R}$ is the spatial coordinate, and $t \in \mathbb{R}$ represents time. This equation is obtained via a systematic asymptotic expansion of the full Euler equations under the assumptions of weak nonlinearity and long wavelengths relative to the water depth, leading to a balance between nonlinear effects and higher-order dispersive corrections [27, 28].

The term u_{tt} captures the inertial response of the fluid, while u_{xx} represents the leading-order linear dispersive propagation. The inclusion of the fourth-order derivative u_{xxxx} and the mixed derivative u_{xxtt} provides refined descriptions of dispersion that are critical for accurately modeling long-wave phenomena in scenarios where simpler models, such as the Korteweg–de Vries or Boussinesq equations, are insufficient [29, 30]. Furthermore, the nonlinear terms $u_t\,u_{xx}$ and $2\,u_x\,u_{xt}$ embody the intricate interactions between the wave's temporal evolution and its spatial curvature, thereby facilitating the modulation of wave amplitude and phase.

In this work, traveling wave solutions of the Benney–Luke equation given in Equation 1 are studied by reducing it to an ordinary differential equation through the transformation $\xi = x - Vt$. To derive explicit analytical solutions, we employ a modified expansion technique that integrates the $(m+\frac{1}{G'})$-method with Jacobi elliptic functions, notably the elliptic sine function $\mathrm{sn}(\xi, m)$. Jacobi elliptic functions generalize trigonometric functions by incorporating elliptic integrals, yielding a family of functions that include the sn, cn (elliptic cosine), and dn (delta amplitude) functions. These functions are inherently periodic and parametrized by an elliptic modulus m (where $0 \leq m \leq 1$), which governs their shape and behavior. For instance, when $m \to 0$, the sn function approaches the standard sine function, producing sinusoidal oscillations; as $m \to 1$, it resembles the hyperbolic tangent function, characteristic of solitary waves. This adaptability makes Jacobi elliptic functions an ideal tool for modeling the diverse wave profiles encountered in the Benney–Luke equation, from gently undulating cnoidal waves to sharply peaked solitary structures. This approach facilitates a systematic balance between the nonlinear and dispersive terms inherent in the equation, yielding a traveling wave solution that encompasses both solitary and periodic characteristics. The incorporation of Jacobi elliptic functions enriches the solution structure, allowing the construction of hybrid waveforms that are capable of capturing the intricate interplay between nonlinearity and dispersion in shallow-water wave phenomena.

By introducing the traveling wave variable $\xi = x - Vt$, The Benney–Luke equation is reduced to an ordinary differential equation of the form

$$(V^2 - 1)U' + (1 - V^2)U''' - \frac{3V}{2}(U')^2 = 0, \tag{2}$$

where $U(\xi) = u(x,t)$ and $U' = \frac{dU}{d\xi}$. It is then proposed that the solution is constructed using the solution

$$U(\xi) = a_{-1}\left(m + \frac{1}{G'(\xi)}\right)^{-1} + a_0 + a_1\left(m + \frac{1}{G'(\xi)}\right) + b_1\,\mathrm{sn}(\xi, m), \tag{3}$$

with the auxiliary function $G(\xi)$ satisfying

$$G''(\xi) + \lambda\,G'(\xi) + \mu\,G(\xi) = 0. \tag{4}$$

By balancing the highest-order derivative U''' and the nonlinear term $(U')^2$, the relation

$$n + 3 = 2(n + 1) \tag{5}$$

is obtained, from which it follows that $n = 1$. For the elliptic contributions arising from $\mathrm{sn}(\xi, m)$, the parameter is set as $p = 1$, so that the total balance degree is $n + p = 2$.
Subsequently, the proposed trivial solution is substituted into the reduced ordinary differential equation. In doing so, the derivatives of the Jacobi elliptic sine function are computed using the standard identities

$$\frac{d}{d\xi}\mathrm{sn}(\xi, m) = \mathrm{cn}(\xi, m)\,\mathrm{dn}(\xi, m), \quad \frac{d^2}{d\xi^2}\mathrm{sn}(\xi, m) = -(1 + m)\,\mathrm{sn}(\xi, m) + 2m\,\mathrm{sn}^3(\xi, m), \tag{6}$$

and the derivatives of the term $\left(m + \frac{1}{G'(\xi)}\right)$ are calculated by differentiating appropriately and applying the auxiliary equation. After substituting and organizing the resulting expression by collecting terms with like powers of $\left(m + \frac{1}{G'(\xi)}\right)^i$ and $\mathrm{sn}^j(\xi, m)$, the method of undetermined coefficients is applied to set the coefficients of each independent term to zero.
For instance, by equating the coefficient of $\mathrm{sn}^3(\xi, m)$ to zero, one obtains the algebraic relation

$$\theta\, b_1^3 + 2c^2 m\, b_1 = 0, \tag{7}$$

which implies, under the assumption of nontrivial b_1, that

$$b_1 = \pm\sqrt{-\frac{2c^2 m}{\theta}}. \tag{8}$$

Similarly, by balancing the coefficient of $\mathrm{sn}(\xi, m)$, the equation

$$-c^2(1 + m)b_1 - \theta\, b_1 + 3\theta\, a_0^2 b_1 = 0 \tag{9}$$

is obtained, leading to the determination of a_0 via

$$a_0 = \pm\sqrt{\frac{c^2(1 + m) + \theta}{3\theta}}. \tag{10}$$

Furthermore, equating the coefficient of the cubic term $\left(m + \frac{1}{G'(\xi)}\right)^3$ to zero yields

$$-2c^2\mu^2\, a_1 + \theta\, a_1^3 = 0, \tag{11}$$

from which it follows that

$$a_1 = \pm\sqrt{\frac{2c^2\mu^2}{\theta}}. \tag{12}$$

Thus, the final form of the solution is expressed as

$$\begin{aligned} u(x,t) &= a_{-1}\left(m+\frac{\lambda+2m\mu}{-\mu+(\lambda+2m\mu)A_1\,e^{-(\lambda+2m\mu)\xi}}\right)^{-1}+a_0 \\ &\quad +a_1\left(m+\frac{\lambda+2m\mu}{-\mu+(\lambda+2m\mu)A_1\,e^{-(\lambda+2m\mu)\xi}}\right)+b_1\,\mathrm{sn}(\xi,m), \end{aligned} \tag{13}$$

where A_1 denotes an integration constant and $\xi = x - Vt$. This solution exemplifies the hybrid structure in which the exponential/polynomial part, derived from the $\left(m+\frac{1}{G'(\xi)}\right)$-expansion method, is combined with the elliptic contribution provided by the Jacobi elliptic function, thereby capturing the complex interplay between nonlinearity and dispersion in the governing equation.

A careful selection of parameters to ensure that the Benney–Luke equation yields mathematically consistent and physically plausible traveling wave solutions. In this context, the traveling wave speed V is typically chosen to reflect the characteristic celerity of shallow-water waves, for example, by invoking the relation $V \sim \sqrt{gh}$, where g denotes the acceleration due to gravity and h represents the characteristic water depth. The elliptic modulus m appearing in the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ is constrained to the interval $0 < m < 1$; values of m near unity favor the emergence of solitary wave profiles, whereas lower values give rise to more oscillatory, cnoidal-type solutions. Concurrently, the auxiliary parameters λ and μ are chosen such that the auxiliary function $G(\xi)$, which satisfies

$$G''(\xi)+\lambda\,G'(\xi)+\mu\,G(\xi)=0, \tag{14}$$

and its derivative $G'(\xi)$ remain uniformly bounded over the domain of interest, thereby ensuring that the composite term

$$m+\frac{1}{G'(\xi)} \tag{15}$$

remains free of singular behavior. The coefficients a_{-1}, a_0, a_1, and b_1 in the solution

$$U(\xi)=a_{-1}\left(m+\frac{1}{G'(\xi)}\right)^{-1}+a_0+a_1\left(m+\frac{1}{G'(\xi)}\right)+b_1\,\mathrm{sn}(\xi,m) \tag{16}$$

are then determined by enforcing a balance between the highest-order derivatives and the nonlinear terms in the reduced ordinary differential equation. For instance, the relation

$$n+3=2(n+1) \tag{17}$$

implies $n = 1$, and subsequent algebraic constraints such as

$$b_1=\pm\sqrt{-\frac{2c^2m}{\theta}},\quad a_0=\pm\sqrt{\frac{c^2(1+m)+\theta}{3\theta}},\quad a_1=\pm\sqrt{\frac{2c^2\mu^2}{\theta}} \tag{18}$$

further restrict the parameter values. By fixing an integration constant (e.g., $A_1 = 1$) and adjusting λ and

μ appropriately, an auxiliary function is obtained,

$$F(\xi) = m + \frac{\lambda + 2m\mu}{-\mu + (\lambda + 2m\mu)A_1\, e^{-(\lambda+2m\mu)\xi}}, \tag{19}$$

which remains bounded and precludes singular behavior in the final solution.

Results and Discussion

In the first case, the parameter values are chosen as $m = 0.5$, $\lambda = -0.2$, and $\mu = 0.1$. With this moderate elliptic modulus, the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ yields a cnoidal wave form characterized by periodic oscillations with smooth peaks. The exponential component, governed by the term $m + \frac{1}{G'(\xi)}$, interacts with the oscillatory behavior in such a way that the parameters λ and μ effectively balance the exponential decay and the inherent oscillations. As a result, the traveling wave solution exhibits sustained amplitude and moderate wavelength, thereby capturing the essential features of the nonlinear dispersive dynamics inherent in the Benney–Luke equation.

The resulting traveling wave solution for first case is given in Figures 1 and 2.

The resulting profile exhibits a cnoidal wave structure, characterized by a series of smooth, gently rounded peaks and troughs. This waveform arises from the interplay between the elliptic contribution, captured by $\mathrm{sn}(\xi, m)$ with $m = 0.5$, and the exponential/polynomial terms derived from the auxiliary function $G(\xi)$. The chosen elliptic modulus $m = 0.5$ places the solution squarely in the cnoidal regime rather than approaching a purely sinusoidal or solitary wave, while the parameters $\lambda = -0.2$ and $\mu = 0.1$ effectively regulate the balance between exponential decay and oscillatory behavior. The moderate amplitude and gentle periodicity of the wave are evident in the 2D plot, indicating that the nonlinear and dispersive effects inherent to the Benney–Luke equation are harmonized in such a way that the wave retains a stable, repeating structure over the displayed domain of ξ.

In the three-dimensional view, the same wave solution is rendered in terms of the original spatial and temporal coordinates x and t. This depiction reveals that the wave maintains its form as it propagates, with each crest and trough traveling at the speed V without noticeable attenuation or distortion. The color scale underscores the amplitude distribution across both x and t, confirming that the wave remains nearly uniform in shape and amplitude over the chosen simulation interval. The presence of several wave crests and troughs, each aligned along the diagonal in the (x, t)-plane, illustrates the steady nature of the traveling wave and reaffirms the balance of nonlinearity and dispersion established by the parameters. The exponential term in the auxiliary function, governed by λ and μ, ensures that the wave profile does not develop spurious oscillations or blow-up phenomena, while the elliptic term, modulated by $m = 0.5$, imparts the gentle yet sustained periodicity.

In the second case, a reduction of the elliptic modulus to $m = 0.3$ is employed. This smaller value renders the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ closer to a sinusoidal profile, thereby inducing higher-frequency oscillations in the solution. Furthermore, the corresponding reduction in the parameters λ and μ softens the exponential decay within the composite function $m + \frac{1}{G'(\xi)}$, leading to a traveling wave solution with lower amplitude and a more pronounced periodic structure. Consequently, the overall wave profile closely resembles that of a modulated sine wave, where the oscillatory behavior dominates due to the reduced nonlinearity relative to the dispersion.

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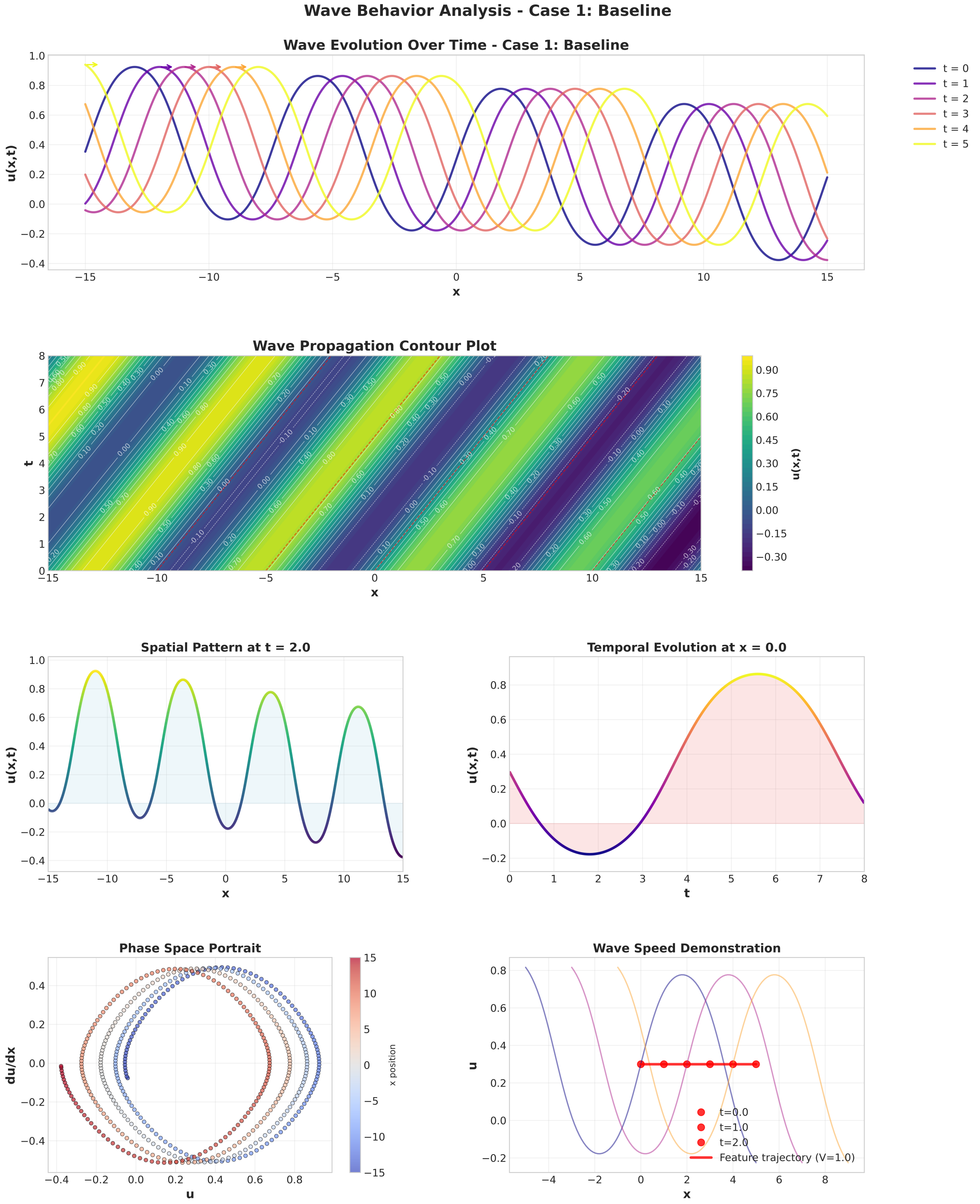

Figure 1. *Wave evolution analysis for Case 1: Baseline, with parameters* $m = 0.5$, $\lambda = -0.2$, $\mu = 0.1$, $A_1 = 1.0$, $a_{-1} = -0.3$, $a_0 = 0.4$, $a_1 = 0.2$, $b_1 = 0.5$, $V = 1.0$

The resulting traveling wave solution for second case is given in Figures 3 and 4.

In the two-dimensional representation, the solution is displayed as a function of $\xi = x - Vt$, with a smaller elliptic modulus $m = 0.3$. This value of m causes the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ to exhibit more sinusoidal characteristics, resulting in wave crests and troughs that appear more uniform in height

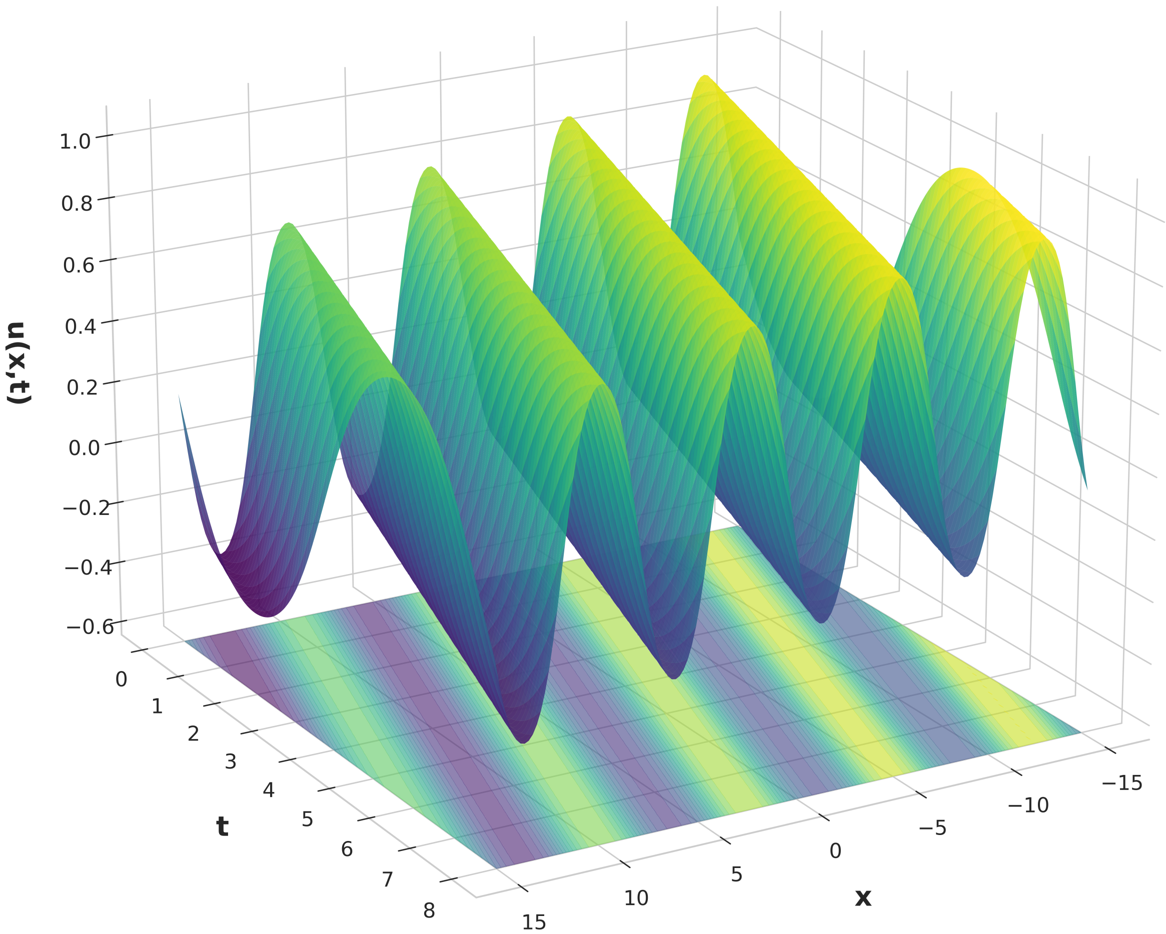

Figure 2. *Traveling Wave solution for Case 1: Baseline, with parameters* $m = 0.5$, $\lambda = -0.2$, $\mu = 0.1$, $A_1 = 1.0$, $a_{-1} = -0.3$, $a_0 = 0.4$, $a_1 = 0.2$, $b_1 = 0.5$, $V = 1.0$

and spacing compared to the baseline case. The oscillations are relatively higher in frequency, and the overall amplitude is somewhat reduced, indicating that the balance between nonlinearity and dispersion now favors a more rapid temporal and spatial variation. The auxiliary parameters λ and μ have been adjusted accordingly, ensuring that the exponential/polynomial contribution derived from the auxiliary function $G(\xi)$ does not overwhelm the oscillatory component, thereby preserving the wave's periodic structure over the plotted domain of ξ.

In the three-dimensional view, the same traveling wave solution is presented in terms of the original coordinates x and t. The color map highlights how the wave maintains its shape and amplitude as it propagates, with each crest moving steadily in time and only slight modulations along the propagation direction. The relatively higher frequency of oscillation is clearly visible, as is the wave's reduced crest height compared to the baseline case. Despite these sharper oscillations, the solution remains stable,

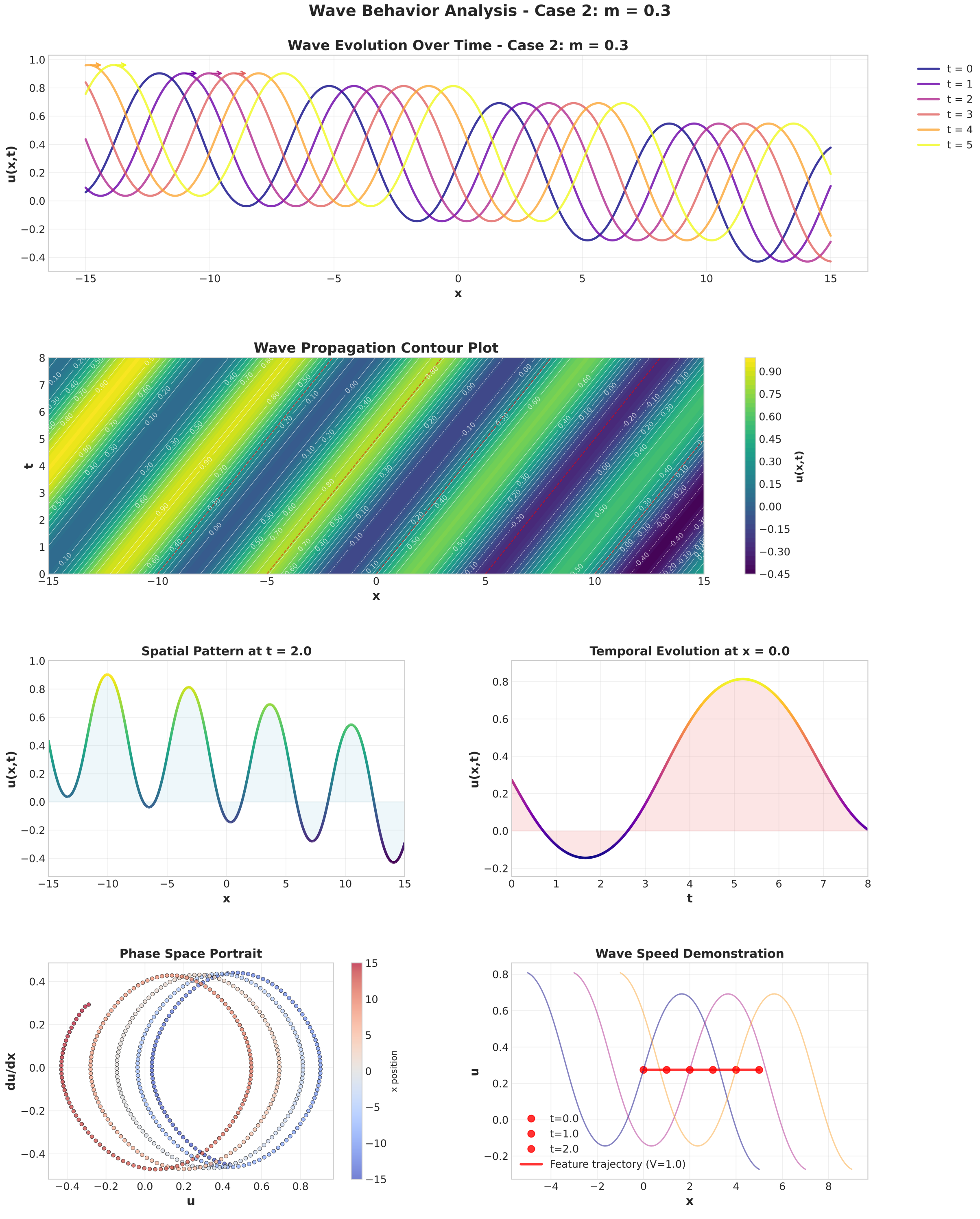

Figure 3. *Wave evolution analysis for Case 2: Lower elliptic modulus, with parameters* $m = 0.3$, $\lambda = -0.15$, $\mu = 0.08$, $A_1 = 1.0$, $a_{-1} = -0.25$, $a_0 = 0.35$, $a_1 = 0.25$, $b_1 = 0.45$, $V = 1.0$

and no blow-up or decay is evident over the selected range of x and t. The interplay between the smaller elliptic modulus and the chosen exponential parameters ensures that the traveling wave retains a consistent sinusoidal-like form, illustrating how a reduction in m can induce a more oscillatory solution to the Benney–Luke equation.

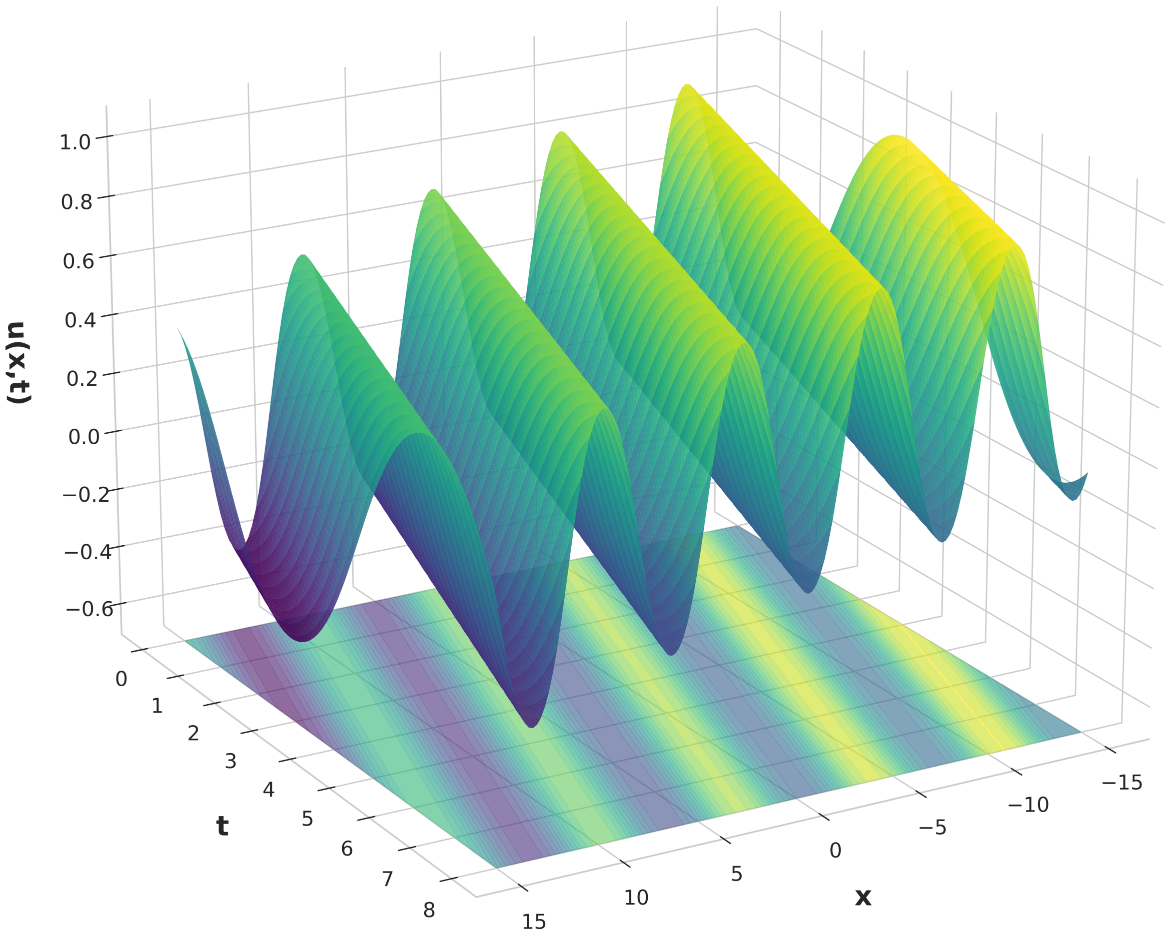

Figure 4. *Traveling Wave solution for Case 2: Lower elliptic modulus, with parameters* $m = 0.3$, $\lambda = -0.15$, $\mu = 0.08$, $A_1 = 1.0$, $a_{-1} = -0.25$, $a_0 = 0.35$, $a_1 = 0.25$, $b_1 = 0.45$, $V = 1.0$

In the third case, the elliptic modulus is increased to $m = 0.7$, causing the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ to approach a solitary wave profile as it tends towards the limiting behavior observed when $m \to 1$. This increase in m produces sharper peaks and broader troughs, indicating a transition from periodic to solitary wave dynamics. Simultaneously, larger values of λ and μ are chosen to enhance the exponential terms, thereby steepening the wavefront and localizing the wave energy. The resulting traveling wave solution thus embodies characteristics typical of a solitary wave, with the interplay between the exponential decay and the elliptic component emphasizing the nonlinear and dispersive balance within the equation.

The resulting traveling wave solution for third case is given in Figures 5 and 6.

In the two-dimensional representation, the solution is plotted as a function of $\xi = x - Vt$ with a higher elliptic modulus $m = 0.7$. This choice of m causes the Jacobi elliptic function $\mathrm{sn}(\xi, m)$ to approach a

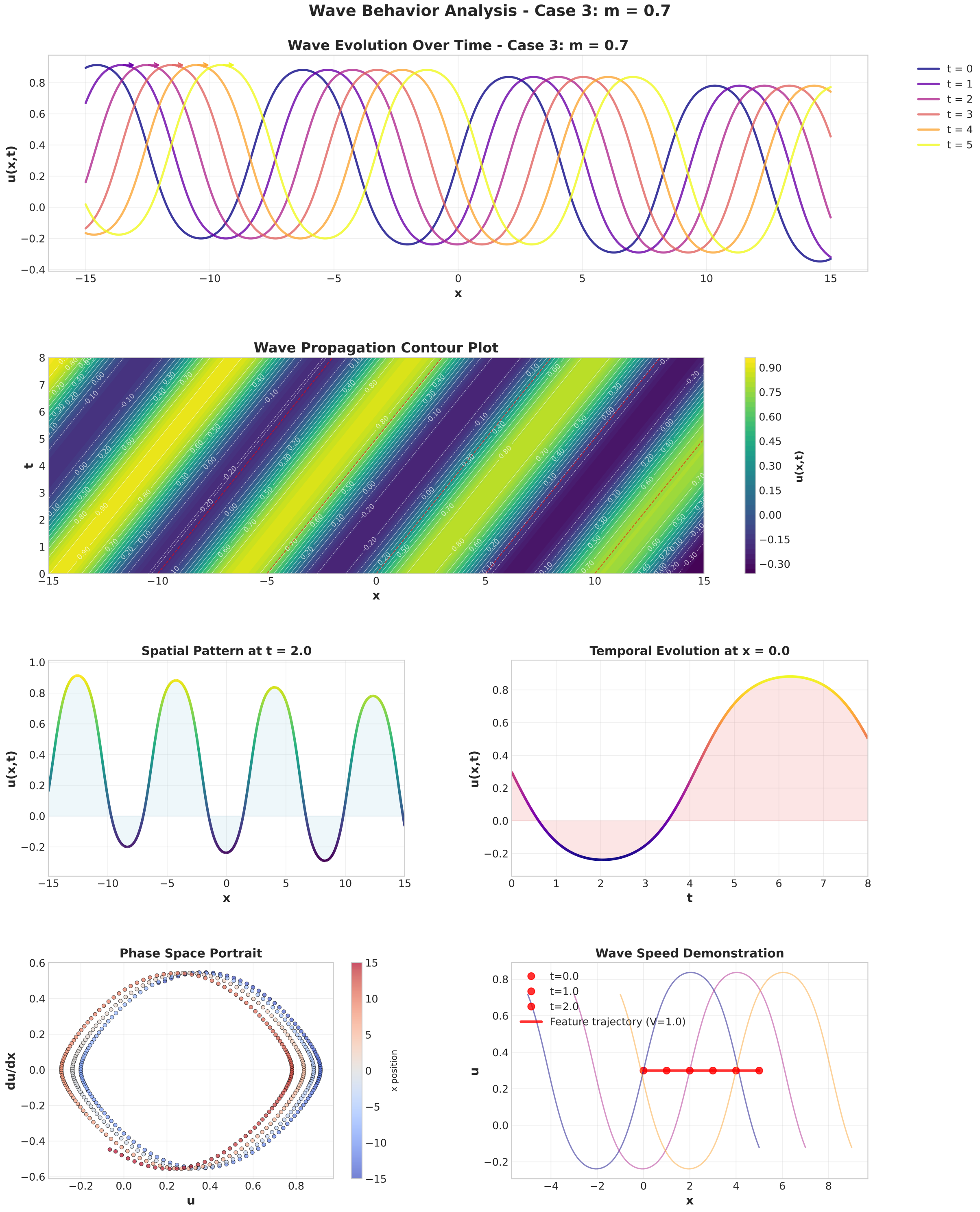

Figure 5. *Wave evolution for Case 3: Higher elliptic modulus, with parameters* $m = 0.7$, $\lambda = -0.25$, $\mu = 0.12$, $A_1 = 1.0$, $a_{-1} = -0.35$, $a_0 = 0.45$, $a_1 = 0.15$, $b_1 = 0.55$, $V = 1.0$

solitary-like profile, manifested in sharper peaks and broader troughs compared to the lower-modulus cases. The exponential and polynomial contributions arising from the auxiliary function $G(\xi)$ are calibrated so that the resulting traveling wave exhibits a more localized structure, where the crests stand out prominently and the wave amplitude remains relatively large over each period. The interaction

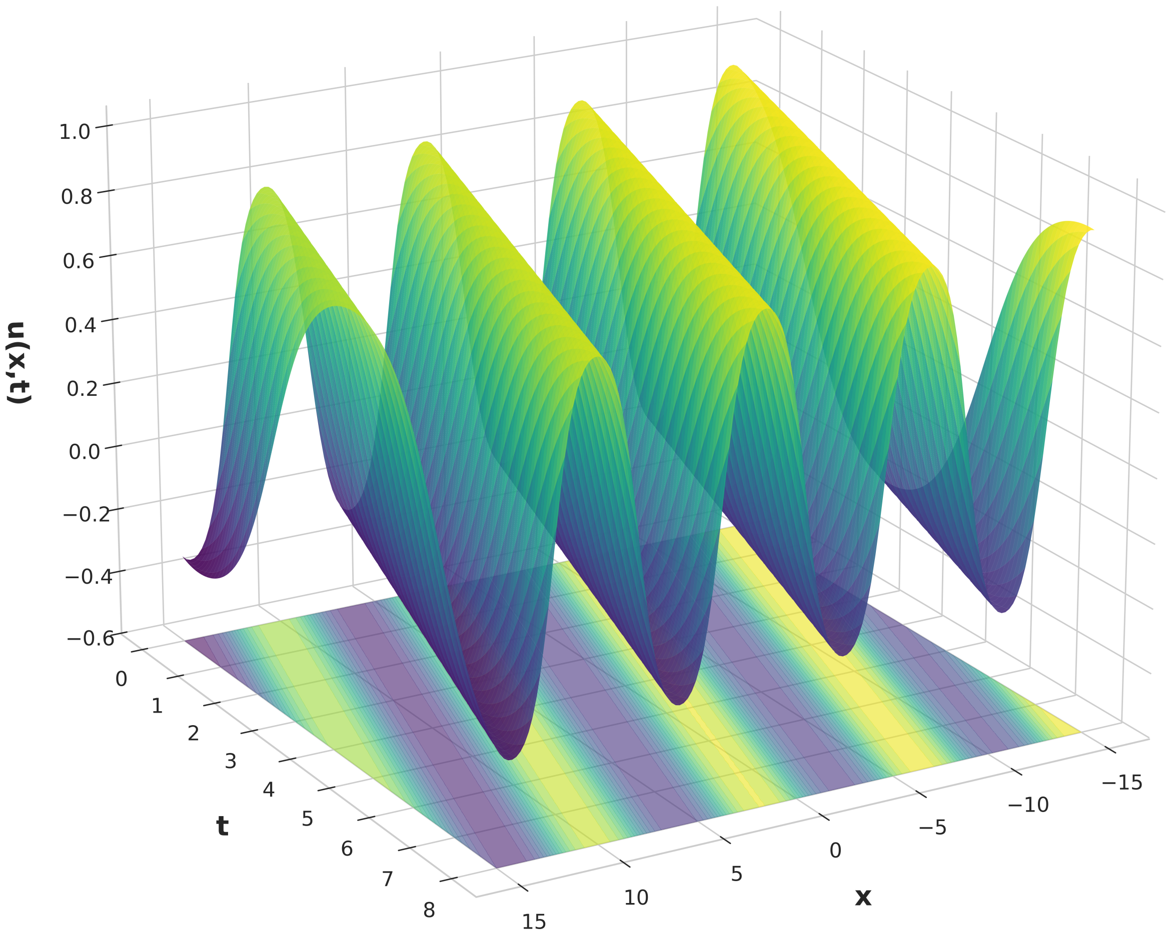

Figure 6. *Traveling Wave solution for Case 3: Higher elliptic modulus, with parameters* $m = 0.7$, $\lambda = -0.25$, $\mu = 0.12$, $A_1 = 1.0$, $a_{-1} = -0.35$, $a_0 = 0.45$, $a_1 = 0.15$, $b_1 = 0.55$, $V = 1.0$

between the stronger nonlinear effects implied by $m = 0.7$ and the dispersive terms ensures that the wave maintains coherence, preventing it from fragmenting into multiple smaller oscillations or decaying into a near-sinusoidal pattern.

In the three-dimensional perspective, the wave is shown evolving over both space x and time t, illustrating the persistence of these pronounced peaks and the gradual flattening of the troughs as the wave propagates. The color shading accentuates the increased amplitude of the primary crests, reinforcing the observation that the solution moves closer to solitary wave characteristics when m is larger. The parameters λ and μ have been selected to enhance the exponential component in such a way that the wavefront remains steep and localized, indicating that the balance between nonlinearity and dispersion is shifted toward sustaining higher crests over a longer spatial extent. Consequently, the wave retains its integrity without developing secondary oscillations, highlighting the crucial role of the elliptic modulus in shaping the

transition between periodic and solitary wave behaviors within the Benney–Luke equation.
In the fourth case, the parameters λ and μ are adjusted to $\lambda = -0.3$ and $\mu = 0.15$ while maintaining the elliptic modulus at $m = 0.5$. The increased magnitudes of $|\lambda|$ and μ amplify the exponential term in the composite function

$$F(\xi) = m + \frac{\lambda + 2m\mu}{-\mu + (\lambda + 2m\mu)A_1\, e^{-(\lambda+2m\mu)\xi}}, \tag{20}$$

resulting in a more rapid spatial decay. This enhanced decay leads to a steeper wave profile with a prominent primary peak and diminished secondary oscillations. The modified balance between dispersion and nonlinearity, as manifested through these adjusted exponential parameters, yields a wave solution that is both more stable and localized, thereby demonstrating the sensitivity of the solution structure to variations in the auxiliary parameters.
The resulting traveling wave solution for fourth case is given in Figures 7 and 8.
In the two-dimensional view, the solution is presented in terms of the traveling wave variable $\xi = x - Vt$ for the elliptic modulus $m = 0.5$ and the auxiliary parameters $\lambda = -0.3$ and $\mu = 0.15$. This choice of λ and μ amplifies the exponential denominator in the term $m + \frac{1}{G'(\xi)}$, producing a faster spatial decay in certain components of the wave. The resulting waveform exhibits peaks that are slightly steeper than in the baseline case, yet the overall amplitude remains moderate and consistent over each period. The reduced prominence of secondary oscillations suggests that the dispersion and nonlinearity are balanced in a manner that emphasizes the primary wave crest, allowing the wave profile to remain stable without developing pronounced ripples between successive crests.
In the three-dimensional representation, the wave is shown propagating in space and time with a steady form, as the diagonal ridges in the (x, t)-plane indicate that the crest velocity is maintained throughout the domain. The color shading underscores the relative steepness of the crests while also illustrating the moderate amplitude of the troughs, confirming that the choice of λ and μ leads to a solution where the exponential component quickly diminishes away from each crest. This rapid decay prevents the accumulation of additional oscillatory energy behind the main wavefront, ensuring that the wave does not exhibit the extended tailing observed in cases with lower values of $|\lambda|$ or μ. Consequently, the solution demonstrates how an increased magnitude of these auxiliary parameters can favor localized, stable waves that retain a cnoidal character but with sharper peaks and reduced secondary undulations.

Conclusion

In this study, we have developed and applied a modified $\left(m + \frac{1}{G'}\right)$-expansion technique, integrated with the properties of Jacobi elliptic functions, to construct explicit traveling wave solutions of the Benney–Luke equation a higher order dispersive model for shallow water wave dynamics. This hybrid approach successfully captures a diverse family of waveforms, ranging from nearly sinusoidal and cnoidal waves to sharply peaked solitary structures, through the modulation of key parameters such as the elliptic modulus m and auxiliary coefficients λ and μ.
The analytical solutions demonstrate how variations in m influence the transition between periodic and localized waveforms: lower values generate smoother, oscillatory profiles, while higher values yield more localized, solitary-like waves with greater amplitude and steeper gradients. The auxiliary

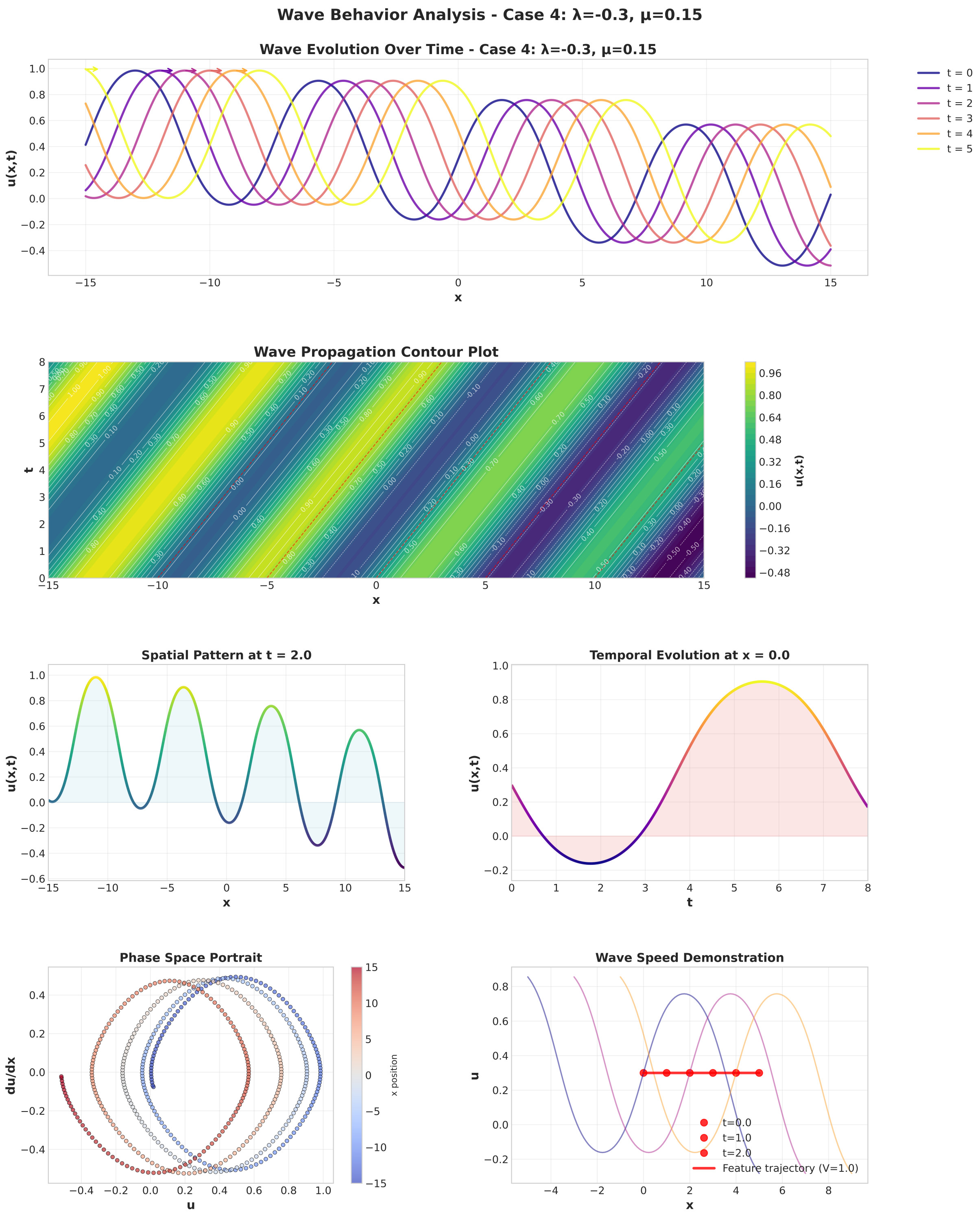

Figure 7. *Traveling Wave solution for Case 4: Variation in* λ *and* μ*, with parameters* $m = 0.5$*,* $\lambda = -0.3$*,* $\mu = 0.15$*,* $A_1 = 1.0$*,* $a_{-1} = -0.4$*,* $a_0 = 0.5$*,* $a_1 = 0.2$*,* $b_1 = 0.5$*,* $V = 1.0$

parameters provide additional degrees of freedom that refine the solution's structure, enabling precise control over features such as amplitude, steepness, and decay behavior. Computational visualizations confirm the consistency and stability of the obtained solutions across different parametric regimes.

The proposed method not only contributes a versatile analytical framework for solving the Benney–Luke

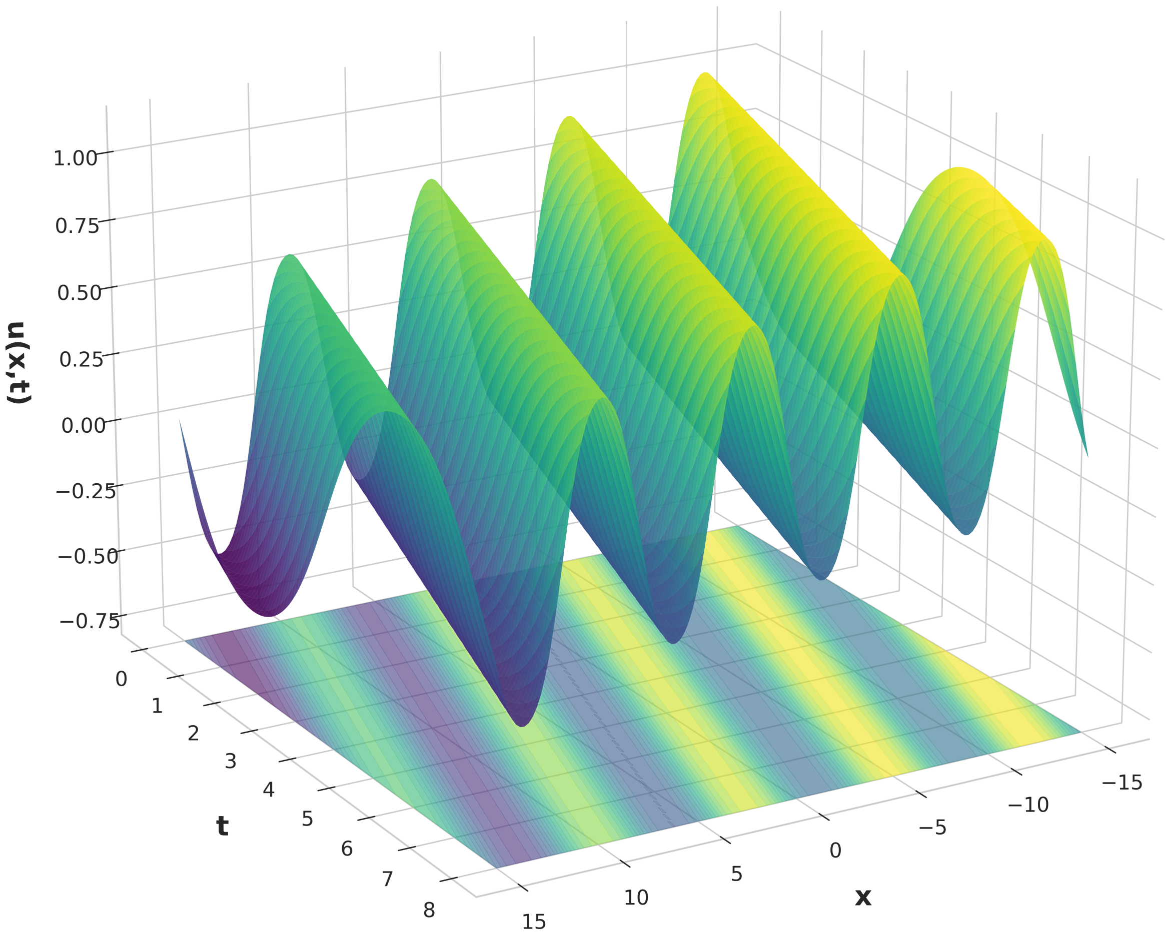

Figure 8. *Traveling Wave solution for Case 4: Variation in* λ *and* μ*, with parameters* $m = 0.5$, $\lambda = -0.3$, $\mu = 0.15$, $A_1 = 1.0$, $a_{-1} = -0.4$, $a_0 = 0.5$, $a_1 = 0.2$, $b_1 = 0.5$, $V = 1.0$

equation but also provides deeper insights into the interplay between nonlinearity and dispersion in shallow-water environments. By systematically mapping the solution space, this study sheds light on the mechanisms governing wave profile evolution and offers a generalized platform for investigating similar nonlinear dispersive systems.

Looking forward, several promising directions for future research emerge. The methodology could be extended to higher-dimensional generalizations of the Benney–Luke equation, where additional spatial variables may yield more complex wave phenomena. Incorporating realistic physical factors such as variable bottom topography, fluid viscosity, or dynamic surface tension would enhance the model's applicability to experimental and field settings.

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Authors Contribution The author read and approved the final manuscript.

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